Survey of numerical stability issues in convergence acceleration

Avram Sidi

Computer Science Department, Technion–Israel Institute of Technology, Haifa 32000, Israel
email: `asidi@cs.technion.ac.il`

Abstract

A cardinal issue that arises in application of convergence acceleration (extrapolation) methods is that of numerical stability (rather lack of it) in floating-point arithmetic. This issue turns out to be critical because numerical instability is inherent, even built in, when convergence acceleration methods are applied to many sequences that occur commonly in practice. It is encountered, for example, when summing power series, Fourier series, or orthogonal polynomial expansions near points of singularity of the limit functions. If extrapolation methods are applied without taking this issue into account, the numerical accuracy they can attain is limited, and eventually destroyed completely, as more terms are added in the process. Therefore, it is important to understand the origin of the problem and to propose practical ways to solve it effectively. For a detailed discussion, see [1, Introduction]. A brief qualitative description of the subject follows:

Let $\{A_m\}$ be a sequence with limit A, and let E_n be approximations that are produced by some extrapolation method applied to $\{A_m\}$. Then, in almost all cases, E_n can be shown to be of the form $E_n = \sum_{i=0}^{K_n} \theta_{ni} A_i$, with $\sum_{i=0}^{K_n} \theta_{ni} = 1$. (Of course, the θ_{ni} depend on the A_i nonlinearly.) In case the A_i have been computed with absolute errors bounded by ϵ, the quantity $\Gamma_n = \sum_{i=0}^{K_n} |\theta_{ni}| \geq 1$ controls the propagation of these errors into E_n, in that, it can be argued that the error in the (computed) E_n is bounded by $\Gamma_n\,\epsilon$. Now, Γ_n may be unbounded as $n \to \infty$, in which case the error in the computed E_n tends to infinity. Even when $\sup_n \Gamma_n < \infty$, Γ_n may be very large and the accuracy attainable by the computed E_n may be quite limited. By a detailed study of the structure of Γ_n and its asymptotic behavior as $n \to \infty$, it becomes possible to design effective ways of applying the extrapolation methods to make Γ_n bounded or smaller, hence improving the quality of the computed E_n substantially.

In this survey, we discuss this issue within the context of several known extrapolation methods and show strategies of improving the performance of these extrapolation methods in the presence of built-in instabilities.

References

[1] A. Sidi, *Practical Extrapolation Methods: Theory and Practice*, Cambridge University Press, Cambridge, 2003.