

# Acceleration of convergence of series via orthogonal polynomials

Dirk Laurie

Department of Mathematical Sciences, University of Stellenbosch, South Africa  
email: dlaurie@na-net.ornl.gov

## Abstract

The proposed method is a variation of the summation method of [Cohen et al.(2000)] which is included in Pari-GP. The method is based on assuming the existence of an integral representation of the terms to be summed.

We wish to sum a series  $a_1, a_2, a_3, \dots$ . Suppose that there exists  $r \in (-1, 1)$  and a weight function  $w$ , non-negative over  $(0, r)$ , such that  $a_k = \int_0^r t^{k-1} w(t) dt$ ,  $k = 1, 2, \dots$ . This is true if  $r^k a_k$  is totally monotonic, i.e. the  $j$ -th difference  $\Delta^j(r^k a_k)$ ,  $j = 0, 1, 2, \dots$ , has the constant sign  $(-1)^j$ ,  $j = 0, 1, 2, \dots$ .

The sum of the series is then given by  $s = \int_0^r (1-t)^{-1} w(t) dt$ . This integral can be approximated as

$$s_{k,n} = \int_0^r \frac{(1-t^k p_n(t)) w(t) dt}{1-t},$$

which can be evaluated exactly as a linear combination of the terms  $a_1, a_2, a_3, \dots$ , with coefficients obtained by polynomial division. The exact error is

$$s_{k,n} = \int_0^r \frac{t^k p_n(t) w(t) dt}{1-t}.$$

[Cohen et al.(2000)] have several suggestions for the polynomials  $p_n$ , including shifted Chebyshev polynomials. The proposed method is to reverse-engineer  $w$  from what is known about the terms  $a_k$ , and then to use polynomials orthogonal over  $(0, r)$  with respect to  $w$ . Even the crude approximation  $w = 1$ , giving the Legendre polynomials, often gives better results than using the Chebyshev polynomials. The three-term recursion formula for the polynomials translates to a rhombus algorithm for the  $s_{k,n}$ .

A particularly favourable example occurs when the terms have geometric-harmonic behaviour, i.e.  $a_k = \frac{r^k}{k+\beta}$ , with  $r$  very close but not equal to  $-1$ . Such a series is usually quite troublesome to general-purpose convergence acceleration methods, e.g. the epsilon, Levin and theta algorithms. We do better because the proposed method can utilize our knowledge of  $\beta$  and  $r$ . The appropriate polynomials are the shifted Jacobi polynomials  $p_n(t) = J_n^{(0, \beta-1)}(x)$ , where  $x = 2t/r - 1$ . When  $r = -0.94$ ,  $\beta = 0.125$ . we get 15 correct digits from 11 terms.

## References

[Cohen et al.(2000)] H. Cohen, F. Rodriguez Villegas, and D. Zagier, *Experimental Mathematics* **9**, 3–12 (2000), ISSN 1058-6458.