Logica intuizionista predicativa LI

$$\begin{array}{c} \text{ax-id} \\ A \vdash A \end{array} \qquad \begin{array}{c} \text{ax-}\bot \\ \bot \vdash \end{array}$$

$$\frac{\Gamma \vdash \Sigma}{\Gamma, \Gamma' \vdash \Sigma}\ \text{in}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma}{\Gamma \vdash \Sigma, \Sigma'}\ \text{in}_{\text{dx}}$$

$$\frac{\Sigma, \Gamma, \Theta, \Gamma', \Delta \vdash \nabla}{\Sigma, \Gamma', \Theta, \Gamma, \Delta \vdash \nabla}\ \text{sc}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Theta, \Delta', \nabla}{\Gamma \vdash \Sigma, \Delta', \Theta, \Delta, \nabla}\ \text{sc}_{\text{dx}}$$

$$\frac{\Sigma, \Gamma, \Gamma, \Delta \vdash A}{\Sigma, \Gamma, \Delta \vdash A}\ \text{cn}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Delta, \nabla}{\Gamma \vdash \Sigma, \Delta, \nabla}\ \text{cn}_{\text{dx}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A\&B}\ \&-\text{F} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A\&B \vdash C}\ \&-\text{re}_1 \qquad \frac{\Gamma, B \vdash C}{\Gamma, A\&B \vdash C}\ \&-\text{re}_2$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee-\text{F} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee-\text{re}_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee-\text{re}_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to-\text{F} \qquad \frac{\Gamma' \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B, \Gamma' \vdash C}\ \to-\text{re}$$

$$\frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x A(x)}\ \forall-\text{F}\ (x \notin VL(\Gamma)) \qquad \frac{\Gamma, A(t) \vdash C}{\Gamma, \forall x\ A(x) \vdash C}\ \forall-\text{re}$$

$$\frac{\Gamma, A(x) \vdash C}{\Gamma, \exists x\ A(x) \vdash C}\ \exists-\text{F}\ (x \notin VL(\Gamma, C)) \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists\ x\ A(x)}\ \exists-\text{re}$$

$$\begin{array}{c} =-\text{ax} \\ \vdash t = t \end{array} \qquad \frac{\Gamma(t) \vdash C(t)}{\Gamma(s), t = s \vdash C(s)}\ =-\text{F}$$

Calcolo intuizionista proposizionale senza indebolimento e contrazione $\mathbf{LI}_p^{abbr}$

$$\begin{array}{cc} \text{ax-id}^* & \text{ax}^*\text{-}\bot \\ \Gamma, A, \Gamma' \vdash A & \Gamma, \bot, \Gamma' \vdash C \end{array}$$

$$\frac{\Sigma, \Gamma, \Theta, \Gamma', \Delta \vdash \nabla}{\Sigma, \Gamma', \Theta, \Gamma, \Delta \vdash \nabla}\ \mathrm{sc_{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Theta, \Delta', \nabla}{\Gamma \vdash \Sigma, \Delta', \Theta, \Delta, \nabla}\ \mathrm{sc_{dx}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \&-\mathrm{F} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \& B \vdash C}\ \&\mathrm{S}$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee-\mathrm{F} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee-\mathrm{re}_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee-\mathrm{re}_2$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ \to -\mathrm{F} \qquad \frac{\Gamma, A \to B \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\ \to -\mathrm{re}^*$$

Regole derivate per LI

$$\begin{array}{cc} \text{ax-id}^* & \text{ax-}\bot^* \\ \Gamma, A, \Gamma' \vdash A & \Gamma, \bot, \Gamma' \vdash \Sigma \end{array}$$

$$\frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg-\mathrm{F} \qquad \frac{\Gamma' \vdash A}{\Gamma, \neg A, \Gamma' \vdash C}\ \neg-\mathrm{re}$$

$$\begin{array}{cc} \neg\text{-ax}_{sx1} & \neg\text{-ax}_{sx2} \\ \Gamma, A, \Gamma', \neg A, \Gamma'' \vdash C & \Gamma, \neg A, \Gamma', A, \Gamma'' \vdash C \end{array}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \& B \vdash C}\ \&-\mathrm{S}$$

$$\begin{array}{cc} \text{rf}^* & \text{sm}^* \\ \vdash t = t & \Gamma, t = s \vdash s = t \end{array}$$

$$\begin{array}{cc} \text{tra}^* & \text{cf}^* \\ \Gamma, s = t, t = u \vdash s = u & \Gamma, t = s \vdash f(t) = f(s) \end{array}$$

$$\begin{array}{c} \text{cp}^* \\ \Gamma, P(t), t = s \vdash P(s) \end{array}$$

Logica classica predicativa LC

$$\begin{array}{cc} \text{ax-id} & \text{ax-}\bot \\ A \vdash A & \bot\vdash \end{array}$$

$$\frac{\Gamma \vdash \Sigma}{\Gamma, \Gamma' \vdash \Sigma}\ \text{in}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma}{\Gamma \vdash \Sigma, \Sigma'}\ \text{in}_{\text{dx}}$$

$$\frac{\Sigma, \Gamma, \Theta, \Gamma', \Delta \vdash \nabla}{\Sigma, \Gamma', \Theta, \Gamma, \Delta \vdash \nabla}\ \text{sc}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Theta, \Delta', \nabla}{\Gamma \vdash \Sigma, \Delta', \Theta, \Delta, \nabla}\ \text{sc}_{\text{dx}}$$

$$\frac{\Sigma, \Gamma, \Gamma, \Delta \vdash \nabla}{\Sigma, \Gamma, \Delta \vdash \nabla}\ \text{cn}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Delta, \nabla}{\Gamma \vdash \Sigma, \Delta, \nabla}\ \text{cn}_{\text{dx}}$$

$$\frac{\Gamma \vdash A, \nabla \quad \Gamma \vdash B, \nabla}{\Gamma \vdash A\&B, \nabla}\ \&-\text{D} \qquad \frac{\Gamma, A \vdash \nabla}{\Gamma, A\&B \vdash \nabla}\ \&-\text{re}_1 \qquad \frac{\Gamma, B \vdash \nabla}{\Gamma, A\&B \vdash \nabla}\ \&-\text{re}_2$$

$$\frac{\Gamma, A \vdash \nabla \quad \Gamma, B \vdash \nabla}{\Gamma, A \vee B \vdash \nabla}\ \vee-\text{F} \qquad \frac{\Gamma \vdash A, \nabla}{\Gamma \vdash A \vee B, \nabla}\ \vee-\text{D}_1 \qquad \frac{\Gamma \vdash B, \nabla}{\Gamma \vdash A \vee B, \nabla}\ \vee-\text{D}_2$$

$$\frac{\Gamma, A \vdash B, \nabla}{\Gamma \vdash A \rightarrow B, \nabla}\ \rightarrow -\text{D} \qquad \frac{\Gamma' \vdash A, \nabla \quad \Gamma, B \vdash \nabla}{\Gamma, A \rightarrow B, \Gamma' \vdash \nabla}\ \rightarrow -\text{re}_c$$

$$\frac{\Gamma \vdash A(x), \Delta}{\Gamma \vdash \forall x A(x), \Delta}\ \forall-\text{D}\ (x \notin VL(\Gamma, \Delta)) \qquad \frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x\ A(x) \vdash \Delta}\ \forall-\text{S}$$

$$\frac{\Gamma, A(x) \vdash \Delta}{\Gamma, \exists x\ A(x) \vdash \Delta}\ \exists-\text{S}\ (x \notin VL(\Gamma, \Delta)) \qquad \frac{\Gamma \vdash A(t), \Delta}{\Gamma \vdash \exists\ x\ A(x), \Delta}\ \exists-\text{D}$$

$$\begin{array}{c} = -\text{ax} \\ \vdash t = t \end{array} \qquad \frac{\Gamma(t) \vdash \Delta(t)}{\Gamma(s), t = s \vdash \Delta(s)}\ = -\text{F}$$

Calcolo classico proposizionale $\mathbf{LC}_p^{abbr}$

$$\overset{\text{ax-id}^*}{\Gamma, A, \Gamma' \vdash \Delta, A, \Delta'} \qquad \overset{\text{ax}^*\text{-}\bot}{\Gamma, \bot, \Gamma' \vdash \nabla}$$

$$\frac{\Sigma, \Gamma, \Theta, \Gamma', \Delta \vdash \nabla}{\Sigma, \Gamma', \Theta, \Gamma, \Delta \vdash \nabla}\ \text{sc}_{\text{sx}} \qquad \frac{\Gamma \vdash \Sigma, \Delta, \Theta, \Delta', \nabla}{\Gamma \vdash \Sigma, \Delta', \Theta, \Delta, \nabla}\ \text{sc}_{\text{dx}}$$

$$\frac{\Gamma \vdash A, \nabla \quad \Gamma \vdash B, \nabla}{\Gamma \vdash A\&B, \nabla}\ \&-\text{D} \qquad \frac{\Gamma, A, B \vdash \nabla}{\Gamma, A\&B \vdash \nabla}\ \&-\text{S}$$

$$\frac{\Gamma, A \vdash \nabla \quad \Gamma, B \vdash \nabla}{\Gamma, A \vee B \vdash \nabla}\ \vee-\text{F} \qquad \frac{\Gamma \vdash A, B, \nabla}{\Gamma \vdash A \vee B, \nabla}\ \vee-\text{D}$$

$$\frac{\Gamma, A \vdash B, \nabla}{\Gamma \vdash A \to B, \nabla}\ \to -\text{D} \qquad \frac{\Gamma \vdash A, \nabla \quad \Gamma, B \vdash \nabla}{\Gamma, A \to B \vdash \nabla}\ \to -\text{S}$$

Regole derivate per LC

$$\frac{\Gamma, A \vdash \nabla}{\Gamma \vdash \neg A, \nabla}\ \neg-\text{F} \qquad \frac{\Gamma \vdash A, \nabla}{\Gamma, \neg A \vdash \nabla}\ \neg-\text{S}$$

$$\overset{\neg\text{-ax}_{sx1}}{\Gamma, A, \Gamma', \neg A, \Gamma'' \vdash \nabla} \qquad \overset{\neg\text{-ax}_{sx2}}{\Gamma, \neg A, \Gamma', A, \Gamma'' \vdash \nabla}$$

$$\overset{\neg\text{-ax}_{dx1}}{\Gamma \vdash \Sigma, A, \Sigma', \neg A, \Sigma''} \qquad \overset{\neg\text{-ax}_{dx2}}{\Gamma \vdash \Sigma, \neg A, \Sigma', A, \Sigma''}$$

$$\overset{\text{rf}^*}{\vdash t = t, \Delta} \qquad \overset{\text{sm}^*}{\Gamma, t = s \vdash s = t, \Delta}$$

$$\overset{\text{tra}^*}{\Gamma, s = t, t = u \vdash s = u, \Delta} \qquad \overset{\text{cf}^*}{\Gamma, t = s \vdash f(t) = f(s), \Delta} \qquad \overset{\text{cp}^*}{\Gamma, P(t), t = s \vdash P(s), \Delta}$$

1 Regole derivate in presenza di composizioni

In LI + comp_{sx}+ comp_{dx} e in LC + comp_{sx}+ comp_{dx}

$$\frac{\Gamma' \vdash A \quad \Gamma, A, \Gamma" \vdash \nabla}{\Gamma, \Gamma', \Gamma'' \vdash \nabla}\ \text{comp}_{sx} \qquad \frac{\Gamma \vdash \Sigma, A, \Sigma" \quad A \vdash \Sigma'}{\Gamma \vdash \Sigma, \Sigma', \Sigma"}\ \text{comp}_{dx}$$

si hanno le seguenti regole derivate:

$$\frac{\Gamma \vdash t = s}{\Gamma \vdash s = t}\ \text{sy-r} \qquad \frac{\Gamma \vdash t = s \qquad \Gamma' \vdash s = u}{\Gamma, \Gamma' \vdash t = u}\ \text{tr-r}$$

Aritmetica di Heyting e Peano

Si ricorda che l'aritmetica di Heyting e di Peano sono ottenute rispettivamente aggiungendo a LI + comp$_{sx}$+ comp$_{dx}$ e a LC + comp$_{sx}$+ comp$_{dx}$ i seguenti assiomi:

$$Ax1. \vdash \forall x\ (\ s(x) \neq 0\ )$$
$$Ax2. \vdash \forall x\ \forall y\ (\ s(x) = s(y) \rightarrow x = y\ )$$
$$Ax3. \vdash \forall x\ (\ x + 0 = x\ )$$
$$Ax4. \vdash \forall x\ \forall y\ (\ x + s(y) = s(x + y)\ )$$
$$Ax5. \vdash \forall x\ (\ x \cdot 0 = 0\ )$$
$$Ax6. \vdash \forall x\ \forall y\ (\ x \cdot s(y) = x \cdot y + x\ )$$
$$Ax7. \vdash A(0)\&(\ \forall x\ A(x) \rightarrow A(s(x))\ ) \rightarrow \forall x\ A(x)$$

ove il numerale n si rappresenta in tal modo

$$n \equiv \underbrace{s(s \ldots (0))}_{\text{n-volte}}$$

e quindi per esempio

$$1 \equiv s(0)$$
$$2 \equiv s(\ s(0)\ )$$