INVERSE FACTORIAL SERIES: A LITTLE KNOWN TOOL FOR THE SUMMATION OF DIVERGENT SERIES

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Approximation and Extrapolation of Convergent and Divergent Sequences and Series
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"Rediscovery" of Factorial Series

▷ In 1985/6, I tried to understand Levin's sequence transformation whose input data are not only sequence elements $\{s_n\}_{n=0}^{\infty}$, but also explicit remainder estimates $\{\omega_n\}_{n=0}^{\infty}$.

▷ Levin's sequence transformation can be constructed via the model sequence

$$\frac{s_n - s}{\omega_n} = \sum_{j=0}^{k-1} \frac{c_j}{(n+\beta)^j}, \quad n \in \mathbb{N}_0, \beta > 0.$$

⇒ The weighted difference operator $\Delta^k (n+\beta)^{k-1}$ (acting on n) produces an explicit expression.

▷ Replacing powers $(n+\beta)^j$ by Pochhammer symbols $(n+\beta)_j$ yields the model sequence

$$\frac{s_n - s}{\omega_n} = \sum_{j=0}^{k-1} \frac{c_j}{(n+\beta)_j}, \quad n \in \mathbb{N}_0, \beta > 0.$$

⇒ Here, $\Delta^k (n+\beta)_{k-1}$ does the job and yields an expression for a sequence transformation.

▷ What are series involving inverse Pochhammer symbols? Are they something useful?

Definition of Factorial Series

▷ Let $\Omega\colon \mathbb{C} \to \mathbb{C}$ be a function which vanishes as $z \to +\infty$. A *factorial series* for $\Omega(z)$ is an expansion whose z-dependence occurs in Pochhammer symbols in the denominator:

$$\Omega(z) = \frac{a_0}{z} + \frac{a_1 1!}{z(z+1)} + \frac{a_2 2!}{z(z+1)(z+2)} + \cdots$$
$$= \sum_{\nu=0}^{\infty} \frac{a_\nu \nu!}{(z)_{\nu+1}}.$$

The separation of the coefficients into a factorial $n!$ and a reduced coefficient a_n often offers formal advantages.

Convergence of Factorial Series

▷ The factorial series for $\Omega(z)$ converges with the possible exception of $z = -m$ with $m \in \mathbb{N}_0$ if the associated Dirichlet series

$$\tilde{\Omega}(z) = \sum_{n=1}^{\infty} \frac{a_n}{n^z}$$

converges.

⇒ The associated Dirichlet series converges as $z \to \infty$ even if $a_n \sim n^\beta$ with $\beta > 0$ as $n \to \infty$.

General Considerations

▷ Stirling apparently became aware about factorial series from the work of the French mathematician Nicole.

▷ However, Stirling used an thus popularized factorial series in his classic book *Methodus Differentialis* (1730).

⇒ Later, factorial series played a major role in finite difference equations. Because of

$$\Delta^k \frac{n!}{(z)_{n+1}} = \frac{(-1)^k (n+k)!}{(z)_{n+k+1}}, \quad k \in \mathbb{N}_0,$$

it is extremely easy to apply the finite difference operator Δ (acting on z) to a factorial series:

$$\begin{aligned} \Delta^k \, \Omega(z) &= \sum_{\nu=0}^{\infty} \Delta^k \frac{a_\nu \nu!}{(z)_{\nu+1}} \\ &= (-1)^k \sum_{\nu=0}^{\infty} \frac{a_\nu (\nu+k)!}{(z)_{\nu+k+1}}. \end{aligned}$$

Finite Generating Functions for Stirling Numbers

- Stirling numbers of the first kind:

$$(z-n+1)_n = \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\, z^{\nu}, \quad n \in \mathbb{N}_0 .$$

- Stirling numbers of the second kind:

$$z^n = \sum_{\nu=0}^{n} \mathbf{S}^{(2)}(n,\nu)\, (z-\nu+1)_{\nu}, \quad n \in \mathbb{N}_0 .$$

Infinite Generating Functions for Stirling Numbers

- Stirling numbers of the first kind:

$$\frac{1}{z^{k+1}} = \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}\, \mathbf{S}^{(1)}(k+\kappa,k)}{(z)_{k+\kappa+1}}, \quad k \in \mathbb{N}_0 .$$

- Stirling numbers of the second kind:

$$\frac{1}{(z)_{k+1}} = \sum_{\kappa=0}^{\infty} \frac{(-1)^{\kappa}\, \mathbf{S}^{(2)}(k+\kappa,k)}{z^{k+\kappa+1}}, \quad k \in \mathbb{N}_0 , \quad |z| > k .$$

Conversion of Inverse Power Series to Factorial Series

▷ $f: \mathbb{C} \to \mathbb{C}$ possesses a formal inverse power series:

$$f(z) \; = \; \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}} \, .$$

▷ Inserting the infinite generating function for $\mathbf{S}^{(1)}(n, \nu)$ yields the following factorial series:

$$f(z) \; = \; \sum_{m=0}^{\infty} \frac{(-1)^m}{(z)_{m+1}} \sum_{\mu=0}^{m} (-1)^{\mu} \, \mathbf{S}^{(1)}(m, \mu) \, c_{\mu} \, .$$

▷ This transformation is purely formal.

⇒ The convergence of the resulting factorial series has to be checked explicitly.

▷ It can happen that the inverse power series diverges factorially, but the factorial series converges.

Stieltjes Functions and Series

▷ **Stieltjes Function**

$$F(z) \;=\; \int_0^\infty \frac{\mathrm{d}\Phi(t)}{z+t}\,, \quad |\arg(z)| < \pi$$

$\Phi(t)$: positive measure on $0 \le t < \infty$.

▷ **Stieltjes Series**

$$F(z) = \sum_{m=0}^{\infty} (-1)^m \, \mu_m / z^{m+1}$$
$$\mu_n = \int_0^\infty t^n \mathrm{d}\Phi(t)$$

▷ We only have to insert the geometric series

$$\sum_{\nu=0}^{\infty} (-t)^\nu / z^{\nu+1} \;=\; 1/(z+t)$$

into the integral representation and integrate term-wise to obtain the Stieltjes series, which may converge or diverge.

▷ Stieltjes series are of considerable theoretical importance. There is a highly developed convergence theory for their Padé approximants.

Waring's Formula

▷ Iterating the expression

$$\frac{1}{z-w} = \frac{1}{z} + \frac{w}{z(z-w)}$$

yields Waring's formula

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w)_n}{(z)_{n+1}}, \quad \mathrm{Re}(z-w) > 0,$$

which was actually derived by Stirling.

▷ Inserting the Waring formula with $w = -t$ into the Stieltjes integral yields:

$$\begin{aligned} F(z) &= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-t)_n}{(z)_{n+1}} \, \mathrm{d}\Phi(t) \\ &= \sum_{n=0}^{\infty} \frac{1}{(z)_{n+1}} \int_0^{\infty} (-t)_n \, \mathrm{d}\Phi(t). \end{aligned}$$

⇒ There is a lot of cancellation in the integral $\int_0^{\infty} (-t)_n \mathrm{d}\Phi(t)$ since $(-t)_n$ has alternating signs.

Factorial Series for Stieltjes Functions

▷ Inserting the finite generating function for $\mathbf{S}^{(1)}(n,\nu)$ into the Stieltjes integral representation yields:

$$F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z)_{n+1}} \times \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu) \int_0^{\infty} t^{\nu} \, \mathrm{d}\Phi(t) \, .$$

▷ Now, we only have to do the moment integrals via $\mu_n = \int_0^{\infty} t^n \mathrm{d}\Phi(t)$ to obtain a factorial series:

$$F(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(z)_{n+1}} \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu) \, \mu_{\nu} \, .$$

▷ The moments μ_n and the Stirling numbers $(-1)^{n-\nu} \mathbf{S}^{(1)}(n,\nu)$ are always *positive*.

⇒ There is a lot of cancellation in the strictly alternating finite sum $\sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\mu_{\nu}$ representing the coefficients of the factorial series.

Factorial Series for the Euler Integral

▷ **Euler Integral**

$$\mathcal{E}(z) \;=\; \int_0^\infty \frac{\mathrm{e}^{-t}\mathrm{d}t}{z+t} \;=\; \mathrm{e}^z\, E_1(z)\;.$$

▷ **Euler Series**

$$\mathcal{E}(z) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n n!}{z^{n+1}}$$

Diverges for every finite z but is asymptotic as $z \to \infty$.

▷ **Factorial Series**

$$\begin{aligned}\mathcal{E}(z) \;&=\; \sum_{n=0}^{\infty} \frac{1}{(z)_{n+1}} \int_0^\infty (-t)_n\, \mathrm{e}^{-t}\mathrm{d}t \\ &=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{(z)_{n+1}} \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\,\nu!\,.\end{aligned}$$

Summation by Cancellation

▷ In the inner sum $\sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\nu!$ there is substantial cancellation:

n	$(-1)^n n!$	$(-1)^n \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\nu!$
0	1	1
1	-1	-1
2	2	1
3	-6	-2
4	24	4
5	-120	-14
6	720	38
7	-5040	-216
8	40320	600
9	-362880	-6240
10	3628800	9552
11	-39916800	-319296
12	479001600	-519312
13	-6227020800	-28108560
14	87178291200	-176474352

⇒ The asymptotic Euler series is summed:

$$\frac{\sum_{n=0}^{14} \frac{(-1)^n}{(5)_{n+1}} \sum_{\nu=0}^{n} \mathbf{S}^{(1)}(n,\nu)\,\nu!}{\exp(5)E_1(5)} = 1.000\,000\,764$$

Conversion of Power Series

▷ Transform $f(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ to an inverse power series in $1/z$:

$$f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\gamma_n}{(1/z)^{n+1}} .$$

⇒ Factorial series in $1/z$:

$$f(z) = \frac{1}{z} \sum_{m=0}^{\infty} \frac{(-1)^m}{(1/z)_{m+1}} \times \sum_{\mu=0}^{m} (-1)^{\mu} \, \mathbf{S}^{(1)}(m,\mu) \, \gamma_\mu .$$

⇒ Use

$$\frac{1}{(1/z)_{m+1}} = \frac{z}{m!} \prod_{k=1}^{m} \frac{z}{z + 1/k}$$

to obtain:

$$f(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \prod_{k=1}^{m} \frac{z}{z + 1/k} \times \sum_{\mu=0}^{m} (-1)^{\mu} \, \mathbf{S}^{(1)}(m,\mu) \, \gamma_\mu .$$

Conversion of Factorial Series to Inverse Power Series

▷ The conversion of factorial series to inverse power series is also possible (although not nearly as useful as the inverse transformation).

▷ $\Omega \colon \mathbb{C} \to \mathbb{C}$ possesses a factorial series:

$$\Omega(z) \; = \; \sum_{n=0}^{\infty} \frac{w_n}{(z)_{n+1}} \, .$$

⇒ Inserting the infinite generating function for $\mathbf{S}^{(2)}(n, \nu)$ yields the following inverse power series:

$$\Omega(z) \; = \; \sum_{m=0}^{\infty} \frac{(-1)^m}{z^{m+1}} \times \sum_{\mu=0}^{m} (-1)^{\mu} \, \mathbf{S}^{(2)}(m, \mu) \, w_{\mu} \, .$$

▷ Again, this transformation is purely formal. Convergence has to be checked explicitly.

Quartic Anharmonic Oscillator

▷ **Hamiltonian**

$$\hat{H}(\beta) = \hat{p}^2 + \hat{x}^2 + \beta\,\hat{x}^4\,.$$

$$\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\,.$$

▷ **Perturbation series** (ground state)

$$E(\beta) = \sum_{n=0}^{\infty} b_n\,\beta^n\,.$$

▷ **Large-index asymptotics**

$$b_n \sim (-A)^n\,\frac{n!}{n^{1/2}}\,, \quad n \to \infty\,.$$

⇒ The perturbation series for $E(\beta)$ diverges factorially for every nonzero coupling constant β and has to be summed to produce numerically useful results.

▷ The quartic anharmonic is a simple, but nevertheless non-trivial model system for many factorially divergent perturbation expansions in quantum theory.

Summation of the Divergent Perurbation Series

▷ The energy shift $\Delta E(\beta)$ defined by

$$E(\beta) = b_0 + \beta \, \Delta E(\beta) = b_0 + \beta \sum_{n=0}^{\infty} b_{n+1} \, \beta^n$$

is known to be a Stieltjes function.

⇒ Padé approximants $[n+j/n]$ with $j = -1, 0, 1, \ldots$ to $\Delta E(\beta)$ computed from the divergent perturbation series converge as $n \to \infty$.

▷ Truncated factorial series in $1/\beta$ for $\Delta E(\beta)$:

$$\Delta E(\beta) \approx \sum_{m=0}^{M} \frac{(-1)^m}{m!} \prod_{k=1}^{m} \frac{\beta}{\beta + 1/k} \times \sum_{\mu=0}^{m} (-1)^{\mu} \, \mathbf{S}^{(1)}(m, \mu) \, b_{\mu+1} \, .$$

Summation Results

▷ "Exact" energy for $\beta = 1/5$:

$$E_{\mathsf{exact}}(1/5) \; = \; 1.118\,292\,654\,367\,039\,154\ldots .$$

▷ Compute $E(1/5)$ via the diagonal Padé approximant [17/17] for the energy shift:

$$E_{\mathsf{PA}}(1/5) \; = \; 1.118\;292\;654\;373\ldots .$$

▷ Compute $E(1/5)$ via the truncated factorial series with $M = 34$ for the energy shift:

$$E_{\mathsf{FS}}(1/5) \; = \; 1.118\;305\ldots .$$

⇒ The factorial series, which does a *linear* transformation of the perturbation series coefficients b_1, b_2, ..., b_M, is less efficient than a highly *nonlinear* Padé approximant using the same number of coefficients.

▷ Further improvements are possible by introducing some *nonlinearity* into the computation scheme for factorial series.

Integral Representations for Factorial Series

▷ The beta function

$$B(x,y) \;=\; \Gamma(x)\Gamma(y)/\Gamma(x+y)\,,$$

which possesses the integral representation

$$B(x,y) \;=\; \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t\,,$$

also satisfies $B(z,n+1) = n!/(z)_{n+1}$.

⇒ A factorial series can be expressed as a series of beta functions:

$$\Omega(z) \;=\; \sum_{n=0}^{\infty} a_n\,B(z,n+1)\,.$$

⇒ A factorial series possesses the integral representation

$$\Omega(z) \;=\; \int_0^1 t^{z-1}\,\varphi_\Omega(t)\,\mathrm{d}t\,, \qquad \mathrm{Re}(z) > 0\,,$$
$$\varphi_\Omega(t) \;=\; \sum_{n=0}^{\infty} a_n\,(1-t)^n\,.$$

Evaluation by Numerical Quadrature

- ▷ The integral representation $\int_0^1 t^{z-1} \varphi_\Omega(t) \mathrm{d}t$ can be used for the numerical evaluation of a function $\Omega(z) = \sum_{n=0}^{\infty} a_n n!/(z)_{n+1}$.

- ▷ A straightforward use of the defining power series $\varphi_\Omega(t) = \sum_{n=0}^{\infty} a_n (1-t)^n$ in the integral representation does not lead to improvements (integration is linear).

- ▷ The truncated power series for $\varphi_\Omega(t)$ can be converted to a Padé approximant in $1-t$ and inserted into the integral representation.

⇒ The numerical quadrature of the Padé approximant [17/17] to the associated power series $\varphi_\Omega(t)$ yields for the ground state energy $E(1/5)$:

$$E_{\text{IntFS}}(1/5) = 1.118\,292\,654\,369\ldots .$$

- ▷ But Borel-Padé is still better:

$$E_{\text{BorPade}}(1/5) = 1.118\,292\,654\,367\,039\,152\ldots .$$