Efficient algorithm for summation of some slowly convergent series

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Woźny's & Nowak's method (WN method): main idea

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- Let $\mathbf{L}^{(\infty)}$ be an linear operator acting on the variable n such that

$$\left.\begin{aligned} &\mathbf{L}^{(\infty)}\left(\sum_{j=0}^{\infty} a_{n+j}\right) = 0, \\ &\mathbf{L}^{(\infty)}(1) \neq 0 \end{aligned}\right\}$$

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- Let the operator $\mathbf{L}^{(m)}$ (acting on n) be an approximation of $\mathbf{L}^{(\infty)}$ in the sense that

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- Let us consider the transformation $Q^{(m)} : \{s_n\} \to \{Q_n^{(m)}\}$, defined by

$$Q_n^{(m)}(s_n) := \frac{\mathbf{L}^{(m)}(s_n)}{\mathbf{L}^{(m)}(1)}$$

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- All we need is an efficient algorithm of constructing operators $\mathbf{L}^{(m)}$

WN method: construction of operators $\mathbf{L}^{(m)}$

- Let $\mathbf{E}$ be the shift operator,

$$\mathbf{E}x_n = x_{n+1}, \qquad \mathbf{E}^k x_n = x_{n+k} \quad (k \in \mathbb{Z})$$

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- Suppose we know the linear difference operator $\mathbf{L}^{(m-1)}$:

$$\mathbf{L}^{(m-1)}\left(\sum_{j=0}^{m-2} a_{n+j}\right) = 0$$

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- Suppose we know the linear difference operator $\mathbf{L}^{(m-1)}$.

- By solving a system of linear equations, one can find operators $\mathbf{P}^{(m)}$, $\mathbf{R}^{(m)}$ such that

$$\mathbf{P}^{(m)}\mathbf{L}^{(m-1)} = \mathbf{R}^{(m)}\widetilde{\mathbf{L}}^{(m-1)},$$

where

$$\widetilde{\mathbf{L}}^{(m-1)} := \mathbf{E}^{m-1}\mathbf{L}^{(1)}\mathbf{E}^{1-m}, \qquad \widetilde{\mathbf{L}}^{(m-1)}(a_{n+m-1}) = 0$$

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- Then

$$\mathbf{L}^{(m)} := \mathbf{P}^{(m)}\mathbf{L}^{(m-1)}$$

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- Notice that

$$\mathbf{L}^{(m)} = \mathbf{P}^{(m)}\mathbf{P}^{(m-1)}\ldots\mathbf{P}^{(1)},$$

  where $\mathbf{P}^{(1)} := \mathbf{L}^{(1)}$

WN method: algorithm of computing $\mathcal{Q}_n^{(m)}$

1. Find the operator $\mathbf{L}^{(1)}$ such that

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4. Compute quantities $N_n^{(m)}$, and $D_n^{(m)}$ $(m = 1, 2, \ldots)$ in the following recurrent way:

$$N_n^{(m)} := \mathbf{P}^{(m)}\left(N_n^{(m-1)}\right),$$

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5. Then

$$\mathcal{Q}_n^{(m)} = \frac{N_n^{(m)}}{D_n^{(m)}}$$

WN method: applications

- Generalized hypergeometric series

$$ {}_{r+1}F_r\left(\begin{matrix}\alpha_1,\ \alpha_2,\ldots,\alpha_{r+1}\\ \beta_1,\ \beta_2,\ldots,\beta_r\end{matrix}\ \middle|\ x\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k\cdots(\alpha_{r+1})_k}{(\beta_1)_k(\beta_2)_k\cdots(\beta_r)_k}\cdot\frac{x^k}{k!}, $$

where $(a)_k := a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol.

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- Basic hypergeometric series

$$ {}_{r+1}\phi_r\left(\begin{matrix}\alpha_1,\ \alpha_2,\ldots,\alpha_{r+1}\\ \beta_1,\ \beta_2,\ldots,\beta_r\end{matrix}\;\middle|\; q;\ x\right) := \sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k(\alpha_2;q)_k\cdots(\alpha_{r+1};q)_k}{(\beta_1;q)_k(\beta_2;q)_k\cdots(\beta_r;q)_k}\cdot\frac{x^k}{(q;q)_k}, $$

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- Some orthogonal polynomial series expansions,

$$ f(x) = \sum_{k=0}^{\infty} c_k P_k(x), $$

where $\{P_k\}$ is a sequence of orthogonal polynomials

WN method: example

$$\sqrt{1+x} = -\frac{4\sqrt{2}}{\pi}\sum_{k=0}^{\infty}{}' \frac{(-1)^k}{4k^2-1}T_k(x) \qquad (x \in [-1,1];\ T_k \text{ – } k\text{th Chebyshev poly.})$$

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$$\sqrt{1+x} = -\frac{4\sqrt{2}}{\pi}\sum_{k=0}^{\infty}{}' \frac{(-1)^k}{4k^2-1}T_k(x) \qquad (x \in [-1,1];\ T_k \text{ – kth Chebyshev poly.})$$

$$\mathbf{L}^{(1)} = (2n-1)(2n-3)\mathbf{E}^{-1} + 2x(4n^2-1)\mathbf{I} + (2n+1)(2n+3)\mathbf{E}$$

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$$\mathbf{P}^{(m)} = (2n-1)\mathbf{E}^{-1} + 2x(2n+2m-1)\mathbf{I} + (2n+4m-1)\mathbf{E}$$

$$\begin{array}{lllllll}
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s_5 & = & Q_5^{(0)} & Q_5^{(1)} & Q_5^{(2)} & Q_5^{(3)} & Q_5^{(4)} \\
s_6 & = & Q_6^{(0)} & Q_6^{(1)} & \mathbf{Q_6^{(2)}} & Q_6^{(3)} \\
s_7 & = & Q_7^{(0)} & Q_7^{(1)} & Q_7^{(2)} \\
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$$x := \frac{9}{10}$$

n	$\mathrm{acc}(s_n)$
100	4.98
500	6.38
1000	8.54

$$\mathrm{acc}(\sigma) := -\log_{10}\left|\frac{\sigma}{\sqrt{1+x}} - 1\right|$$

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$$[s_1, s_2, \ldots, s_{31}] \quad \Rightarrow \quad \mathcal{Q}_{16}^{(15)}$$

Paweł Woźny, University of Wrocław, Poland

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$$[s_1, s_2, \ldots, s_{31}] \quad \Rightarrow \quad \mathrm{acc}\left(\mathcal{Q}_{16}^{(15)}\right) = 32.46$$

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WN method: problem

- Let us consider the more general expansion:

$$\cos(q \arccos x) = -\frac{2q \sin q\pi}{\pi} \sum_{k=0}^{\infty}{}' \frac{(-1)^k}{k^2 - q^2} T_k(x) \qquad (x \in [-1, 1];\ q \notin \mathbb{Z}),$$

which corresponds to the previous example with $q = \frac{1}{2}$

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- To use our method, we have to construct—symbolically or numerically—operators $\mathbf{P}^{(m)}$ one after another in the mentioned way

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- To use our method, we have to construct—symbolically or numerically—operators $\mathbf{P}^{(m)}$ one after another in the mentioned way.

- It is possible, however, it is not a good idea. Why? Let us see . . .

WN method: construction of operators $\mathbf{L}^{(m)}$ again

- Suppose we know the linear difference operator $\mathbf{L}^{(m-1)}$ such that

$$\mathbf{L}^{(m-1)}\left(\sum_{j=0}^{m-2} a_{n+j}\right) = 0$$

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- By solving a system of linear equations, one can find operators $\mathbf{P}^{(m)}$, $\mathbf{R}^{(m)}$ such that

$$\mathbf{P}^{(m)}\mathbf{L}^{(m-1)} = \mathbf{R}^{(m)}\widetilde{\mathbf{L}}^{(m-1)},$$

where

$$\widetilde{\mathbf{L}}^{(m-1)} := \mathbf{E}^{m-1}\mathbf{L}^{(1)}\mathbf{E}^{1-m}, \qquad \widetilde{\mathbf{L}}^{(m-1)}(a_{n+m-1}) = 0$$

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Problem

- $r := \text{ord}\left(\mathbf{L}^{(1)}\right) = \text{ord}\left(\mathbf{P}^{(m)}\right), \qquad \text{ord}\left(\mathbf{R}^{(m)}\right) = r(m-1) \qquad (m = 2, 3, \ldots)$

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- We have to solve the system of $m \cdot r$ linear equations in each step

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We need a faster algorithm of constructing operators $\mathbf{P}^{(2)}$, $\mathbf{P}^{(3)}, \ldots$

WN method: new algorithm of constructing operators $\mathbf{L}^{(m)}$

1. Find the operator $\mathbf{P}^{(1)}$ such that $\mathbf{P}^{(1)}a_n = 0$, and set $\mathbf{Q}^{(1)} := \mathbf{P}^{(1)}$

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2. By solving a system of linear equations, find recursively the operators $\mathbf{P}^{(m)}$, and $\mathbf{Q}^{(m)}$ satisfying the following indentity:

$$\mathbf{P}^{(m)}\mathbf{Q}^{(m-1)} = \mathbf{Q}^{(m)}\widetilde{\mathbf{P}}^{(m-1)} \qquad (m = 2, 3, \ldots),$$

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Theorem

$$\mathbf{L}^{(m)}\left(\sum_{j=0}^{m-1} a_{n+j}\right) = 0 \qquad (m = 1, 2, \ldots)$$

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Remark

- $r :=$ ord $\left(\mathbf{P}^{(1)}\right) =$ ord $\left(\mathbf{P}^{(m)}\right) =$ ord $\left(\mathbf{Q}^{(m)}\right) \qquad (m = 2, 3, \ldots)$

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- Now, we have to solve the system of $2r$ linear equations in each step

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$$\mathbf{P}^{(m)}\mathbf{Q}^{(m-1)} = \mathbf{Q}^{(m)}\widetilde{\mathbf{P}}^{(m-1)} \qquad (m = 2, 3, \ldots),$$

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- Now, we have to solve the system of $2r$ linear equations in each step (previous approach: $m \cdot r$ linear equations)

Proof

- $\mathbf{P}^{(m)}\mathbf{Q}^{(m-1)} = \mathbf{Q}^{(m)}\widetilde{\mathbf{P}}^{(m-1)}, \qquad \widetilde{\mathbf{P}}^{(m-1)} := \mathbf{E}\mathbf{P}^{(m-1)}\mathbf{E}^{-1}$

Proof

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- $\mathbf{L}^{(m)} := \mathbf{P}^{(m)}\mathbf{P}^{(m-1)} \ldots \mathbf{P}^{(1)}$

Proof

- $\mathbf{P}^{(m)}\mathbf{Q}^{(m-1)} = \mathbf{Q}^{(m)}\widetilde{\mathbf{P}}^{(m-1)}, \qquad \widetilde{\mathbf{P}}^{(m-1)} := \mathbf{E}\mathbf{P}^{(m-1)}\mathbf{E}^{-1}.$
- $\mathbf{L}^{(m)} := \mathbf{P}^{(m)}\mathbf{P}^{(m-1)}\ldots\mathbf{P}^{(1)}, \qquad \mathbf{L}^{(m)}\left(\sum_{j=0}^{m-1} a_{n+j}\right) = 0$

Proof

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$$\mathbf{L}^{(m+1)}\left(\sum_{j=0}^{m} a_{n+j}\right)$$

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$$\mathbf{L}^{(m+1)}\left(\sum_{j=0}^{m} a_{n+j}\right) = \mathbf{P}^{(m+1)}\mathbf{L}^{(m)}\left(\sum_{j=0}^{m-1} a_{n+j} + a_{n+m}\right)$$

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$$= \mathbf{P}^{(m+1)}\dots\mathbf{P}^{(2)}\mathbf{P}^{(1)}(a_{n+m})$$

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Proof

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$$= \mathbf{P}^{(m+1)}\dots\mathbf{P}^{(4)}\mathbf{Q}^{(3)}\widetilde{\mathbf{P}}^{(2)}\widetilde{\mathbf{P}}^{(1)}(a_{n+m})$$

Proof

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The case $\text{ord}\left(\mathbf{P}^{(1)}\right) = 1$

- If $\text{ord}(\mathbf{P}^{(1)}) = 1$ then our method is equivalent to Wynn's ε-algorithm. More precisely,

$$Q_n^{(m)} = \varepsilon_{2m}^{n}$$

Paweł Woźny, University of Wrocław, Poland

New algorithm (numerical version): case $\operatorname{ord}\left(\mathbf{P}^{(1)}\right) = 2$

- $\mathbf{P}^{(m)} := \mathbf{E}^{-1} + p_0^{(m)}(n)\mathbf{I} + p_1^{(m)}(n)\mathbf{E}, \qquad \mathbf{Q}^{(m)} := \mathbf{E}^{-1} + q_0^{(m)}(n)\mathbf{I} + q_1^{(m)}(n)\mathbf{E}$

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$$\Downarrow$$

$$\begin{bmatrix} \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & \otimes & \otimes \\ \otimes & \otimes & & \\ & & \otimes & \otimes \end{bmatrix} \begin{bmatrix} p_0^{(m)}(n) \\ q_0^{(m)}(n) \\ p_1^{(m)}(n) \\ q_1^{(m)}(n) \end{bmatrix} = \begin{bmatrix} \otimes \\ \otimes \\ \otimes \\ \end{bmatrix}$$

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- Suppose, we know values $p_0^{(1)}(n), p_1^{(1)}(n), q_0^{(1)}(n), q_1^{(1)}(n)$ for $n = 2, 3, \ldots, N-1$, where $3|N$

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- Using the above system of linear equations, one can find values $p_0^{(m)}(n)$, $p_1^{(m)}(n)$, $q_0^{(m)}(n)$, $q_1^{(m)}(n)$ for $m = 2, 3, \ldots, N/3$, and $n = m+1, m+2, \ldots, N-2m+1$

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- Thuse we can compute the transformations $\mathcal{Q}_n^{(m)}$ for $m = 2, 3, \ldots, N/3$, and $n = m+1, m+2, \ldots, N-2m+1$

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- $N = 12$:

$$
\begin{aligned}
s_1 &= \mathcal{Q}_1^{(0)} \\
s_2 &= \mathcal{Q}_2^{(0)} \quad \mathcal{Q}_2^{(1)} \\
s_3 &= \mathcal{Q}_3^{(0)} \quad \mathcal{Q}_3^{(1)} \quad \mathcal{Q}_3^{(2)} \\
s_4 &= \mathcal{Q}_4^{(0)} \quad \mathcal{Q}_4^{(1)} \quad \mathcal{Q}_4^{(2)} \quad \mathcal{Q}_4^{(3)} \\
s_5 &= \mathcal{Q}_5^{(0)} \quad \mathcal{Q}_5^{(1)} \quad \mathcal{Q}_5^{(2)} \quad \mathcal{Q}_5^{(3)} \quad \boldsymbol{\mathcal{Q}_5^{(4)}} \\
s_6 &= \mathcal{Q}_6^{(0)} \quad \mathcal{Q}_6^{(1)} \quad \mathcal{Q}_6^{(2)} \quad \mathcal{Q}_6^{(3)} \\
s_7 &= \mathcal{Q}_7^{(0)} \quad \mathcal{Q}_7^{(1)} \quad \mathcal{Q}_7^{(2)} \quad \mathcal{Q}_7^{(3)} \\
s_8 &= \mathcal{Q}_8^{(0)} \quad \mathcal{Q}_8^{(1)} \quad \mathcal{Q}_8^{(2)} \\
s_9 &= \mathcal{Q}_9^{(0)} \quad \mathcal{Q}_9^{(1)} \quad \mathcal{Q}_9^{(2)} \\
s_{10} &= \mathcal{Q}_{10}^{(0)} \quad \mathcal{Q}_{10}^{(1)} \\
s_{11} &= \mathcal{Q}_{11}^{(0)} \quad \mathcal{Q}_{11}^{(1)} \\
s_{12} &= \mathcal{Q}_{12}^{(0)}
\end{aligned}
$$

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- $N = 12 \quad \Rightarrow \quad \mathcal{Q}_5^{(4)}$.

Remark. The case when $\text{ord}\left(\mathbf{P}^{(1)}\right) = r$ $(r = 3, 4, \ldots)$ can be consider in a similar way.

Examples

$$\cos(q \arccos x) = -\frac{2q \sin q\pi}{\pi} \sum_{k=0}^{\infty}{}' \frac{(-1)^k}{k^2 - q^2} T_k(x) \qquad (x \in [-1, 1];\ q \notin \mathbb{Z}),$$

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$$\mathbf{P}^{(1)} = ((n-1)^2 - q^2)\mathbf{E}^{-1} + 2x(n^2 - q^2)\mathbf{I} + ((n+1)^2 - q^2)\mathbf{E}$$

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$$q := 3.33, \qquad [s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad \mathcal{Q}_{12}^{(11)}$$

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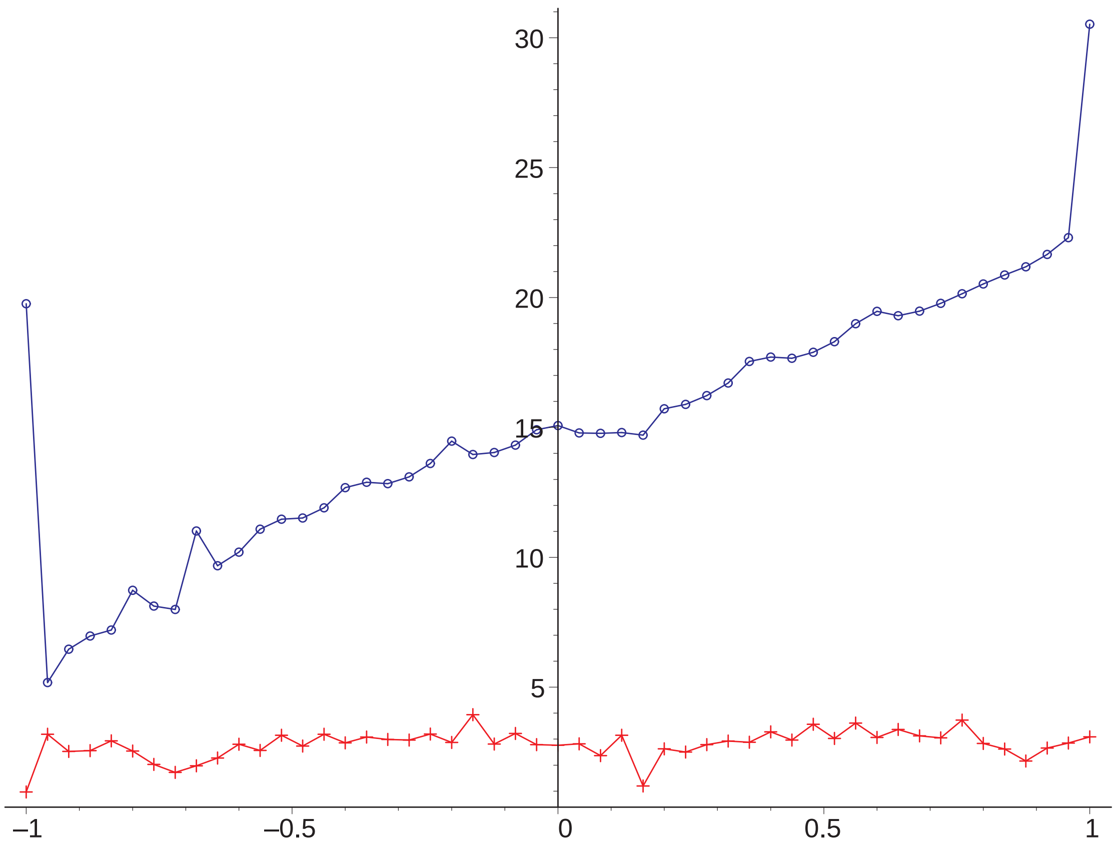

$\mathrm{acc}\left(\mathcal{Q}_{12}^{(11)}\right)$

$\mathrm{acc}(s_{33})$

$$x \in [-1, 1], \qquad q := 3.33, \qquad [s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad \mathcal{Q}_{12}^{(11)}$$

Examples

$$(1-x)^{\rho} = 2^{\rho}\frac{\Gamma(\alpha+\rho+1)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+\rho+2)}\sum_{k=0}^{\infty} c_k P_k^{(\alpha,\beta)}(x),$$

where

$$c_k := \frac{(2k+\alpha+\beta+1)(-\rho)_k(\alpha+\beta+1)_k}{(\alpha+1)_k(\alpha+\beta+\rho+2)_k}$$

and $x \in (-1,1)$, $\alpha, \beta > -1$, $-\rho < \min(\alpha+1, \frac{\alpha}{2}+\frac{3}{4})$

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$$\rho := -3.33, \quad \alpha := 5.3, \quad \beta := 1.3, \qquad [s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad \mathcal{Q}_{12}^{(11)}$$

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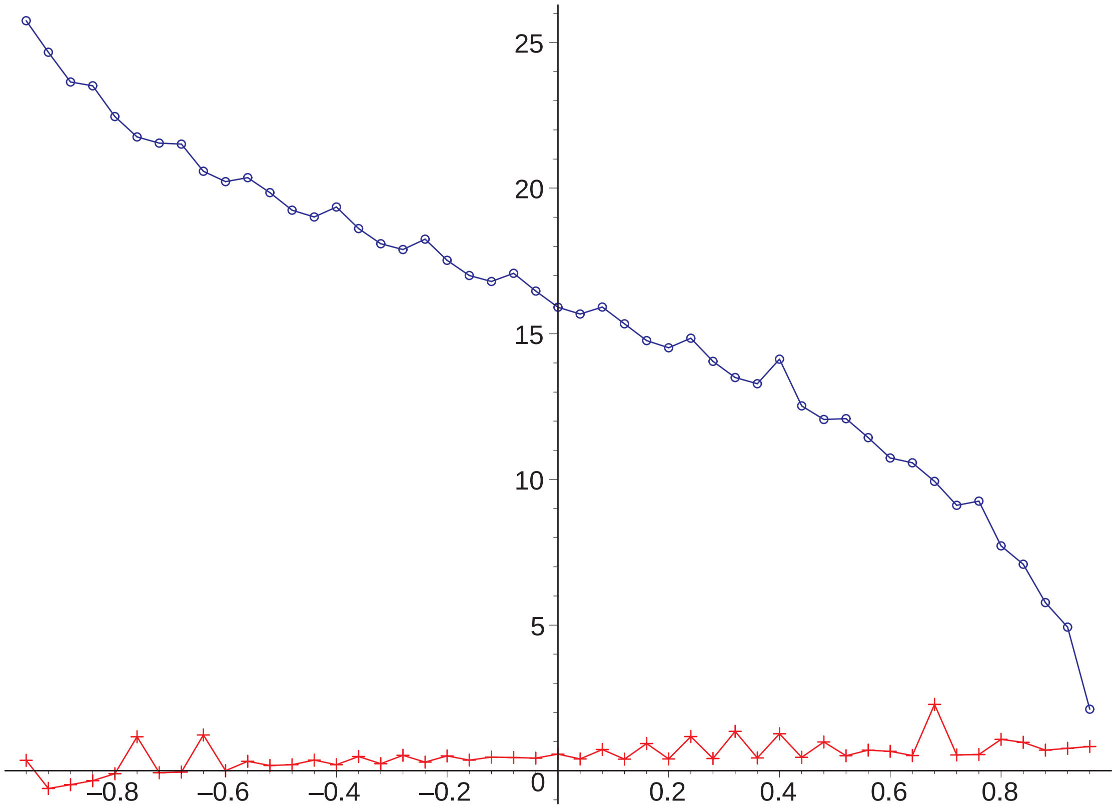

$\mathrm{acc}\left(Q_{12}^{(11)}\right)$

$\mathrm{acc}(s_{33})$

$$x \in (-1,1),\ \rho := -3.33,\ \alpha := 5.3,\ \beta := 1.3, \qquad [s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad Q_{12}^{(11)}$$

Examples

$$\frac{1}{2}\left(\ln\left(1+x\right)\right)^2 = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}x^{k+1}}{k+1}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right) \qquad (x^2<1)$$

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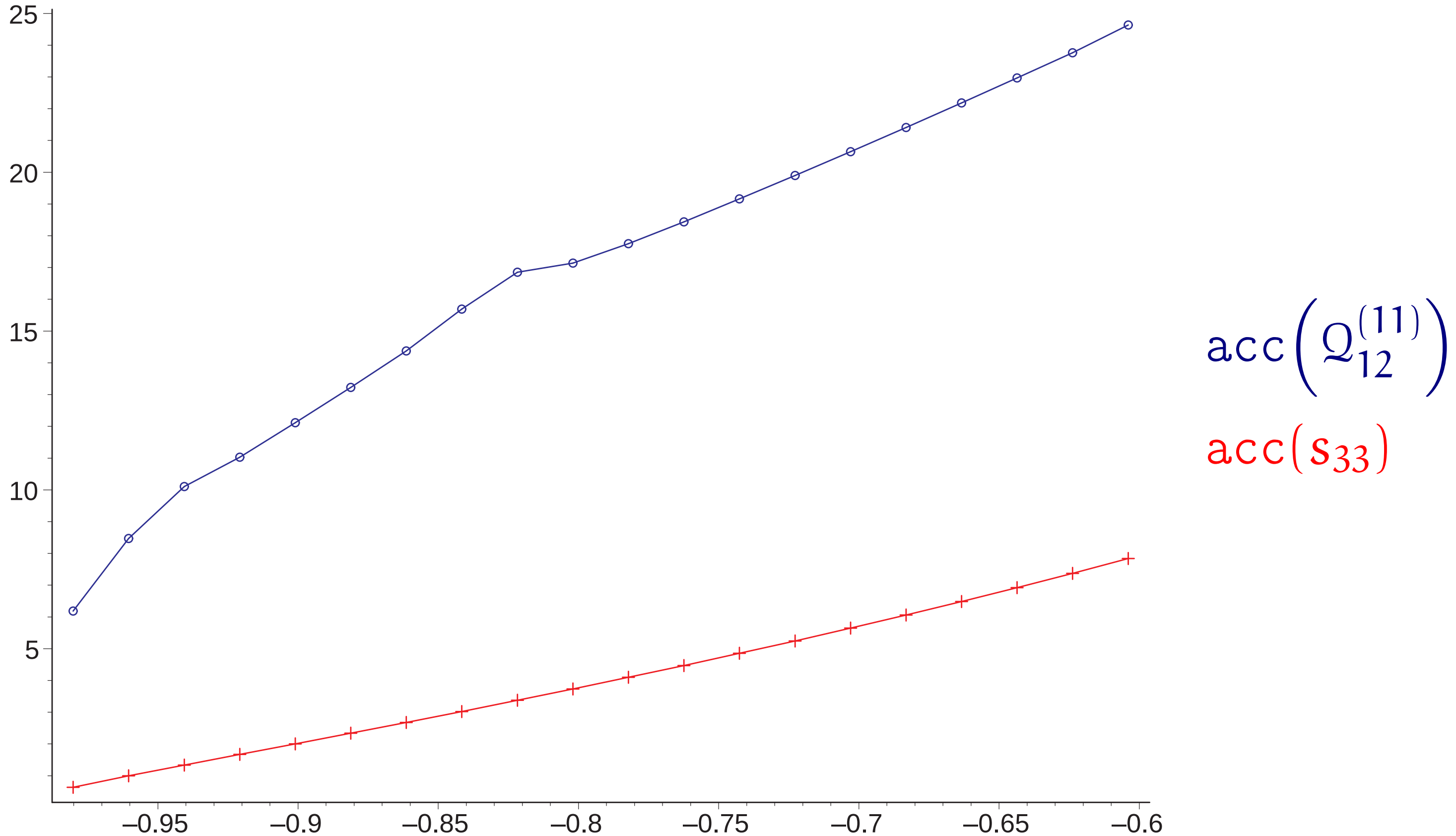

$$x \in (-1, -0.6), \qquad [s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad Q_{12}^{(11)}$$

Examples

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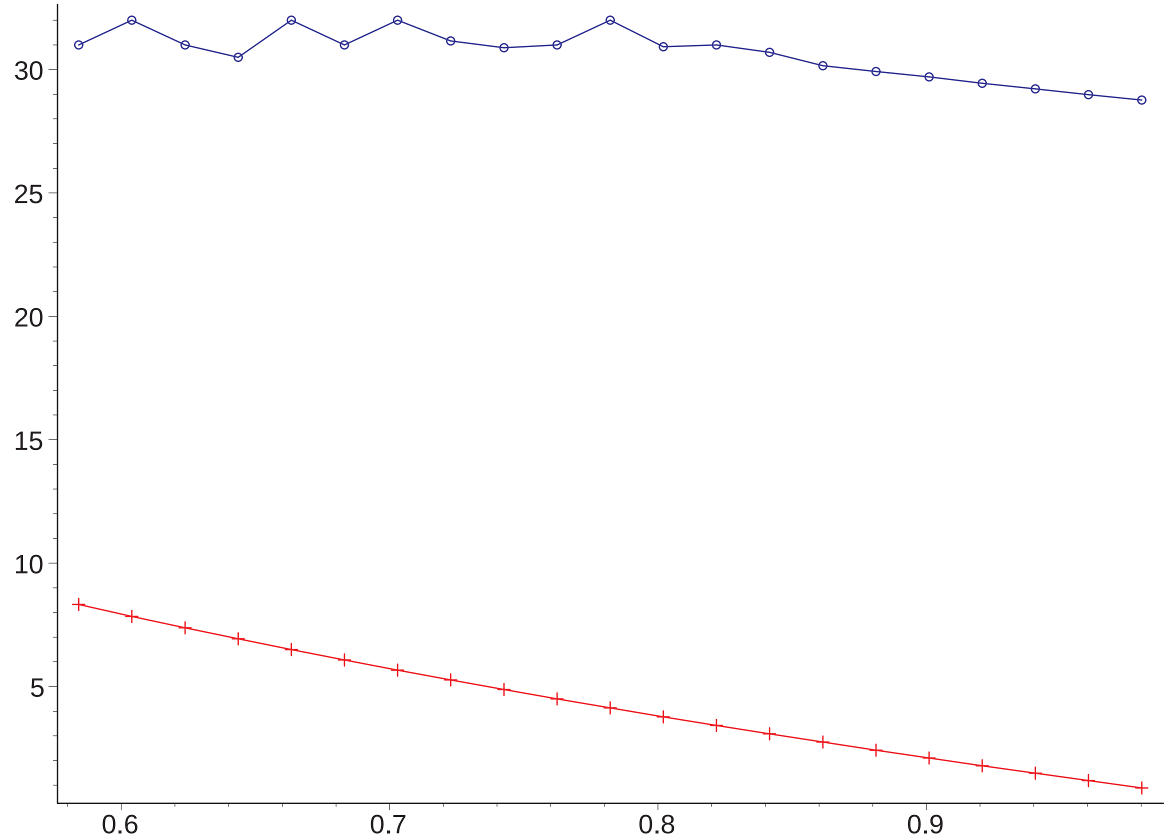

$\mathrm{acc}\left(\mathcal{Q}_{12}^{(11)}\right)$

$\mathrm{acc}(s_{33})$

$$x\in(0.6,1), \qquad [s_1,s_2,\ldots,s_{33}] \quad\Rightarrow\quad \mathcal{Q}_{12}^{(11)}$$

Examples

$$0.0185355891429332393\ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+33)\sqrt{k+1}}$$

Examples

$$0.018535589142933 2393\ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+33)\sqrt{k+1}}$$

$$\mathbf{P}^{(1)} = \sqrt{n}(n+32)\mathbf{E}^{-1} + 2\sqrt{n+1}(n+33)\mathbf{I} + \sqrt{n+2}(n+34)\mathbf{E}$$

Examples

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$$[s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad Q_{12}^{(11)}$$

Examples

$$0.0185355891429332393\ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+33)\sqrt{k+1}}$$

$$\mathbf{P}^{(1)} = \sqrt{n}(n+32)\mathbf{E}^{-1} + 2\sqrt{n+1}(n+33)\mathbf{I} + \sqrt{n+2}(n+34)\mathbf{E}$$

n	$\mathrm{acc}(s_n)$
100	1.7
500	2.65
1000	3.08
10000	4.57

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$$[s_1, s_2, \ldots, s_{33}] \quad \Rightarrow \quad \mathrm{acc}\left(\mathcal{Q}_{12}^{(11)}\right) = 28.65$$

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