

Approximation and extrapolation of convergent and divergent sequences and series

CIRM Luminy
September 28, 2009 - October 2, 2009

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On the convergence of rational Ritz values, with applications to rational interpolation of rational functions

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Abstract

We are interested in the question which poles of a (large degree) rational function with real poles and positive residuals (a Markov function) are well detected by a small degree rational interpolant, and how we can monitor by the choice of the real interpolation points such approximation properties, especially in the more delicate situation where interpolation points lie within the convex hull of the poles. Using classical orthogonality relations related to rational interpolation, we may rephrase this question as how well the zeros of a rational orthogonal function for a discrete scalar product approach the support of orthogonality. Initially, this question was motivated by the mathematically equivalent question of how well rational Ritz values approach the spectrum of a hermitian matrix [1].

In the present talk we suggest some asymptotic answer in terms of logarithmic potential theory, by generalizing work of Kuijlaars [2] (for Padé approximants at ∞ or polynomial Ritz values).

References

- [1] B. Beckermann, S. Güttel, R. Vandebril, On the convergence of rational Ritz values, University of Lille (2009), submitted.
http://math.univ-lille1.fr/bbecker/abstract/BB_SG_RV.pdf
- [2] A.B.J. Kuijlaars, Which eigenvalues are found by the Lanczos method?, SIAM Journal on Matrix Analysis and Applications, 22 (2000), pp. 306-321.

Froissart Doublets: Unequivocal Signal-Noise Separation by Padé-Based Quantifications of Time Signals Applied to Cancer Diagnostics through Magnetic Resonance Spectroscopy *

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Abstract

Magnetic resonance spectroscopy (MRS) is a key diagnostic modality in oncology due to detection of early changes in biomolecules as a salient feature of cancer. However, progress in MRS is hampered by the lack of mathematically reliable spectral analysis for extraction of quantifiable information from scanned tissue. A number of fitting algorithms are available, but they are all inadequate because of subjectivity, non-uniqueness and inability to separate physical (genuine) from unphysical (spurious, noisy) information. The fast Padé transform (FPT), which is a non-linear, polynomial quotient from the Padé approximant, bridges this gap by unambiguously resolving and quantifying tightly overlapped and nearly degenerate resonances [1-3]. The FPT uses pole-zero cancellation (Froissart doublets) to unequivocally distinguish true from spurious resonances. Spurious resonances are recognized by their twofold signature: (i) coincidence of the zeros of the numerator and denominator polynomials, (ii) the zero-valued amplitudes. The task is to identify and discard spurious resonances from the final results of spectral analysis. The computation is carried out by gradually and systematically increasing the degree of the Padé polynomials. As this degree changes, the reconstructed spectral parameters and spectra fluctuate until all the physical resonances stabilize. We will thoroughly illustrate the signal-noise separation by means of the general concept of Froissart doublets for noise-corrupted magnetic resonance time signals reminiscent of clinically encoded data employed for cancer diagnostics.

References

- [1] Dž. Belkić, *Quantum Mechanical Signal Processing and Spectral Analysis*, Institute of Physics Publishing, Bristol, 2005.
- [2] Dž. Belkić, *Exponential convergence rate (the spectral resolution) of the fast Padé transform for exact quantification in magnetic resonance spectroscopy*, *Phys. Med. Biol.* **51** (2006), no. 24, 6483–6512.
- [3] Dž. Belkić, K. Belkić, *The general concept of signal-noise separation (SNS): mathematical aspects and implementation in magnetic resonance spectroscopy*, *J. Math. Chem.* **45** (2009), no. 3, 563–597.

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\mathcal{PT} symmetry*

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Abstract

The average quantum physicist on the street believes that a quantum-mechanical Hamiltonian must be Dirac Hermitian (symmetric under combined matrix transposition and complex conjugation) in order that the energy eigenvalues are real and that time evolution is unitary. However, the Hamiltonian $H = p^2 + ix^3$, for example, which is clearly not Dirac Hermitian, has a real positive discrete spectrum and generates unitary time evolution, and thus it defines a fully consistent quantum mechanics. Evidently, the axiom of Dirac Hermiticity is too restrictive. The Hamiltonian $H = p^2 + ix^3$ is not Dirac Hermitian, but it is \mathcal{PT} symmetric; that is, it is symmetric under combined space reflection \mathcal{P} and time reversal \mathcal{T} . In general, if a Hamiltonian H is not Dirac Hermitian but has an unbroken \mathcal{PT} symmetry, there is a procedure for determining the adjoint operation under which H is Hermitian. (One should not assume that the adjoint operation that interchanges bra and ket vectors in the Hilbert space of states is the Dirac adjoint. This would be like postulating a priori what the metric $g^{\mu\nu}$ in curved space is before solving Einstein's equations.)

In the past year, new table-top experiments have been performed that allow one to observe the transition between theories having a broken and an unbroken \mathcal{PT} symmetry.

In the past a number of interesting quantum theories, such as the Lee model and the Pais-Uhlenbeck model, were abandoned because they were thought to have an incurable disease. The symptom of the disease was the appearance of ghost states (states of negative norm). The cause of the disease was that the Hamiltonians for these models were inappropriately treated as if they were Dirac Hermitian. The disease can be cured because the Hamiltonians for these models are \mathcal{PT} symmetric, and one can calculate exactly and in closed form the appropriate adjoint operation under which each Hamiltonian is Hermitian. When this is done, one can see immediately that there are no ghost states and that these models are perfectly acceptable quantum theories. Thus, generalizing the requirement of Dirac Hermiticity to \mathcal{PT} symmetry allows for the possibility of new kinds of quantum theories.

\mathcal{PT} -symmetric quantum theories may be viewed as extensions of ordinary quantum theories into the complex domain. \mathcal{PT} quantum theories can be better understood when the associated conventional classical-mechanical theories are extended into the complex domain as well. We will show that by extending classical mechanics into the complex domain, the classical-mechanical theories that one obtains share many of the features of ordinary quantum mechanics.

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From numerical quadrature to Padé approximation

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Abstract

The paper reviews the relation between Padé-type approximants of a power series and interpolatory quadrature formulas with free nodes, and that between Padé approximants and Gaussian quadrature methods.

Quadrature methods are well-known. They are used for obtaining an approximate value of a definite integral, and are described in any book of numerical analysis. In this talk, we will show that Padé-type approximants could be interpreted as quadrature formulas with free nodes for the special function $g(x) = 1/(1 - xt)$, and that Padé approximants are, in fact, Gaussian quadratures for the same function g . Thus Kronrod procedure and anti-Gaussian quadrature formulas could be used for estimating their accuracy. Then, Padé approximation for series of functions will be discussed. The talk will end by some perspectives for future researches.

The convergence of Fourier-Takenaka-Malmquist series*

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Abstract

The Takenaka-Malmquist basis is defined as

$$\mathcal{B}_0 = 1, \quad \mathcal{B}_n(z) = \frac{\sqrt{1 - |\alpha_n|^2}}{1 - \bar{\alpha}_n z} \prod_{k=1}^{n-1} \frac{z - \alpha_k}{1 - \bar{\alpha}_k z}, = \overline{\mathcal{B}_{-n}(1/\bar{z})} \quad n = 1, 2, \dots, \quad z \in \mathbb{C}$$

where $\{\alpha_k\}_1^\infty$ is a sequence of points in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. These form an orthogonal system with respect to the Lebesgue measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. See for example [2].

If $f(\omega)$ is a 2π -periodic function, we discuss the convergence of the truncated Fourier series

$$f_n(\omega) = \sum_{k < |n|} a_k \mathcal{B}_k(e^{i\omega})$$

to $f(\omega)$ as $n \rightarrow \infty$.

Some of this work is published in [1].

References

- [1] A. Bultheel and P. Carrette. Algebraic and spectral properties of general Toeplitz matrices. *SIAM J. Control Optim.*, 41(5):1413–1439, 2003.
- [2] J.L. Walsh. *Interpolation and approximation*, volume 20 of *Amer. Math. Soc. Colloq. Publ.* Amer. Math. Soc., Providence, Rhode Island, 3rd edition, 1960. First edition 1935.

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Padé approximants and the prediction of non-perturbative parameters in particle physics *

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Abstract

The theory of the strong interactions in particle physics is a gauge theory based on the $SU(3)$ group, for which a Lagrangian formulation exists in terms of quarks and gluons. However, at ordinary energies the theory is strongly coupled and the observable states are hadrons, composite states of quarks and gluons. Only at very high energies the coupling gets small enough such that one can use perturbative methods. This very limited amount of information makes it very difficult to make predictions on the parameters of the theory, which typically depend on the full range of energies. There is however an approximation to the theory based on a generic $SU(N)$ gauge group taking the limit $N \rightarrow \infty$. Interestingly, in the $SU(N \rightarrow \infty)$ theory one can prove that correlators are meromorphic [1]. This fact makes the theory of Padé approximants especially suited to extract information on the physical parameters of the $SU(N \rightarrow \infty)$ theory. I will discuss some applications (with different degrees of rigorousness) of the theory of Padé approximants in the determination of non-perturbative parameters in particle physics [2, 3]. I will make special emphasis on the open problems and present limitations of the approach.

References

- [1] E. Witten, *Baryons In The $1/N$ Expansion*, Nucl. Phys. B **160** (1979) 57.
- [2] M. Knecht, S. Peris and E. de Rafael, *A new approach to weak amplitudes in large $N(C)$ QCD*, Nucl. Phys. Proc. Suppl. **86** (2000) 279 [arXiv:hep-ph/9910396].
- [3] A. A. Migdal, *Multicolor QCD As Dual Resonance Theory*, Annals Phys. **109** (1977) 365.

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Positive rational interpolatory quadrature formulas on the unit circle and the interval *

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Abstract

In this talk we present a relation between rational Gauss-type (Gaussian, Gauss-Radau and Gauss-Lobatto) quadrature formulas that approximate integrals of the form

$$J_{\sigma}(F) = \int_{-1}^1 F(x)\sigma(x)dx,$$

and rational Szegő quadrature formulas that approximate integrals of the form

$$I_{\omega}(F) = \int_{-\pi}^{\pi} F(e^{i\theta})\omega(\theta)d\theta.$$

The functions σ and ω are assumed to be weight functions on $[-1, 1]$ and $[-\pi, \pi]$, respectively, and are related by

$$\omega(\theta) = \sigma(\cos \theta)|\sin \theta|.$$

Part of the content of this talk extend to the rational case some of the results obtained in [1]-[2]. We finally include some illustrative numerical examples.

References

- [1] A. Bultheel, L. Daruis and P. González-Vera, *A connection between quadrature formulas on the unit circle and the interval* $[-1, 1]$, Journal of Computational and Applied Mathematics 132(1) (2002), 1–14.
- [2] A. Bultheel, L. Daruis and P. González-Vera, *Positive interpolatory quadrature formulas and para-orthogonal polynomials*, Journal of Computational and Applied Mathematics 179 (2005), 97–119.

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Accelerating the convergence of continued fraction representations

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Abstract

A lot of well-known constants as well as elementary and special functions in mathematics, physics and engineering enjoy very nice continued fraction representations [9, 7, 10]. In addition, most of these fractions are limit-periodic. There is a lot of literature describing algorithms for the evaluation of these constants or functions making use of their continued fraction representations [8, 6, 4, 5, 3, 2].

The tail or remainder term of a convergent series representation converges to zero. But remarkably, the tail of a convergent continued fraction representation does itself not need to converge at all. A suitable approximation of the usually disregarded continued fraction tail may speed up the convergence of the continued fraction approximants. This idea is further elaborated in this talk [1].

References

- [1] F. Backeljauw, S. Becuwe and A. Cuyt, *Validated Evaluation of Special Mathematical Functions*, Lecture Notes in Computer Science **5144** (2008), 206–216.
- [2] I.J. Thompson and A.R. Barnett, *COULCC: a continued-fraction algorithm for Coulomb functions of complex order with complex arguments*, Comput. Phys. Comm. **36** (1985), 363–372.
- [3] I.A. Stegun and R. Zucker, *Automatic computing methods for special functions. Part II. The exponential integral $E_n(x)$* , J. Res. Nat. Bur. Standards **78B** (1974), no. 4, 199–205.
- [4] Y.L. Ratis and P. Fernández de Córdoba, *A code to calculate (high order) Bessel functions based on the continued fractions method*, Comput. Phys. Comm. **76** (1993), 381–388.
- [5] A. Gil and J. Segura, *Parabolic cylinder functions of integer and half-integer orders for nonnegative arguments*, Comput. Phys. Comm. **115** (1998), 69–86.
- [6] A. Gil and J. Segura, *Evaluation of Legendre functions of argument greater than one*, Comput. Phys. Comm. **105** (1997), 273–283.
- [7] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clarke, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.
- [8] A. Gil, J. Segura and N.M. Temme, *Numerical Methods for Special Functions*, Society for Industrial and Applied Mathematics, Philadelphia, 2007.
- [9] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables*, U.S. Government Printing Office, NBS, Washington, D. C., 1964.
- [10] A. Cuyt, V. Brevik Petersen, B. Verdonk, H. Waadeland and W.B. Jones, *Handbook of Continued Fractions for Special Functions*, Springer, Berlin, 2008.

Block QD Algorithm

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Abstract

The LR algorithm applied to a $n \times n$ tridiagonal matrix A and the qd algorithm are closely connected (see [1]). The LU decomposition of A is recursively used after having multiplied U by L at the previous step. If A is a positive definite symmetric matrix, if $E^{(k)}Q^{(k)}$ denotes the k th decomposition and if $q_i^{(k)}$ denotes the diagonal entry in row i in $Q^{(k)}$, then it was proved (see [2]) that

$$\begin{aligned} q_1^{(k)} &< q_1^{(k+1)}, & \forall k, \\ q_n^{(k)} &> q_n^{(k+1)}, & \forall k. \end{aligned}$$

This result was extended to the case of a positive definite symmetric band matrix of half width ℓ (that is to say $A_{i,j} = 0$ if $|i - j| > \ell$) such that $A_{i,j} \neq 0 \forall i, j$ such that $|i - j| = \ell$ (see [3]).

When $\ell = 2$, the characteristic determinant of A can be written as a six term recurrence relation. But, when $\ell > 2$, the expansion of the characteristic determinant as a recurrence relation is inextricable. On the other hand any $n\ell \times n\ell$ symmetric band matrix A with a half width ℓ can always be considered as a $n \times n$ tridiagonal block matrix whose the size of the blocks are $\ell \times \ell$. Then it corresponds to a Jacobi block matrix for some matrix orthogonal polynomials. The block LR (or block qd) algorithm can be applied to such a matrix if $A_{i,j} \neq 0$ if $|i - j| = \ell$. Then we prove that the eigenvalues $\lambda_i^{(k)}$, $i = 1, \dots, \ell$, of the first diagonal block $Q_1^{(k)}$ and those $\lambda_i^{(k)}$, $i = (n - 1)\ell + 1, \dots, n\ell$, of the last diagonal block $Q_n^{(k)}$ are such that

$$\begin{aligned} \lambda_i^{(k)} &< \lambda_i^{(k+1)}, & i = 1, \dots, \ell, \forall k, \\ \lambda_i^{(k)} &> \lambda_i^{(k+1)}, & i = (n - 1)\ell + 1, \dots, n\ell, \forall k. \end{aligned}$$

References

- [1] C. Brezinski, *Padé-Type Approximation and General Orthogonal Polynomials*, ISNM vol. 50, Birkhäuser, Basel, 1980.
- [2] A. Draux, *Improvement of the formal and numerical estimation of the constant in some Markov-Bernstein inequalities*, Numerical Algorithms **24** (2000), 31-58.
- [3] A. Draux, B. Moalla and M. Sadik, *Generalized qd algorithm and Markov-Bernstein inequalities for Jacobi weight*, to appear in Numerical Algorithms (2009).

The ambiguities in the determination of field correlators represented by asymptotic perturbation series*

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Abstract

Starting from the divergence pattern of perturbation expansions in Quantum Field Theory and the (assumed) asymptotic character of the series, we address the problem of ambiguity of a function determined by the perturbation expansion. We consider functions represented by an integral of the Laplace-Borel type along a general contour in the Borel complex plane. Proving a modified form of the Watson lemma (called Lemma 2 in the following), we obtain a large class of functions having the same asymptotic perturbation expansion, differing from each other by the angle of validity of the expansion. Remarkable correlations between the strength of the bounds on the remainder and the size of the angles of validity are obtained. Imposing weak conditions both on the Borel transform $B(u)$ of the expanded function and on the integration contour, our Lemma 2 reveals a great ambiguity of the resummation procedures having the same asymptotic expansion. The form and length of the contour do not affect the expansion, contributing only to the exponentially suppressed remainder. Applications to perturbative QCD are discussed, using the particular case of the Adler function.

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100 years of improvements of bounding properties of Padé approximants to the Stieltjes functions

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Abstract

This story starts from the Stieltjes work at the end of 19th century. In his work T.J.Stieltjes introduced the functions defined as follows

$$z \in \mathbb{C} \setminus]-R, \infty[: \quad f(z) = \int_0^{1/R} \frac{d\mu(t)}{1+tz}, \quad d\mu \geq 0 \quad (1)$$

now called Stieltjes functions. At the same time H.Padé studied the rational approximations $\frac{P_m}{Q_n} =: [m/n]$ of functions defined by:

$$f(z) - \frac{P_m(z)}{Q_n(z)} = O(z^{m+n+1})$$

now called Padé approximants. Because the convergents (or approximants) of continued fractions are, in fact, the Padé approximants, then the convergence of continued fractions can be equivalently called convergence of Padé approximants. To prove this convergence to the functions (1) Stieltjes used the following inequalities:

$$\begin{aligned} x \in]-R, 0[: \quad & 0 \leq [0/0] \leq [0/1] \leq \dots \leq [n-1/n] \leq [n/n] \leq \dots \leq f \\ x \in [0, \infty[: \quad & 0 \leq [0/1] \leq [n-1/n] \dots \leq f \leq \dots \leq [n/n] \leq \dots \leq [0/0] \end{aligned} \quad (2)$$

Inequality (2) can be rewritten as follows:

$$x \in]-R, 0[: \quad 0 \leq f - [n/n] \leq f - [n-1/n] \quad (3)$$

becoming inequality between the errors of contiguous Padé approximants to Stieltjes function. In 1979 J.G. and Alphonse Ph.Magnus remarked that the orders of the left hand side of the inequality (3) and of the right hand side are different: $(2n+1)$ for the first and $2n$ for the second. Finally, after about ten years of different tentatives to prove it (with A.Ph. Magnus and also Jaime Vinuesa), the method of continued fractions appeared as the best tool to study the Padé approximant errors. The inequality (3) was optimized as follows:

$$x \in]-R, 0[: \quad 0 \leq f(x) - [n/n](x) \leq -\frac{x}{R} \left(f(x) - [n-1/n](x) \right).$$

and the similar order equilibrated inequalities between the contiguous entries were proved in the full Padé error table.

The discovered method being very efficient, the optimal inequalities between the errors of two-point (0 and ∞) Padé approximant errors for the Stieltjes functions were proved in 2002, and for the N -point ($N > 2$) Padé approximant errors in 2004 by J.G. with the polish team (M.Pindor, St.Tokarzewski, J.J.Telega). The two-sided bounds of Stieltjes functions, from top and from below, by N -point Padé approximants were also obtained. We hope then the same method can be successfully applied to obtain the inequalities for the approximants of the new type of continued fractions, U-fractions, introduced recently (OPSFA9, 2007) by A.Ph. Magnus, S. Tokarzewski and J.G.

Computation of Rational Szegő-Lobatto Quadrature Formulas *

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Abstract

Szegő quadrature formulas are analogs of Gauss quadrature rules when dealing with the approximate integration of periodic functions, since they exactly integrate trigonometric polynomials of as high degree as possible, or more generally Laurent polynomials which can be viewed as rational functions with poles at the origin and infinity. When more general rational functions with prescribed poles on the extended complex plane not on the unit circle are considered to be exactly integrated, the so called “Rational Szegő Quadrature Formulas” appear. In this talk, and as a continuation of earlier papers ([1], [2]), some computational aspects concerning these quadratures are analyzed when one or two nodes are previously fixed on the unit circle.

References

- [1] C. Jagels, L. Reichel, *Szegő-Lobatto quadrature rules*, Journal of Computational and Applied Mathematics 200 (2007), 116–126.
- [2] A. Bultheel, P. González-Vera, E. Hendriksen, O. Njåstad *Rational quadrature formulas on the unit circle with prescribed node and maximal domain of validity*. IMA Journal of Numerical Analysis. To appear (2009).

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Asymptotic expansions and Mellin-Barnes representation *

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Abstract

The Mellin-Barnes (MB) representation of integrals is a powerful tool in asymptotic analysis. We show here a few of its applications, basing our presentation on quantum field theory examples. Perturbative approaches of quantum field theory imply the evaluation of Feynman diagrams, which are in general complicated multidimensional integrals. We first show how the MB representation of Feynman diagrams with one scale, when combined with the converse mapping theorem, allows the simplification of the analytic evaluation, to an arbitrary order, of their asymptotic expansions in powers and in logs of this scale [1]. Then, we present the extension of the method to Feynman diagrams containing several scales [2]. For this, we need multidimensional complex analysis and especially multidimensional residues theory. This is illustrated by evaluating analytically a specific class of 5-loop quantum electrodynamics contributions to the anomalous magnetic moment of the muon. The MB representation may also deal with non-perturbative aspects of quantum field theory. Using a method mixing MB representation and Borel resummation, we show that it is possible to rewrite, in terms of non-perturbative series, the tail of the (divergent) formal power series which constitutes the perturbative expansion of an arbitrary N -point function for the simple case of zero-dimensional ϕ^4 field theory [3]. An exponentially improved asymptotic expansion is first obtained, but one may then go further and we show that a structure of interwoven subleading nonperturbative series emerges (a hyperasymptotic expansion) whose coefficients are related to the perturbative ones by an interesting resurgence phenomenon. The MB representation allows our results to be automatically valid for a wide range of the phase of the complex coupling constant, including values for which the perturbative expansion is not Borel summable.

References

- [1] S. Friot, D. Greynat and E. de Rafael, Phys. Lett. B **628** (2005) 73
- [2] J. P. Aguilar, D. Greynat and E. De Rafael, Phys. Rev. D **77** (2008)
- [3] S. Friot, D. Greynat, *To be published*

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Some results on integrable algorithms *

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Abstract

It is known that the discrete integrable systems have played an important role in the field of numerical analysis, especially in convergence acceleration methods and matrix eigenvalue algorithms. The connection between convergence acceleration algorithms and discrete integrable systems is a subject whose interest is rapidly growing among workers in the field. In this talk, several known facts on integrable systems are briefly reviewed and some results on integrable numerical algorithms are reported.

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Bender-Wu formulas and generalized nonanalytic expansions for odd anharmonic oscillators

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Abstract

Since the seminal investigations of Bender and Wu in the 1970s [1], there has been one unresolved problem in the theory of anharmonic oscillators: What is the leading-order factorial growth of perturbative coefficients, for an arbitrary energy level of an arbitrary odd anharmonic oscillator? This very question has been answered by Bender and Wu for even oscillators, but the same basic question had been left unanswered for the odd case. We present results for dispersion relations and for generalized nonanalytic expansions that describe the energy levels, thereby answering a few of the remaining questions, and sustain the results by numerical verification [2].

References

- [1] C. M. Bender and T. T. Wu, Phys. Rev. Lett. **27**, 461 (1971).
- [2] U. D. Jentschura, A. Surzhykov and J. Zinn-Justin Phys. Rev. Lett. **102**, 011601 (2009).

Convergence of ray sequences of Padé approximants for ${}_2F_1(a, 1; c; z)$, for $c > a > 0^*$

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Abstract

The Padé table of ${}_2F_1(a, 1; c; z)$ is normal for $c > a > 0$ (cf. [2]). For $m \geq n - 1$ and $c \notin \mathbb{Z}^-$, the denominator polynomial $Q_{mn}(z)$ in the $[m/n]$ Padé approximant $P_{mn}(z)/Q_{mn}(z)$ for ${}_2F_1(a, 1; c; z)$ and the remainder term $Q_{mn}(z){}_2F_1(a, 1; c; z) - P_{mn}(z)$ were explicitly evaluated by Padé (cf. [1], [3] or [4]). We show that for $c > a > 0$ and $m \geq n - 1$, the poles of $P_{mn}(z)/Q_{mn}(z)$ lie on the cut $(1, \infty)$. We deduce that the sequence of approximants $P_{mn}(z)/Q_{mn}(z)$ converges to ${}_2F_1(a, 1; c; z)$ as $m \rightarrow \infty$, $n/m \rightarrow \rho$ with $0 < \rho \leq 1$, uniformly on compact subsets of the unit disc $|z| < 1$ for $c > a > 0$.

References

- [1] G.A. Baker, *Essentials of Padé Approximants*, Academic Press, New York, 1975.
- [2] M.G. de Bruin, *Some classes of Padé tables whose upper halves are normal*, *Nieuw Arch. Wisk.*, **25(3)** (1977), no. 4, 148–160.
- [3] H. Padé, *Recherches sur la convergence des développements en fractions continues d'une certaine catégorie de fonctions*, *Ann. Sc. École Norm. Sup.*, **214** (1907), no. 4, 341–400.
- [4] H. van Rossum, *Systems of orthogonal and quasi-orthogonal polynomials connected with the Padé table II*, *Nederl. Akad. Wetensch. Proc.*, **58** (1955), 526–534.

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Converting divergent weak-coupling expansions into exponentially fast convergent strong-coupling expansions

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Abstract

With the help of a variational method developed in the textbooks [1, 2] we convert the partial sums of order N of divergent series expansions $\sum_{n=0}^N a_n g^n$ of a function $f(g)$ into partial sums $\sum_{n=0}^N b_n g^{-\omega n}$ with some $\omega > 0$. The results of the latter partial sums converge against the exact value of $f(g)$ for g larger than some g_0 , with an error that decreases like $e^{-\text{const.} \times N}$.

References

- [1] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, World Scientific, Singapore, 2009 (<http://www.physik.fu-berlin.de/~kleinert/b5>).
- [2] H. Kleinert and V. Schulte-Frohlinde, *Critical Phenomena in Φ^4 -Theory*, World Scientific, Singapore, 2001 (<http://www.physik.fu-berlin.de/~kleinert/b8>).

Exact computation of Simultaneous Rational Approximants

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Abstract

In exact computing environments such as Maple and Mathematica problems often have symbolic parameters. As such a typical domain for computation is an integral domain (such as $\mathbb{Q}[a_1, \dots, a_k]$) rather than a field. In such environments growth of coefficients in intermediate computations are a central concern. For methods that involve elimination intermediate growth can be controlled by removing greatest common divisors at each step. Fraction-free computation is an elimination process which controls coefficient growth in intermediate computations while at the same time avoids expensive greatest common divisor computations.

In this talk we give a new, fast algorithm for solving the simultaneous Padé approximation problem. The algorithm is fraction-free and is intended for computation in exact arithmetic. The algorithm gives significant improvement on previous fraction-free methods, in particular when solved via the use of vector Hermite-Padé approximation using the FFFG order basis algorithm previously done by the authors. The improvements are both in terms of bit complexity and in reduced size of the intermediate quantities. The primary technique takes advantage of certain duality properties of Hermite-Padé and Simultaneous-Padé approximation problems.

References

- [1] B. Beckermann and G. Labahn. *Recursiveness in matrix rational interpolation problems*. Journal of Computational and Applied Math. **77** (1997), 5–34.
- [2] B. Beckermann and G. Labahn. *Fraction-free computation of matrix rational interpolants and matrix GCDs*. SIAM J. Matrix Analysis and its Applications, **22** (2000), no. 1, 114–144
- [3] K. Mahler. *Perfect systems.*, Compositio Math. **19**, (1968), 95–166

Acceleration of convergence of series via orthogonal polynomials

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Abstract

The proposed method is a variation of the summation method of [Cohen et al.(2000)] which is included in Pari-GP. The method is based on assuming the existence of an integral representation of the terms to be summed.

We wish to sum a series a_1, a_2, a_3, \dots . Suppose that there exists $r \in (-1, 1)$ and a weight function w , non-negative over $(0, r)$, such that $a_k = \int_0^r t^{k-1} w(t) dt$, $k = 1, 2, \dots$. This is true if $r^k a_k$ is totally monotonic, i.e. the j -th difference $\Delta^j(r^k a_k)$, $j = 0, 1, 2, \dots$, has the constant sign $(-1)^j$, $j = 0, 1, 2, \dots$.

The sum of the series is then given by $s = \int_0^r (1-t)^{-1} w(t) dt$. This integral can be approximated as

$$s_{k,n} = \int_0^r \frac{(1-t^k p_n(t)) w(t) dt}{1-t},$$

which can be evaluated exactly as a linear combination of the terms a_1, a_2, a_3, \dots , with coefficients obtained by polynomial division. The exact error is

$$s_{k,n} = \int_0^r \frac{t^k p_n(t) w(t) dt}{1-t}.$$

[Cohen et al.(2000)] have several suggestions for the polynomials p_n , including shifted Chebyshev polynomials. The proposed method is to reverse-engineer w from what is known about the terms a_k , and then to use polynomials orthogonal over $(0, r)$ with respect to w . Even the crude approximation $w = 1$, giving the Legendre polynomials, often gives better results than using the Chebyshev polynomials. The three-term recursion formula for the polynomials translates to a rhombus algorithm for the $s_{k,n}$.

A particularly favourable example occurs when the terms have geometric-harmonic behaviour, i.e. $a_k = \frac{r^k}{k+\beta}$, with r very close but not equal to -1 . Such a series is usually quite troublesome to general-purpose convergence acceleration methods, e.g. the epsilon, Levin and theta algorithms. We do better because the proposed method can utilize our knowledge of β and r . The appropriate polynomials are the shifted Jacobi polynomials $p_n(t) = J_n^{(0, \beta-1)}(x)$, where $x = 2t/r - 1$. When $r = -0.94$, $\beta = 0.125$. we get 15 correct digits from 11 terms.

References

[Cohen et al.(2000)] H. Cohen, F. Rodriguez Villegas, and D. Zagier, *Experimental Mathematics* **9**, 3–12 (2000), ISSN 1058-6458.

On the perfectness of Nikishin systems*

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Abstract

In 1980, E.M. Nikishin introduced general systems of Markov type functions which are called Nikishin systems. They have attracted great attention in connection with the study of the convergence of their simultaneous Hermite-Padé approximation and the asymptotic properties of the common denominator of these rational approximants. Multi-indices for which the common denominator of the corresponding Hermite-Padé approximation has maximum degree are called normal. The concept of normal index for general systems of functions was introduced by K. Mahler. Systems for which all multi-indices are normal receive the name of perfect systems. We prove that Nikishin systems are perfect and give some applications of this result.

References

- [1] K. Mahler, *Perfect systems*, Compositio Math. **19** (1968), 95–166
- [2] E.M. Nikishin, *On simultaneous Padé approximants*, Math. USSR Sb. **41** (1980), 409–425.

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Convergence of sequences of linear fractional transformations

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Abstract

Linear fractional transformations occur in a number of mathematical areas. Many of these applications depend on convergence properties for sequences of such transformations. Therefore we can find convergence criteria for such sequences in many different settings, such as for instance operator theory, group theory, continued fractions, orthogonal polynomials and similar recurrence theory, dynamical systems, moment problems etc. It is therefore no surprise that convergence theorems are discovered and rediscovered throughout the literature.

In this talk we try to unify some of these results and even extend some of them, in order to give a better picture of what is actually known.

Numerical evaluation of Oscillatory Integrals*

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Abstract

Asymptotic expansions of the form

$$\int_a^b F(x)e^{ikG(x)} dx \sim e^{ika} \sum \alpha_n k^{-\mu_n} + e^{ikb} \sum \beta_n k^{-\nu_n}$$

are derived. F and G are of the form $\tilde{F}(x)(x-a)^\lambda$ and $\tilde{G}(x)(x-a)^s$, respectively, where \tilde{F} and \tilde{G} are real and analytic on $[a, b]$. $\{\mu_n\}$ and $\{\nu_n\}$ are monotonic increasing sequences and α_n and β_n depend on the local behavior of F and G at a and at b .

This derivation is based on inverse functions and does not involve steepest descent or stationary phase.

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Smoothing the Gibbs phenomenon using Padé-Hermite approximants

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Abstract

In order to reduce the Gibbs phenomenon exhibited by the partial Fourier sums of a periodic function f , defined on $[-\pi, \pi]$, discontinuous at 0, Driscoll and Fornberg [1] suggested the construction of a class of approximants which incorporate the knowledge of that singularity. More precisely, their approach is the following one: let g_2 denote the series such that $f(t) = \Re(g_2(e^{it}))$. Then, the goal is to approach g_2 on the unit circle (and more precisely its real part). It is typical that the singularity of the function f , located at 0 say, corresponds to a logarithmic singularity for g_2 , then located at 1, and that this function g_2 is analytic in the complex plane, with a branch cut that can be taken as the interval $[1, \infty)$. Defining $g_1(z) = \log(1 - z)$, we may consider the problem of determining polynomials p_0, p_1, p_2 such that

$$p_0(z) + p_1(z)g_1(z) + p_2(z)g_2(z) = \mathcal{O}(z^{n_0+n_1+n_2+2}) \quad (z \rightarrow 0)$$

where n_j denotes the degree of p_j , $j = 0, 1, 2$. We can then propose the *Hermite-Padé approximant*

$$\Pi_{\vec{n}}(z) = -\frac{p_0(z) + p_1(z)g_1(z)}{p_2(z)}, \quad (1)$$

to approximate g_2 . Note that when $p_1(z) = 0$ (or formally $n_1 = -1$) we recover the usual Padé approximant of g_2 of type (n_0, n_2) .

Convincing numerical experiments have been obtained by Driscoll and Fornberg, but no error estimates have been proven so far. In this talk we obtain rates of convergence of sequences of Hermite-Padé approximants for a class of functions known as *Nikishin systems*. Our theoretical findings and numerical experiments confirm that particular sequences of Hermite-Padé approximants (diagonal and row sequences, as well as linear HP approximants) are more efficient than the more elementary Padé approximants, particularly around the discontinuity of the goal function f .

References

- [1] T. Driscoll, B. Fornberg, *A Padé-based algorithm for overcoming the Gibbs phenomenon*, Numer. Algorithms **26** (2001), 77-92.

Method of summation of some slowly convergent series

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Abstract

A new method of summation of slowly convergent series is proposed. It may be successfully applied to the summation of generalized and basic hypergeometric series, as well as some classical orthogonal polynomial series expansions. In some special cases, our algorithm is equivalent to as Wynn's epsilon algorithm, Weniger transformation [3, §8.2] or the technique recently introduced by Čížek, Zamastil and Skála [1]. In the case of trigonometric series, our method is very similar to the Homeier's \mathcal{H} transformation, while in the case of orthogonal series — to the \mathcal{H} transformation; see [2, §4.2.9, §4.2.11]. Two iterated methods related to the proposed method are considered. Some theoretical results and several illustrative numerical examples are given.

References

- [1] J. Čížek, J. Zamastil, and L. Skála, *New summation technique for rapidly divergent perturbation series. Hydrogen atom in magnetic field*, Journal of Mathematical Physics **44** (2003), no. 3, 962–968.
- [2] H. H. H. Homeier, *Scalar Levin-type sequence transformations*, Journal of Computational and Applied Mathematics **122** (2000), 81–147.
- [3] E. J. Weniger, *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Computer Physics Reports **10** (1989), 189–371.

Studying the asymptotic structure of solutions of hydrodynamical equations

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Abstract

It is well known that analytic properties of functions of complex variables are tightly connected to the asymptotic behavior of the Taylor or Fourier coefficients. Thus, one approach to studying the analytic properties of solutions of partial differential equations (e.g. of the ones encountered in hydrodynamics) consists of determining the asymptotic structure of their Fourier or Taylor coefficients.

We have applied the so called asymptotic extrapolation method developed by van der Hoeven [1], after testing it on the example of the inviscid Burgers (or Riemann) equation [2], to determine the asymptotic structure of the Fourier coefficients of solutions of the two-dimensional Euler equation

$$\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla p, \quad \nabla \cdot \vec{v} = 0, \quad (1)$$

and of the one-dimensional Burgers equation with modified dissipation

$$\partial_t u + u \partial_x u = -\mu \mathcal{D}u, \quad (2)$$

where the dissipation term $\mu \mathcal{D}u$ is given in the Fourier representation by $k^{2\alpha} \hat{u}$ or $e^k \hat{u}$.

In the case of the Euler equation [3] as well as in the case of the Burgers equation we were able to determine the asymptotic structure of the solutions using high precision arithmetics. Our numerical findings are partially supported by theory [4, 5].

References

- [1] J. van der Hoeven, *On asymptotic extrapolation*, Journal of Scientific Computation, in press, 2009.
- [2] W. Pauls and U. Frisch, *A Borel transform method for locating singularities of Taylor and Fourier series*, J. Stat. Phys. **127** (2007), 1095–1119.
- [3] W. Pauls, *On complex singularities of the Euler equation*, submitted to Physica D, 2009.
- [4] C. Bardos, U. Frisch, W. Pauls, S. S. Ray and E. S. Titi, *Entire solutions of hydrodynamical equations with exponential dissipation*, Communications in Mathematical Physics, in press, 2009.
- [5] A. Gilbert and W. Pauls, *Complex manifolds for the Euler equations: a hierarchy of ODEs and the case of vanishing angle in two dimensions*, submitted to Nonlinearity, 2009.

Large- N_c quantum chromodynamics and rational approximants*

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Abstract

Quantum Chromodynamics is the fundamental theory of the interactions of quarks and gluons which underlies, among other things, Nuclear Physics. Mathematically, this is a gauge theory based on the “color” quantum number, N_c , following the structure of the $SU(N_c)$ Lie group. Although in the real world $N_c = 3$, ever since the pioneering work of Ref. [1], it has been widely appreciated that it is very useful to study this theory in a power series in $1/N_c$ around $N_c = \infty$. This expansion is similar to an expansion in the number of components of a field, like e.g. in the CP^N theories. In the limit $N_c \rightarrow \infty$, QCD is described in terms of Green’s functions which are meromorphic, with infinite isolated poles located on the real axis. The position of these poles corresponds to the value of the mass of the physical particle present in the spectrum of the Hamiltonian. Despite continuous efforts, no solution to QCD at $N_c = \infty$ has been found. On the other hand, in recent years, it has been proposed that saturating these infinite sums with just *one* term could be a reasonable approximation, leading to a significant phenomenological success[2] which encompasses the old and celebrated Vector Meson Dominance approximation from the 60’s. I would like to point out with the help of model calculations [3] that this saturation with a finite number of poles can be best understood within the theory of Pade Approximants, explaining different results which have appeared in the recent literature.

References

- [1] G. ’t Hooft, *A Two-Dimensional Model For Mesons*, Nucl. Phys. B **75** (1974) 461; E. Witten, *Baryons In The $1/N$ Expansion*, Nucl. Phys. B **160** (1979) 57.
- [2] M. Knecht and E. de Rafael, *Patterns of spontaneous chiral symmetry breaking in the large $N(c)$ limit of QCD-like theories*, Phys. Lett. B **424**, 335 (1998) [arXiv:hep-ph/9712457]; S. Peris, M. Perrottet and E. de Rafael, *Matching long and short distances in large- $N(c)$ QCD*, JHEP **9805**, 011 (1998) [arXiv:hep-ph/9805442];
- [3] P. Masjuan and S. Peris, *A Rational Approach to Resonance Saturation in large- N_c QCD*, JHEP **0705**, 040 (2007) [arXiv:0704.1247 [hep-ph]].

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Implementations of the Levin-Weniger convergence accelerator and applications to problems in physics

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Abstract

Levin sequence transformations [1] as generalized in [2], are useful tools for the summation of slowly convergent series appearing in atomic variational calculations (as an example, see [3]). The necessity of high precision results for the upper bound estimates has the consequence of dealing with a high number of summations that have to be done accurately. Therefore, special care must be paid in the balance between precision and computational time.

For the numerical series that appear in the evaluation of the multi-electron integrals required, it is not known a general expansion for large n of the difference between the infinite and the partial sum up to n . The generalized Levin formula for the convergence acceleration requires a choice for the remainder estimates ω_n . In this work we will propose and additional choice of ω_n which comes from the use of the Euler-McLaurin formula for series for which the general term behaves asymptotically as $n^{-\alpha}$ and compare to the Levin u -transform for different types of series of physical interest. In particular, we will find some examples where the present choice is exact.

In addition to this, two strategies will also be presented in this work for reducing the well known unstabilities of the convergence accelerators due to precision losses: (i) using a rearrangement of terms for the numerical evaluation and (ii) by means of an arbitrary precision implementation of the Levin sequence transformation in C++ using a free multi-precision library (MPFR). Some numerical tests will be shown.

References

- [1] D. Levin, *Development of non-linear transformations for improving convergence of sequences*, Int. J. Comput. Math. **3**, 371 (1973).
- [2] E.J. Weniger, *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Comput. Phys. Rep. **10**, 189 (1989).
- [3] I. Porrás and F.W. King, *Evaluation of some integrals for the atomic three-electron problem using convergence accelerators*, Phys. Rev. A **49**, 1637 (1994).

An extended procedure for extrapolation to the limit

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Abstract

In this talk, a new procedure for the extrapolation to the limit of slowly convergent sequences, and the summation of convergent and divergent series is proposed. It is a generalization of the E -algorithm which is the most general extrapolation algorithm known so far. Some properties of the kernel of this transformation are given. Particular cases, such as Drummond's transform and extensions are given. These transformations are related to Padé and Padé-type approximants.

Convergent and divergent series, solutions of the Prolate Spheroidal differential equation

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Abstract

The prolate spheroidal wave functions, $\{\varphi_{n,\sigma,\tau}\}$, constitute an orthonormal basis of the space of σ -bandlimited functions on the real line, i. e. functions whose Fourier transforms have support on the interval $[-\sigma, \sigma]$. They can be characterized as the eigenfunctions of a differential operator of order 2:

$$(\tau^2 - t^2)\varphi''_{n,\sigma,\tau} - 2t\varphi'_{n,\sigma,\tau} - \sigma^2 t^2 \varphi_{n,\sigma,\tau} = \mu_{n,\sigma,\tau} \varphi_{n,\sigma,\tau}.$$

In this talk we will present some new results obtained on the formal solutions of this equation. For this purpose, we specialize to particular values of the two parameters: $\sigma = \tau = 1$. We use the MAPLE package DESIR to compute the formal solutions in the neighborhood of the singularities (the regular ones ± 1 , and the irregular one, infinity) and to do some numerical experiments: computation of Stokes matrices [1] and of monodromy. This leads to the conjecture that the following properties are equivalent:

- μ is an eigenvalue of the differential operator $L = (t^2 - 1)\frac{d^2}{dt^2} + 2t\frac{d}{dt} + t^2$;
- the series solutions near ± 1 of the equation $L(y) = \mu y$ are entire functions (and so, eigenfunctions);
- the series appearing in the solutions near infinity of the equation $L(y) = \mu y$ are convergent;
- the Stokes phenomenon of the operator $L - \mu$ at infinity is trivial;
- the monodromy around $[-1, 1]$ of the operator $L - \mu$ is trivial.

The second part of the talk will give the proof of the conjecture.

References

- [1] Fauvet F., Richard-Jung F., Thomann J., *Automatic computation of Stokes matrices*, Numerical Algorithms, volume 50, number 2, feb. 2009, online at <http://dx.doi.org/10.1007/s11075-008-9223-6>
- [2] Ramis J.P., *Séries divergentes et théories asymptotiques. (French) [Divergent series and asymptotic theories]* Bull. Soc. Math. France 121 (1993), Panoramas et Synthèses, suppl., 74 pp.
- [3] Walter G., Soleski T., *A friendly Method of Computing Prolate Spheroidal Wave Functions and Wavelets*, Comp. Harmonic Anal. **19**, 432-443 (2005).

Rational approximations to values of the Gamma function at rational points

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Abstract

I will show how to obtain sequences of rational approximations that rapidly converge to any value $\Gamma(a/b)$, $a/b \in \mathbb{Q} \setminus \mathbb{Z}$ by means of Padé approximants applied to divergent power series similar to Euler's series $\sum_{n=0}^{\infty} n!x^n$. A similar method enables me to obtain rational approximations to any of the numbers $\gamma + \log(x)$, where γ is Euler's constant and $x \in \mathbb{Q}$, $x > 0$.

References

- [1] T. Rivoal, *Rational approximations for values of derivatives of the Gamma function*, 37 pages, to appear in the Transactions of the AMS.
- [2] T. Rivoal, *Approximations rationnelles des valeurs de la fonction Gamma aux rationnels*, 13 pages, submitted.

Generalized techniques in numerical integration

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Abstract

Integration by parts is one of the most popular techniques in the analysis of integrals. The product of the technique is usually a divergent series formed from evaluating boundary terms [1]; however, sometimes the remaining integral is also evaluated [2].

Due to the successive differentiation and antidifferentiation required to form the series or the remaining integral, the technique in its raw form is difficult to apply to problems more complicated than the simplest. In this talk, we explore a generalized and formalized integration by parts to create equivalent representations to some challenging integrals. Coupled with sequence transformations and/or extrapolation methods, we assess each of the different methods that can be formed from our formalized integration by parts.

As a demonstrative archetype, we examine the infinite-range Fresnel integrals. We also examine some peculiarities with the methods applied to the Twisted Tail. And lastly, we examine the most challenging molecular integrals that arise in molecular structure calculation using exponential type functions as a basis set of atomic orbitals.

References

- [1] E. J. Weniger, Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, *Comp. Phys. Rep.*, 10, 189–371, 1989.
- [2] H. Safouhi, The properties of sine, spherical Bessel and reduced Bessel functions for improving convergence of semi-infinite very oscillatory integrals: the evaluation of three-center nuclear attraction integrals over B functions, *J. Phys. A: Math. Gen.*, 34, 13, 2801–2818, 2001.

Padé Theory and phenomenology of resonance poles *

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Abstract

We use Padé approximants for the description of hadronic matrix elements. We analyze resonant amplitudes in the elastic region. By means of the Montessus de Ballore's theorem we are able to extract properties of the amplitude in a model independent way. For instance, we will show how it is possible to obtain the resonance pole mass and width without relying in any particular hadronic model.

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Survey of numerical stability issues in convergence acceleration

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Abstract

A cardinal issue that arises in application of convergence acceleration (extrapolation) methods is that of numerical stability (rather lack of it) in floating-point arithmetic. This issue turns out to be critical because numerical instability is inherent, even built in, when convergence acceleration methods are applied to many sequences that occur commonly in practice. It is encountered, for example, when summing power series, Fourier series, or orthogonal polynomial expansions near points of singularity of the limit functions. If extrapolation methods are applied without taking this issue into account, the numerical accuracy they can attain is limited, and eventually destroyed completely, as more terms are added in the process. Therefore, it is important to understand the origin of the problem and to propose practical ways to solve it effectively. For a detailed discussion, see [1, Introduction]. A brief qualitative description of the subject follows:

Let $\{A_m\}$ be a sequence with limit A , and let E_n be approximations that are produced by some extrapolation method applied to $\{A_m\}$. Then, in almost all cases, E_n can be shown to be of the form $E_n = \sum_{i=0}^{K_n} \theta_{ni} A_i$, with $\sum_{i=0}^{K_n} \theta_{ni} = 1$. (Of course, the θ_{ni} depend on the A_i nonlinearly.) In case the A_i have been computed with absolute errors bounded by ϵ , the quantity $\Gamma_n = \sum_{i=0}^{K_n} |\theta_{ni}| \geq 1$ controls the propagation of these errors into E_n , in that, it can be argued that the error in the (computed) E_n is bounded by $\Gamma_n \epsilon$. Now, Γ_n may be unbounded as $n \rightarrow \infty$, in which case the error in the computed E_n tends to infinity. Even when $\sup_n \Gamma_n < \infty$, Γ_n may be very large and the accuracy attainable by the computed E_n may be quite limited. By a detailed study of the structure of Γ_n and its asymptotic behavior as $n \rightarrow \infty$, it becomes possible to design effective ways of applying the extrapolation methods to make Γ_n bounded or smaller, hence improving the quality of the computed E_n substantially.

In this survey, we discuss this issue within the context of several known extrapolation methods and show strategies of improving the performance of these extrapolation methods in the presence of built-in instabilities.

References

- [1] A. Sidi, *Practical Extrapolation Methods: Theory and Practice*, Cambridge University Press, Cambridge, 2003.

On the JWKB expansion and Borel summability, with particular attention to modifications of the radial Schrödinger equation

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Abstract

The JWKB method is generally characterized as an asymptotic expansion for the logarithm of the wave function in powers of \hbar . In applications to the radial Schrödinger equation, retention of \hbar in the centrifugal potential leads to ambiguity. Kramers [1] himself implicitly took advantage of the ambiguity by modifying the potential, replacing $\hbar^2 l(l+1)/2mr^2$ by $\hbar^2(l+1/2)^2/2mr^2$, to get better results. Following Kramers, many modifications of the centrifugal potential have been proposed, which can best be understood in terms of a two- \hbar analysis, one \hbar for expansion, the other for the centrifugal potential, to be set equal at the end to recover the physical problem [2].

When provable, Borel summability greatly enhances the significance of a JWKB expansion, making it equivalent to an exact solution and providing a method for obtaining accurate numerical results. Proof of Borel summability of the some of the two- \hbar JWKB expansions for the radial Coulomb problem will given.

References

- [1] H. A. Kramers, *Wave mechanics and half-integral quantisation*, Z. Phys. **39** (1926), no. 10/11, 828–840.
- [2] Tatsuya Koike and Harris J. Silverstone, *Rereading Langer's influential 1937 JWKB paper: the unnecessary Langer transformation; the two \hbar 's*, (submitted for publication).

Strong coupling asymptotics of the β -function in ϕ^4 theory and QED

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Abstract

The well-known algorithm for summing of divergent series is based on the Borel transformation in combination with the conformal mapping (Le Guillou and Zinn-Justin, 1977). Modification of this algorithm allows to determine a strong coupling asymptotics of the sum of the series through the values of the expansion coefficients. Application of the algorithm to the β -function of φ^4 theory leads to the asymptotics $\beta(g) = \beta_\infty g^\alpha$ at $g \rightarrow \infty$, where $\alpha \approx 1$ for space dimensions $d = 2, 3, 4$. The natural hypothesis arises, that asymptotic behavior is $\beta(g) \sim g$ for all d . Consideration of the "toy" zero-dimensional model confirms the hypothesis and reveals the origin of this result: it is related with a zero of a certain functional integral. Generalization of this mechanism to the arbitrary space dimensionality leads to the linear asymptotics of $\beta(g)$ for all d . The same idea can be applied for QED and gives asymptotics $\beta(g) = g$, where g is the running fine structure constant. Relation to the "zero charge" problem is discussed.

Analysis of divergent series by Euler–Maclaurin summation

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Abstract

In this presentation, we will consider expressions of the form

$$S(\delta) = \sum_{s=0}^{\infty} (s+x)^n E(s, \delta), \quad n \in \mathbb{N}, \quad (1)$$

where x is a fixed real parameter, and the function E is such that the series converges for $\delta > 0$, but $E(s, 0) = E_0$ (a constant). Series of this type are of great importance in the study of wave scattering by periodic structures [1, 2], and it is often necessary to determine the nature of the singularity of the function $S(\delta)$ at the point $\delta = 0$. For the case where $E(s, \delta) = e^{-s\delta}$, Nørlund [3, p. 53] used Euler–Maclaurin summation [4] to obtain

$$\lim_{\delta \rightarrow 0^+} \left[\sum_{s=0}^{\infty} (s+x)^n e^{-(s+x)\delta} - \int_0^{\infty} s^n e^{-s\delta} ds \right] = -\frac{B_{n+1}(x)}{(n+1)}, \quad (2)$$

where $B_n(\cdot)$ is the Bernoulli polynomial of order n . This can be used as a means of expanding the series on the left-hand side in negative powers of δ , because the integral can be evaluated exactly in terms of the gamma function.

We will show how (2) can be generalised to account for other forms of the function E . The resulting formula is remarkably robust, in the sense that there are many cases where it is valid, despite the fact that the Euler–Maclaurin formula, from which it is derived, is not.

References

- [1] V. Twersky. *Elementary function representation of Schlömilch series*, Arch. Rational Mech. Anal. **8** (1961) 323–332.
- [2] C. M. Linton. *Lattice sums for the Helmholtz equation*. Submitted*.
- [3] N. E. Nørlund. *Vorlesungen über Differenzenrechnung*, Springer Verlag, Berlin, 1924[†].
- [4] T. M. Apostol. *An elementary view of Euler’s summation formula*, Amer. Math. Monthly, **106** (1999), no. 5, 409–418.

*Preprint: www.lboro.ac.uk/departments/ma/research/preprints/papers09/09-04.pdf

[†]Available online at <http://resolver.sub.uni-goettingen.de/purl?PPN373206070>

Unusual convergence behaviour of certain rational interpolants

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Abstract

In [2, 3, 5] we discuss a rational interpolation procedure that generalises near-best polynomial interpolation in Chebyshev points. For a given (fixed) sequence of (real or complex) poles $\{\alpha_1, \alpha_2, \dots\}$ outside $[-1, 1]$, the corresponding interpolation points are zeros or maxima of so-called Chebyshev rational functions that were introduced in [1, 4].

Underlying these interpolation points are certain conformal maps $g(x)$ defined implicitly by

$$g^{-1}(y) = (J \circ B_N^{1/N} \circ J^{-1})(y),$$

where $J(x) = (x + x^{-1})/2$ is the Joukowski map, $B_N(x)$ is the finite Blaschke product

$$B_N(x) = \prod_{k=1}^N \frac{x - \beta_k}{1 - \overline{\beta_k}x},$$

and $\beta_k = J^{-1}(\alpha_k)$.

In this talk we discuss a different interpolation procedure based on these maps, which exhibits very unusual convergence behaviour.

References

- [1] K. Deckers, J. Van Deun, and A. Bultheel. Rational Gauss-Chebyshev quadrature formulas for complex poles outside $[-1, 1]$. *Math. Comput.*, 77(262):967–983, 2008.
- [2] J. Van Deun. Eigenvalue problems to compute almost optimal points for rational interpolation with prescribed poles. *Numer. Algorithms*, 45:89–99, 2007.
- [3] J. Van Deun. Electrostatics and ghost poles in near best fixed pole rational interpolation. *Electron. Trans. Numer. Anal.*, 26:439–452, 2007.
- [4] J. Van Deun, A. Bultheel, and P. González Vera. On computing rational Gauss–Chebyshev quadrature formulas. *Math. Comput.*, 75(253):307–326, 2006.
- [5] J. Van Deun, K. Deckers, A. Bultheel, and J. A. C. Weideman. Algorithm 882: Near-best fixed pole rational interpolation with applications in spectral methods. *ACM Trans. Math. Software*, 35(2):14:1–14:21, 2008.

Inverse factorial series: a little known tool for the summation of divergent series

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Abstract

Let $\Omega: \mathbb{C} \rightarrow \mathbb{C}$ be a function which vanishes as $z \rightarrow +\infty$. A *factorial series* for $\Omega(z)$ is an expansion involving Pochhammer symbols:

$$\Omega(z) = \frac{a_0}{z} + \frac{a_1 1!}{z(z+1)} + \frac{a_2 2!}{z(z+1)(z+2)} + \dots = \sum_{\nu=0}^{\infty} \frac{a_\nu \nu!}{(z)_{\nu+1}}. \quad (1)$$

Factorial series were already used in Stirling's classic book *Methodus Differentialis* (1730). Later, they were used quite a lot in the context of finite difference equations. But in recent years, factorial series have largely been neglected, which in my opinion is not justified: Factorial series have many interesting features which have not yet been exploited properly.

Factorial series occur in the theory of Stirling numbers. By means of these Stirling numbers, it is possible to transform inverse power series and factorial series into each other by means of comparatively simple algebraic operations. In particular, it is often possible to convert a *factorially divergent* inverse power series into a *convergent* factorial series.

By a simple change of argument, we obtain in this way a somewhat unusual expansion for a function $f(z)$ defined by a formal and thus possibly divergent power series:

$$f(z) = \sum_{n=0}^{\infty} \gamma_n z^n = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \prod_{k=1}^m \frac{z}{z+1/k} \sum_{\mu=0}^m (-1)^\mu \mathbf{S}^{(1)}(m, \mu) \gamma_\mu. \quad (2)$$

Here, $\mathbf{S}^{(1)}(m, \mu)$ is a Stirling number of the first kind. If the power series coefficients γ_n have strictly alternating signs, then the value of the inner sum $\sum_{\mu=0}^m (-1)^\mu \mathbf{S}^{(1)}(m, \mu) \gamma_\mu$ is usually much smaller than the values of its terms, and (2) can be used for the summation of factorially divergent alternating power series.

Alternatively, a function $\Omega(z)$ represented by a factorial series can also be computed via the following integral representation:

$$\Omega(z) = \int_0^1 t^{z-1} \varphi_\Omega(t) dt, \quad \Re(z) > 0, \quad (3a)$$

$$\varphi_\Omega(t) = \sum_{n=0}^{\infty} a_n (1-t)^n. \quad (3b)$$

If approximations to $\varphi_\Omega(t)$ are converted to Padé approximants, we obtain something resembling the well known *Borel-Padé* summation method.

Efficient algorithm for summation of some slowly convergent series

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Abstract

The \mathcal{Q} transformation, introduced recently in [1], may serve us a good tool for summation of slowly convergence series. As was shown in the mentioned paper, this approach can be easily applied to the case of generalized and basic hypergeometric series, as well as some orthogonal polynomial expansions. It is closely related to the famous Wynn's epsilon algorithm, Weniger's or Homeier's transformations, and the method introduced by Čížek, Zamastil and Skála.

However, it is difficult to use the algorithm proposed in [1]—because of its high complexity, and some other restrictions—in the case of arbitrary series. In this talk, we propose another realization of the \mathcal{Q} transformation, resulting in obtaining a simpler and faster algorithm. Notice that it can implemented in a symbolic or numerical version.

References

- [1] P. Woźny, R. Nowak, *Method of summation of some slowly convergent series*, submitted.

Divergent series in quantum mechanics

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Abstract

This talk is concerned with two themes: how to obtain the large-order behavior of the divergent series (LOBDS) and how to use it for the summation of the series.

As shown explicitly for the first time in [1] the large-order behavior of the perturbation coefficients for wide class of the eigenvalue problems can be through dispersion relations reduced to the WKB calculation of the tunneling through potential barrier. Several recent improvements in constructing the WKB approximation will be described, namely the following.

First, it is not necessary to deal with inadequacy of the WKB approximation at the classical turning points [2]. Second, the WKB approximation can be formulated directly as an expansion in pertinent coupling constant without any reference to quantum-classical correspondence [2]. Third, for multidimensional problems the WKB approximation can be formulated in such a way that the calculation can be reduced to the quadratures and at the same time the method yields systematic approximation to the probability flux [3]. Applying this method to Zeeman effect in hydrogen we were able for the first time to carry out the multidimensional WKB approximation explicitly beyond the leading order [4].

The use of LOBDS for the summation of the series will be discussed next. We show that first LOBDS naturally leads to the form of the sequence transformation [5]. Second LOBDS yields the singularities of the Borel transform [6], third it improves the summation to the smallest term and Padé summation for Stieltjes series.

References

- [1] C. M. Bender and T. T. Wu, Phys. Rev. D **7**, 1620 (1973).
- [2] J. Zamastil, J. Čížek and L. Skála, Phys. Rev. Lett. **84**, 5683 (2000). See also J. Zamastil, J. Čížek and L. Skála, Phys. Rev. A **63**, 022107 (2001).
- [3] J. Zamastil, Phys. Rev. A **72**, 024101 (2005). See also J. Zamastil, V. Špirko, J. Čížek, L. Skála, and O. Bludský, Phys. Rev. A **64**, 042101 (2001).
- [4] J. Zamastil and L. Skála, J. Math. Phys. **47**, 022106 (2006).
- [5] J. Čížek, J. Zamastil and L. Skála, J. Math. Phys. **44**, 962 (2003).
- [6] J. C. Le Guillou and J. Zinn-Justin Eds. *Large-Order Behaviour of Perturbation Theory*, North-Holland, Amsterdam, 1990.

Order-dependent mapping: summation of divergent series

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Abstract

Order-dependent mapping has been proposed in [1] as a simple method to sum some class of divergent series that appear frequently in physics. The method has found since many applications and I shall review a few.

References

- [1] R. Seznec and J. Zinn-Justin, *Summation of divergent series by order dependent mappings: Application to the anharmonic oscillator and critical exponents in field theory*, J. Math. Phys **20** (1979), no. 7, 1398–1408.

Asymptotic expansions of Euler series truncation errors via Bell polynomials

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Abstract

By using the approach recently proposed by Weniger [1], an asymptotic analysis of the remainder obtained by truncating the Euler series (ES henceforth) to the n th-order term is presented. In particular, closed-form expansions of the remainder, both in inverse powers and in inverse rising factorials of n , are found. Such expansions involve the Bell polynomials[2], well known in combinatorics[3], and two families of polynomials closely related to them. Recurrence rules are derived for these polynomials, as well as the closed-form expressions of the corresponding generating functions.

References

- [1] E. J. Weniger, *Asymptotic approximations to truncation errors of series representations for special functions*, in *Algorithms for Approximation Proceedings of the 5th International Conference*, Chester, July 2005, arXiv.org e-Print archive, math. CA/0511074v1.
- [2] E.T. Bell, "Exponential polynomials," *Ann. of Math.* **35** (1934), 258–277.
- [3] J. Riordan, *An Introduction to Combinatorial Analysis* Wiley, New York, 1980.

Numerical advances from rational approximation in modelling economic time series data

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Abstract

In this poster, we present a chronological summary of the results we have obtained in modelling economic and financial time-series data by using numerical methods associated with the rational approximation. In economics, the systematic treatment of data to obtain specific properties from long (or short) data series is a main objective. Therefore, the use of rational models and related numerical methods can be useful to help to predict the behaviour of relevant economic variables with a certain degree of certainty. Some numerical methods, closely related to theoretical research in Pad approximation, have been proposed to identify some type of rational structure associated with economic data in different contexts (financial, marketing, farming, energy and water consumption). First, considering the causal theory in time domain we present two alternatives to improve the fit and forecasting of classic time series models: The matrix theory and the non-causal theory. Finally, we contribute with some recent results in scale domain (wavelets) to suitable economic data.

Padé approximant in complex points revisited

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Abstract

It is "well known" that the interlaced zeros and poles of Padé approximants describe the position of cuts of considered function f . More, the Padé approximants choose automatically this position, "in principal" in the direction joining the point of developpement of f and its ramification point. The "well known" property was studied by J.S.R.Chisholm and A.C.Genz et M.Pusterla forty years ago for $\ln(1-z)$ function at the complex points, but the "well known" results produced by the authors are false. A number of numerical examples show that the positions of zeros and poles deviate from the supposed "well known" position. We show also, that only N -point Padé approximants computed with pairs of points, complex and complex conjugate, leads to the traditional position of cuts.

References

- [1] G. A. Baker Jr., *Essentials of Padé approximants*, Academic Press, 1975, New York
- [2] J. S. R. Chisholm, A. C. Genz, M. Pusterla, *A method for computing Feynman amplitudes with branch cuts*, J. Comp. Appl. Math., **2**, 1976, 73–76
- [3] J. K. Gelfgren, *Multipoint Padé approximants converging to functions of Stieltjes type*, Padé Approximation and its Application Amsterdam, **888**, 1981, 179–207. Springer, Berlin
- [4] J. Gilewicz, *Approximants de Padé*, Lectures Notes in Mathematics, **667**, 1978, Springer, Berlin
- [5] J. Gilewicz, M. Pindor, J. J. Telega, S. Tokarzewski, *N -point Padé Approximants and two-sided estimates of errors on the real axis for Stieltjes functions*, Journal of Computational and Applied Mathematics, **178**, 2005, 247–602.
- [6] J. Karlsson, E. B. Saff, *Singularities of functions determined by the poles of Padé approximants*, Padé Approximation and its Application Amsterdam **888**, 1981, 239–254. Springer, Berlin
- [7] S. Klarsfeld, *Padé approximant and related methods for computing boundary values on cuts*, Padé Approximation and its Application Amsterdam **888**, 1981, 255–262. Springer, Berlin
- [8] J. Nuttall, *Location of poles of Padé approximants to entire functions*, Rational Approximation and interpolation, **1105**, 1984, 354–363, Springer, Berlin

Asymptotic behaviours and general recurrence relations

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Abstract

The study of the polynomials defined by the General recurrence relations of the form $P_{-1}(z) \equiv 0$, $P_0(z) \equiv 1$ and

$$\forall k \geq 0 \quad P_{k+1}(z) = (z - a_k)P_k(z) - \sum_{i=\max(-1, k-q)}^{k-1} b_{i+1}^{[k]} P_i(z),$$

where a_k , $b_j^{[k]}$ are complex numbers and q the order of the recurrence, is an important thing for the applications - see already the wide literature on the subject when $q = 1$ (the Three-term recurrence relation case); some authors have also obtained some interesting results for $q > 1$.

The aim of this talk is the presentation of some asymptotic behaviours for the polynomials generated by some higher recurrence relations (e.g. $q > 1$). We have already given some results in [1,2,3].

References

- [1] Leopold. E, *Perturbed recurrence relations, II the general case*, Numer.Algor. **44** (2007), 347–366.
- [2] Leopold. E, *Perturbed recurrence relations, III the general case - some new applications*, Numer.Algor. **48** (2008), 383–402.
- [3] Leopold. E, *Some asymptotic behaviours for perturbed general recurrence relations*, submitted.

Recurrence relations for multiple orthogonal polynomials of classical weights by a generating function

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Abstract

The multiple orthogonal polynomials (multiple OPS) are historically much related to simultaneous Padé approximants. Nowadays it is tried to find many properties of multiple OPS considering as a natural extension of ordinary orthogonal polynomials [3,4]. For example, differential equations for classical multiple OPS such as Jacobi-Piñeiro polynomials, multiple Bessel polynomial, multiple Laguerre I and II polynomials, and multiple Hermite polynomials are given [1].

For the multiple Hermite polynomials, the multiple Laguerre I and multiple Laguerre II polynomials, the author found the generating functions by Cauchy integral formula in order to get properties such as recurrence relations and differential equations. See [2] and references therein for details.

In this presentation we introduce a method to find a generating function for classical multiple OPS including Jacobi-Piñeiro polynomials and the multiple Bessel polynomials, and then obtain new recurrence relations.

References

- [1] A. I. Aptekarev, A. Branquinho, and W. Van Assche, *Multiple orthogonal polynomials for classical weights*, Trans. Amer. Math. Soc. **355** (2001), 3887–3914.
- [2] D. W. Lee, *Properties of multiple Hermite and multiple Laguerre polynomials by the generating function*, Integral Transforms Spec. Func. **18** (2007), 855–869.
- [3] E. M. Nikishin and V. N. Sorokin, *Rational Approximation and Orthogonality*, American Mathematical Society, Rhode Island, 1991.
- [4] W. Van Assche and E. Coussement, *Some classical multiple orthogonal polynomials*, J. Comp. Appl. Math. **127** (2001), 317–347.

Quantum Appell polynomials and their quadratic decomposition *

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Abstract

By performing the quadratic decomposition of H_q -Appell sequences, where H_q is the lowering (or annihilating) operator defined by $H_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}$, another lowering operator $\mathcal{L}_{q;\varepsilon}$ (with $\varepsilon^2 = 1$) arises, since the two polynomial sequences lying in the principal diagonal are $\mathcal{L}_{q;\varepsilon}$ -Appell. Triggered by this result, after developing the concept of the $\mathcal{L}_{q;\varepsilon}$ -Appell sequences, all the orthogonal $\mathcal{L}_{q;\varepsilon}$ -Appell sequences are sought, which outcome was the *Little q -Laguerre* polynomial sequences - they are indeed the unique ones fulfilling both properties. These latter are not only H_q but also $\mathcal{L}_{q;\varepsilon}$ -classical sequences in Hahn's sense, which opens up the problem of finding all the orthogonal sequences $\{P_n\}_{n \geq 0}$ such that their orthogonality is preserved by the operator $\mathcal{L}_{q;\varepsilon}$.

References

- [1] Y. Ben Cheikh, *On obtaining dual sequences via quasi-monomiality*, Georgian Math. J. **9** (2002), 413–422.
- [2] M. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge Univ. press, UK, 2005.
- [3] L. Khérijji, P. Maroni, *The H_q -classical orthogonal polynomials*, Acta Appl. Math. **71** (2002), 49–115.
- [4] Ana F. Loureiro, Pascal Maroni, *Quadratic decomposition of Appell sequences*, Expo. Math. **26** (2008), no. 2, 177–186.

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Extrapolation Methods: a tool for accelerating real life problems

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Abstract

The mathematical model \mathcal{P} of a real life problem is, typically, a set of complicated non-linear differential equations. The corresponding numerical solution is obtained solving iteratively a, possible, infinite sequence of simpler non linear problems \mathcal{P}_i , $i = 1, 2, \dots$ which approximate better and better the original one. The idea of this iterative process is well known: problem \mathcal{P}_i uses the solution produced by problem \mathcal{P}_{i-1} to generate the next approximation of the final solution of the original problem. This algorithm proceeds until some convergence criterion is satisfied.

From a computational point of view, each step of this iterative step may be time consuming and the entire process may require a high number of steps. Thus, it is interesting to investigate the possibility to accelerate the convergence process. In this paper we show the improvements gained using some acceleration techniques in three engineering fields.

First, we show the acceleration improvement obtained applying the behavior of the convergence process of the steady state Navier-Stokes equations. Second, we show the behavior of the convergence process in the simulation of a MOSFET, one of the most important electronic devices, with and without the application of some kind of polynomial acceleration on the Gummel map. Third, we study the solution of a simple non-linear passive electronic net consisting of some diodes and resistors both using Aitken and a polynomial acceleration.

References

- [1] R. Bertelle, M.R. Russo, *An approach to the Gummel map by vector extrapolation methods*, Numer. Algorithms **45** (2007), no. 1-4, 331–343.
- [2] C. Brezinski, M. Redivo Zaglia, *Extrapolation methods. Theory and practice.*, North–Holland Publishing Co., Amsterdam, 1991.
- [3] Jbilou, K., Sadok, H., *Some results about vector extrapolation methods and related fixed-point iterations*, Journal of Computational and Applied Mathematics **36**, 385–398 (1991).
- [4] Schilders, W.H.A., Gough, P.A., Whight, K., *Extrapolation techniques for improved convergence in semiconductor device simulation*, Proceedings of the NASECODE VIII Conference, Boole Press, Dublin, (1992).