

Analysis of divergent series by Euler–Maclaurin summation

Ian Thompson¹, Chris Linton²

¹Department of Mathematical Sciences, Loughborough University, Loughborough, Leics. UK
email: i.thompson@lboro.ac.uk

²Department of Mathematical Sciences, Loughborough University, Loughborough, Leics. UK
email: c.m.linton@lboro.ac.uk

Abstract

In this presentation, we will consider expressions of the form

$$S(\delta) = \sum_{s=0}^{\infty} (s+x)^n E(s, \delta), \quad n \in \mathbb{N}, \quad (1)$$

where x is a fixed real parameter, and the function E is such that the series converges for $\delta > 0$, but $E(s, 0) = E_0$ (a constant). Series of this type are of great importance in the study of wave scattering by periodic structures [1, 2], and it is often necessary to determine the nature of the singularity of the function $S(\delta)$ at the point $\delta = 0$. For the case where $E(s, \delta) = e^{-s\delta}$, Nørlund [3, p. 53] used Euler–Maclaurin summation [4] to obtain

$$\lim_{\delta \rightarrow 0^+} \left[\sum_{s=0}^{\infty} (s+x)^n e^{-(s+x)\delta} - \int_0^{\infty} s^n e^{-s\delta} ds \right] = -\frac{B_{n+1}(x)}{(n+1)}, \quad (2)$$

where $B_n(\cdot)$ is the Bernoulli polynomial of order n . This can be used as a means of expanding the series on the left-hand side in negative powers of δ , because the integral can be evaluated exactly in terms of the gamma function.

We will show how (2) can be generalised to account for other forms of the function E . The resulting formula is remarkably robust, in the sense that there are many cases where it is valid, despite the fact that the Euler–Maclaurin formula, from which it is derived, is not.

References

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