

Convergent and divergent series, solutions of the Prolate Spheroidal differential equation

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Abstract

The prolate spheroidal wave functions, $\{\varphi_{n,\sigma,\tau}\}$, constitute an orthonormal basis of the space of σ -bandlimited functions on the real line, i. e. functions whose Fourier transforms have support on the interval $[-\sigma, \sigma]$. They can be characterized as the eigenfunctions of a differential operator of order 2:

$$(\tau^2 - t^2)\varphi''_{n,\sigma,\tau} - 2t\varphi'_{n,\sigma,\tau} - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}.$$

In this talk we will present some new results obtained on the formal solutions of this equation. For this purpose, we specialize to particular values of the two parameters: $\sigma = \tau = 1$. We use the MAPLE package DESIR to compute the formal solutions in the neighborhood of the singularities (the regular ones ± 1 , and the irregular one, infinity) and to do some numerical experiments: computation of Stokes matrices [1] and of monodromy. This leads to the conjecture that the following properties are equivalent:

- μ is an eigenvalue of the differential operator $L = (t^2 - 1)\frac{d^2}{dt^2} + 2t\frac{d}{dt} + t^2$;
- the series solutions near ± 1 of the equation $L(y) = \mu y$ are entire functions (and so, eigenfunctions);
- the series appearing in the solutions near infinity of the equation $L(y) = \mu y$ are convergent;
- the Stokes phenomenon of the operator $L - \mu$ at infinity is trivial;
- the monodromy around $[-1, 1]$ of the operator $L - \mu$ is trivial.

The second part of the talk will give the proof of the conjecture.

References

- [1] Fauvet F., Richard-Jung F., Thomann J., *Automatic computation of Stokes matrices*, Numerical Algorithms, volume 50, number 2, feb. 2009, online at <http://dx.doi.org/10.1007/s11075-008-9223-6>
- [2] Ramis J.P., *Séries divergentes et théories asymptotiques. (French) [Divergent series and asymptotic theories]* Bull. Soc. Math. France 121 (1993), Panoramas et Synthèses, suppl., 74 pp.
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