Inverse factorial series: a little known tool for the summation of divergent series

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Abstract

Let $\Omega\colon \mathbb{C} \to \mathbb{C}$ be a function which vanishes as $z \to +\infty$. A *factorial series* for $\Omega(z)$ is an expansion involving Pochhammer symbols:

$$\Omega(z) \; = \; \frac{a_0}{z} + \frac{a_1 1!}{z(z+1)} + \frac{a_2 2!}{z(z+1)(z+2)} + \cdots \; = \; \sum_{\nu=0}^{\infty} \frac{a_\nu \nu!}{(z)_{\nu+1}} \, . \tag{1}$$

Factorial series were already used in Stirling's classic book *Methodus Differentialis* (1730). Later, they were used quite a lot in the context of finite difference equations. But in recent years, factorial series have largely been neglected, which in my opinion is not justified: Factorial series have many interesting features which have not yet been exploited properly.

Factorial series occur in the theory of Stirling numbers. By means of these Stirling numbers, it is possible to transform inverse power series and factorial series into each other by means of comparatively simple algebraic operations. In particular, it is often possible to convert a *factorially divergent* inverse power series into a *convergent* factorial series.

By a simple change of argument, we obtain in this way a somewhat unusual expansion for a function $f(z)$ defined by a formal and thus possibly divergent power series:

$$f(z) \; = \; \sum_{n=0}^{\infty} \gamma_n \, z^n \; = \; \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \prod_{k=1}^{m} \frac{z}{z+1/k} \sum_{\mu=0}^{m} (-1)^\mu \, \mathbf{S}^{(1)}(m,\mu) \, \gamma_\mu \, . \tag{2}$$

Here, $\mathbf{S}^{(1)}(m,\mu)$ is a Stirling number of the first kind. If the power series coefficients γ_n have strictly alternating signs, then the value of the inner sum $\sum_{\mu=0}^{m}(-1)^\mu \mathbf{S}^{(1)}(m,\mu)\gamma_\mu$ is usually much smaller than the values of its terms, and (2) can be used for the summation of factorially divergent alternating power series.

Alternatively, a function $\Omega(z)$ represented by a factorial series can also be computed via the following integral representation:

$$\Omega(z) \; = \; \int_0^1 t^{z-1} \, \varphi_\Omega(t) \, \mathrm{d}t \, , \qquad \Re(z) > 0 \, , \tag{3a}$$

$$\varphi_\Omega(t) \; = \; \sum_{n=0}^{\infty} a_n \, (1-t)^n \, . \tag{3b}$$

If approximations to $\varphi_\Omega(t)$ are converted to Padé approximants, we obtain something resembling the well known *Borel-Padé* summation method.