ORDER-DEPENDENT MAPPINGS

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For order-dependent mappings, the original article is:
R. Seznec and J. Zinn-Justin, *Journal of Math. Phys.* 20 (1979) 1398.

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J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford University Press, 1989), fourth edition: International Series of Monographs on Physics 113, 1054 pp. (2002).

The initial motivation: Perturbative quantum field theory

In quantum field theory, the main analytic tool is the perturbative expansion. As an illustration we shall consider the important example of the ϕ^4 quantum field theory. In the statistical formulation, one considers the Euclidean action $\mathcal{S}$, local functional of the field $\phi(x)$, $x \in \mathbb{R}^d$,

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}\left(\partial_\mu \phi(x)\right)^2 + \tfrac{1}{2} r \phi^2(x) + \frac{g}{4!}\phi^4(x)\right].$$

The partition function is then given by

$$\mathcal{Z} = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{S}(\phi)}\,.$$

Perturbation theory here amounts to expanding in powers of the positive parameter g.

The case $d = 0$ corresponds to a simple integral.

The case $d = 1$ corresponds to the quantum quartic anharmonic oscillator.

$d > 1$ corresponds to quantum field theory.

The dimensions $d = 2, 3$ are especially relevant to classical statistical physics and the theory of phase transitions. $d = 4$ is relevant to relativistic quantum field theory and the so-called Higgs mechanism.

For $d > 1$, the difficulty of evaluating the successive perturbative terms increases very rapidly. Moreover, questions like regularization and renormalization arise. Therefore, the calculation of renormalization group functions in the $d = 3$ $(\phi^2)^2$ field theory up order g^7 (Nickel) is an exceptional achievement.

Large order behaviour

The field integral, in the ϕ^4 field theory, $g = 0$ corresponds to a singularity since the integral is not defined for $g < 0$. The perturbative series is divergent. For $d < 4$, the large order behaviour can be derived from a steepest descent calculation. For any physical observable f, the results have the general structure

$$f_k \underset{k\to\infty}{\propto} (-1)^k k^b a^k k!\,,$$

where a depends only on d and b is a half-integer that depends on the observable. The coefficient $A = 1/a$ has the value

$$d = 0: \ \ A = 3/2\,,$$

$$d = 1: \ \ A = 8 \ \text{(Bender–Wu)}\,.$$

$$d = 2: \ \ A = 35.10268957367896(1) \ \text{ZJ.}$$

$$d = 3: \ \ A = 113.38350781527714(1) \ \text{ZJ.}$$

For $d = 4$, to the contribution coming from the steepest descent calculation, a contribution due to the large momentum singularities of Feynman diagrams has in general to be added.

Finally, notice that for $d < 4$, Borel summability has been proved. For an early review, see J. Zinn-Justin, *Perturbation series at large orders in quantum mechanics and field theories: application to the problem of resummation*, *Phys. Rept.* 70 (1981) 109-167.

It follows that when is the expansion parameter is not small, summation is indispensable. We have explored a number of methods. I present here one such a method, based on Order-Dependent Mappings (ODM).

Order-dependent mapping

The order-dependent mapping (ODM) summation method is based on some knowledge of the analytic properties of the function which has been expanded. It applies both to convergent and divergent series, although it is mainly useful in the latter case.

Let $f(z)$ be an analytic function that has the Taylor series expansion

$$f(z) = \sum_{\ell=0} f_\ell z^\ell .$$

(the $=$ sign has to understood in the sense of series expansion.)

When the Taylor series has a finite radius of convergence, to continue the function in the whole domain of analyticity, one can map the domain onto a circle, while preserving the origin.

In a case of a divergent series, one adds to the domain of analyticity a disk $|z| < r$ of variable radius r and applies a similar mapping. Of course, the transformed series is still divergent. Then, as a empirical rule, for a divergent series, one is instructed to keep adding terms until the modulus of last term is minimum. By choosing order by order the radius r, one can manage to set the minimum always just at the last calculated term.

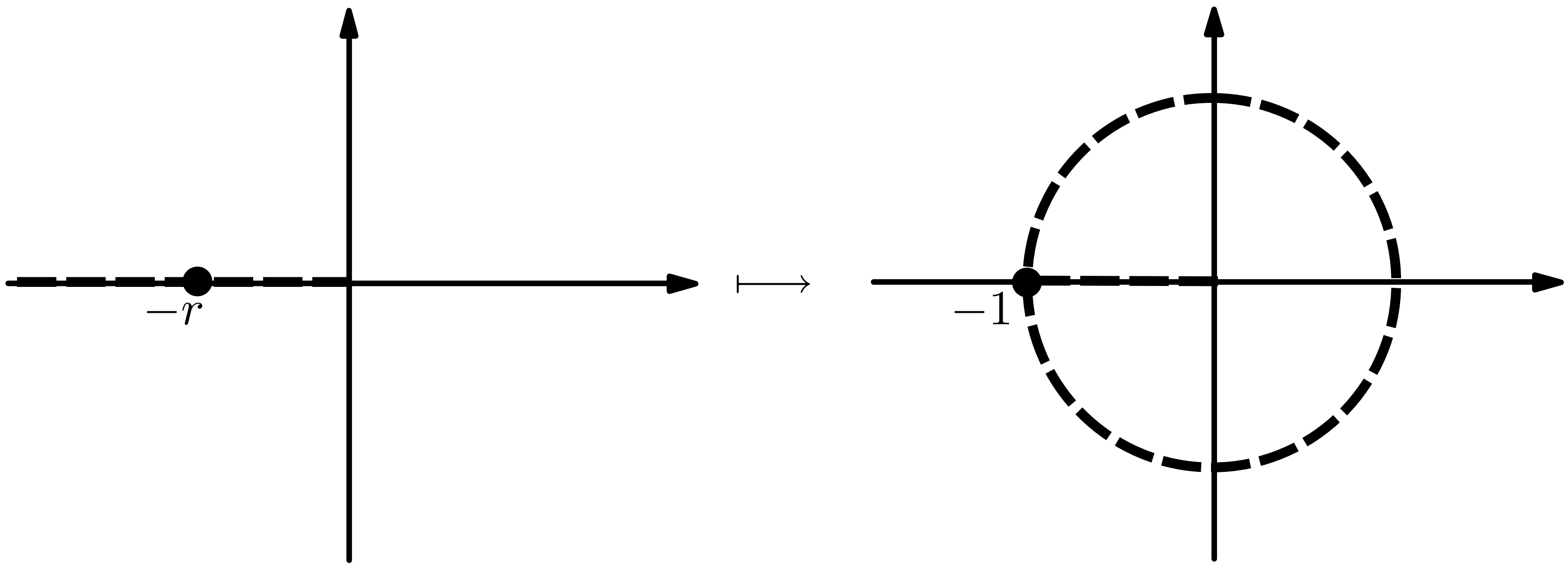

Fig. 1 Mapping: example of a function analytic in a cut-plane.

In what follows, we consider only functions analytic in a sector and mappings $z \mapsto \lambda$ of the form

$$z = \rho\zeta(\lambda), \quad \zeta(\lambda) = \lambda + O\left(\lambda^2\right),$$

where $\zeta(\lambda)$ is an explicit analytic function and ρ an order by order adjustable parameter.

Although the transformed series is still divergent at ρ fixed, we shall verify on a few examples that, by adjusting ρ order by order (here, we limit ourselves to Borel summable examples) one can construct a convergent algorithm.

After the transformation, f is given by a Taylor series in λ of the form

$$f\left(z(\lambda)\right) = \sum_{k=0} P_k(\rho)\lambda^k,$$

where the coefficients $P_k(\rho)$ are polynomials of degree k in ρ. Since the result is independent of the parameter ρ, the parameter can be chosen freely.

The k-th approximant is constructed in the following way: one truncates the expansion at order k and chooses ρ as to cancel the last term. Since $P_k(\rho)$ has k roots (real or complex) we choose for ρ the largest possible root (in modulus) ρ_k for which $P_k'(\rho)$ is small. This leads to a sequence of approximants

$$f^{(k)}(z) = \sum_{\ell=0}^{k-1} P_\ell(\rho_k)\lambda^\ell(\rho_k, z) \quad \text{with} \quad P_k(\rho_k) = 0.$$

In the case of convergent series, it is expected that ρ_k has a non-vanishing limit. By contrast, for divergent series it is expected that ρ_k goes to zero for large k as

$$\rho_k = O(f_k)^{-1/k}.$$

The intuitive idea here is that ρ_k corresponds to a 'local' radius of convergence.

Since ρ_k goes to zero, the function $\zeta(\lambda)$ must diverge for a finite value of λ. Below, we choose $\lambda = 1$ by convention.

Remark.

In the case of real functions when the relevant zeros are complex it is often convenient to choose minima of the polynomials P_k, which satisfy

$$P'_k(\rho_k) = 0\,,$$

choosing, in general, the largest zero for which P_k is small. Other mixed criteria can also be used.

Application: The simple integral $d = 0$

Then,

$$Z(g) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x \; \mathrm{e}^{-x^2/2 - gx^4/4!} \,.$$

Analytic properties suggest that the optimal mapping is given by setting

$$g = \frac{\rho\lambda}{(1-\lambda)^2} \text{ and } Z(g) = (1-\lambda)^{1/2} f(\lambda).$$

With this choice, the first approximant is variational.

Then, f has an expansion of the form

$$f(\lambda) = \sum_k P_k(\rho)[\lambda(g)]^k \,.$$

Convergence can be studied analytically. As expected, one finds

$$\rho_k \sim \frac{R}{k} \text{ with } R = 4.526638\ldots \;\; (R/A = 3.017759).$$

At g fixed, λ converges to 1. More precisely,

$$\lambda = 1 - \sqrt{R/kg} + O(1/k) \;\Rightarrow\; \lambda^k \sim \mathrm{e}^{-\sqrt{Rk/g}}\,.$$

Finally,

$$P_k(\rho_k) \propto \mathrm{e}^{-3k/R} = (0.515\ldots)^k.$$

The ODM approximants thus converge geometrically on the entire Riemann surface.

The quartic anharmonic oscillator: $d = 1$

The path integral corresponds to the quantum Hamiltonian

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}x^2 + \frac{g}{4!}x^4 .$$

As an example, we consider the perturbative expansion of the ground state energy. Variational arguments and scaling suggest the mapping

$$g = \frac{\rho\lambda}{(1-\lambda)^{3/2}} , \qquad E = \frac{\epsilon}{(1-\lambda)^{1/2}} . \tag{1}$$

Then,

$$\epsilon = \sum_k P_k(\rho)[\lambda(g)]^k .$$

Large order behaviour and a steepest calculation lead to the prediction

$\rho_k \sim R/k$ with $R = 32.25\ldots \quad (R/A = 4.031)$, which agrees with numerical data. Again λ converges to 1:

$$\lambda = 1 - \left(\frac{R}{kg}\right)^{2/3} + O(k^{-4/3}) \;\Rightarrow\; \lambda^k \sim \mathrm{e}^{-\mu k^{1/3}/g^{2/3}} \text{ with } \mu = R^{2/3} = 10.13(1).$$

Finally,

$$P_{k+1}(\rho_k) \propto \mathrm{e}^{-9.7k^{1/3}}\,.$$

The error at order k is thus of order $\mathrm{e}^{-k^{1/3}(9.7+\mu g^{-2/3})}$. One finds convergence in the domain

$$0.96 + |g|^{-2/3}\cos(\tfrac{2}{3}\mathrm{Arg}\, g) > 0\,,$$

a domain that contains a section of the first Riemann sheet and extends to the second Riemann sheet for $|g|$ large enough.

ϕ^4 field theory in $d = 3$ dimensions

Due to UV divergences and a needed regularization, scaling arguments are no longer applicable to determine an appropriate mapping. A more sophisticated study suggests a mapping of the form

$$g = \frac{\rho\lambda}{(1-\lambda)^{1/\omega}}$$

but the difficulty is that the exponent ω ($\omega = 0.80(1)$ from Borel transformation and mapping, Guida–ZJ 1998) has to be calculated from a series itself. The results obtained in this way are consistent with those obtained from Borel transformation and mapping, but the empirical errors are more difficult to determine. Also the expected rate of convergence is of order $\mathrm{e}^{-\mathrm{const}.k^{1-\omega}} = \mathrm{e}^{-\mathrm{const}.k^{0.2}}$, which is rather slow.

Table

Series for the exponent ω summed by ODM in the ϕ_3^4 field theory with $\omega_{\mathrm{i}} = 0.79$ and $\mathrm{d}\omega_{\mathrm{cal.}}/\mathrm{d}\omega_{\mathrm{i}} = -0.6$.

k	2	3	4	5	6
ω_k	0.552	0.754	0.711	0.767	0.759

Conclusion

Since it was proposed, the ODM method has found a number of useful applications in physics (sometimes under different names). Convergence proofs has been given for specific examples (Guida–Konishi–Suzuki), however, to my knowledge, systematic mathematical investigations are still lacking and would be most welcome.