

100 years of improvements of bounding properties of Padé approximants to the Stieltjes functions

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Abstract

This story starts from the Stieltjes work at the end of 19th century. In his work T.J. Stieltjes introduced the functions defined as follows

$$z \in \mathbb{C} \setminus]-R, \infty[: \quad f(z) = \int_0^{1/R} \frac{d\mu(t)}{1+tz}, \quad d\mu \geq 0 \quad (1)$$

now called Stieltjes functions. At the same time H. Padé studied the rational approximations $\frac{P_m}{Q_n} =: [m/n]$ of functions defined by:

$$f(z) - \frac{P_m(z)}{Q_n(z)} = O(z^{m+n+1})$$

now called Padé approximants. Because the convergents (or approximants) of continued fractions are, in fact, the Padé approximants, then the convergence of continued fractions can be equivalently called convergence of Padé approximants. To prove this convergence to the functions (1) Stieltjes used the following inequalities:

$$\begin{aligned} x \in]-R, 0[: \quad & 0 \leq [0/0] \leq [0/1] \leq \dots \leq [n-1/n] \leq [n/n] \leq \dots \leq f \\ x \in [0, \infty[: \quad & 0 \leq [0/1] \leq [n-1/n] \dots \leq f \leq \dots \leq [n/n] \leq \dots \leq [0/0] \end{aligned} \quad (2)$$

Inequality (2) can be rewritten as follows:

$$x \in]-R, 0[: \quad 0 \leq f - [n/n] \leq f - [n-1/n] \quad (3)$$

becoming inequality between the errors of contiguous Padé approximants to Stieltjes function. In 1979 J.G. and Alphonse Ph. Magnus remarked that the orders of the left hand side of the inequality (3) and of the right hand side are different: $(2n+1)$ for the first and $2n$ for the second. Finally, after about ten years of different tentatives to prove it (with A.Ph. Magnus and also Jaime Vinuesa), the method of continued fractions appeared as the best tool to study the Padé approximant errors. The inequality (3) was optimized as follows:

$$x \in]-R, 0[: \quad 0 \leq f(x) - [n/n](x) \leq -\frac{x}{R} \left(f(x) - [n-1/n](x) \right).$$

and the similar order equilibrated inequalities between the contiguous entries were proved in the full Padé error table.

The discovered method being very efficient, the optimal inequalities between the errors of two-point (0 and ∞) Padé approximant errors for the Stieltjes functions were proved in 2002, and for the N -point ($N > 2$) Padé approximant errors in 2004 by J.G. with the polish team (M. Pindor, St. Tokarzewski, J.J. Telega). The two-sided bounds of Stieltjes functions, from top and from below, by N -point Padé approximants were also obtained. We hope then the same method can be successfully applied to obtain the inequalities for the approximants of the new type of continued fractions, U-fractions, introduced recently (OPSFA9, 2007) by A.Ph. Magnus, S. Tokarzewski and J.G.