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ARTICLE

Effectiveness and Continuity in Intuitionistic Quasi-toposes of Assemblies

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Abstract

It is well known that over Heyting Arithmetic with finite types, the effective principle of the formal Church Thesis, stating that all number-theoretic functional relations are computable, is inconsistent with Brouwer's intuitionistic principles on the continuum, in particular the Fan theorem.
Here, we build two arithmetic quasi-toposes, validating on the one hand, Brouwer's continuity principles, including the Fan theorem, and on the other hand, a restricted form of Church's Thesis, called *Type-theoretic Church Thesis* and written TCT, expressing that all morphisms of the considered quasi-topos are computable.
One quasi-topos is constructed by formalizing the category of assemblies **Asm** within Hyland's Effective topos using Intuitionistic Zermelo-Fraenkel Set Theory **IZF** extended with Brouwer's continuity principles as our meta-theory. The other quasi-topos is obtained as an elementary quotient completion, in the same intuitionistic meta-theory.
While in previous work by the first author with F. Pasquali and G. Rosolini, it has been shown that these two quasi-toposes are equivalent when working within the classical **ZFC** Set Theory, here, we show that this is no longer the case when working within **IZF**.
We also observe that the aforementioned inconsistency is resolved in such quasi-toposes by the non-validity of the axiom of unique choice on the natural numbers, and that no non-trivial topos can validate the effective principle TCT together with Brouwer's continuity principles.

Keywords: Quasi-topos; Realizability; Tripos Theory; Brouwer's Intuitionism; Church's Thesis.

To Pino Rosolini

All we can say to express our gratitude to him
is that there are not enough words to express it.

1. Introduction

This paper provides a contribution to the foundation of constructive mathematics by showing the existence of constructive quasi-toposes, validating an effective principle à la Church and continuity principles à la Brouwer.

To understand the relevance of our contribution, we recall that there is a large variety of different, and often mutually incompatible, approaches to constructive mathematics. Broadly, constructive mathematics can be conceived as mathematics developed by replacing the usual classical underlying logical reasoning with Brouwer's intuitionistic logic. Significant examples of approaches to constructive mathematics that are incompatible with classical mathematics include Markov's constructivism and Brouwer's intuitionism (see for example Troelstra and van Dalen (1988); Bridges and Richman (1987)). Markov's constructivism emphasizes the effectiveness of constructive functions by presupposing the validity of the formal Church Thesis CT, that all number-theoretic functional relations are computable. Instead, Brouwer's intuitionism privileges the continuity of constructive functions on the continuum by presupposing the validity of Bar Induction, a local continuity principle called LCP which implies that all functions between real numbers are continuous (see Troelstra and van Dalen (1988)), and an instance of the axiom of choice AC_{Baire} on the Baire space.

These two mathematical approaches are not only incompatible with classical mathematics but also incompatible between each other: it is well known from Troelstra and van Dalen (1988); Dummett (2000) that over Heyting Arithmetic with finite types, Brouwer's Fan Theorem, which is a consequence of Bar Induction, is inconsistent with the formal Church Thesis CT.

Here, we want to show that it is possible to reconcile Brouwer's intuitionistic principles with a weakened form of effectiveness without necessarily renouncing to both effective and continuity principles altogether (except for some form of choice) as advocated by Bishop (1967). Actually, Bishop aimed to develop constructive mathematics in a way that was compatible also with classical one. Such an aim was pushed even further in the works Maietti and Sambin (2005); Maietti (2009) with the introduction of the Minimalist Foundation. Indeed, such a foundation was built purposely to formalize mathematics compatibly with the most relevant constructive and classical mathematical approaches, including classical predicativism (see Maietti and Sabelli (2024); Contente and Maietti (2026)).

In this work, we prove that there exists a much less radical reconciliation between Markov's constructivism and Brouwer's intuitionism than Bishop's one by employing category theory. Indeed, we build two quasi-toposes which validate Brouwer's continuity principles Bar Induction, LCP, and AC_{Baire}, together with an effective principle called *Type-theoretic Church Thesis*, for short TCT, which states that all morphisms of the considered quasi-topos are computable. Logically, this means that if Heyting Arithmetic with finite types is enriched with a type of power objects, Brouwer's continuity principles are consistent with the principle TCT stating that only number-theoretic functions arising as lambda-terms of the function type are computable. TCT is weaker than CT because it does not imply that all number-theoretic functions are computable. And the mentioned extension of Heyting Arithmetic with finite types is nothing else that the generic calculus of what we call here *arithmetic CC-strong triposes*. Furthermore, it is important to underline that all this presupposes that Brouwer's *choice sequences* are identified with *number-theoretic functional relations* as advocated in Maietti and Sambin (2013).

The two built quasi-toposes are *intuitionistic*, since they are built by adopting Intuitionistic Zermelo-Fraenkel set theory **IZF** from Friedman (1977), extended with Brouwer's continuity principles as meta-theory. The first quasi-topos, called $\mathbf{Asm}_i$, is an intuitionistic rendering of the quasi-topos of assemblies **Asm**, originally defined as a full subcategory of the Effective topos in Hyland (1982). As a by-product, $\mathbf{Asm}_i$ is no longer boolean as **Asm**, where by *boolean quasi-topos* we mean a quasi-topos whose internal logic of *strong subobjects* satisfies classical logic. The second quasi-topos is built as the elementary quotient completion $\mathcal{Q}_{\mathbb{P}\Gamma}$ of the strong subobject doctrine $\mathbb{P}\Gamma$ of the intuitionistic rendering of partitioned assemblies in Robinson and Rosolini (1990).

In Maietti et al. (2019) it was shown that these two quasi-toposes are equivalent when adopting the classical **ZFC** set theory, as meta-theory. Here, we show this is no longer the case when working within **IZF**. In particular, the intuitionistic rendering of **Asm** looses some projectivity properties.

Finally, we notice that no non-trivial elementary topos (whether formalized in a classical or intuitionistic meta-theory) can model TCT and Brouwer's continuity principles altogether. The reason is that any topos validates the Axiom of Unique Choice AC!, which strengthens TCT to become CT, which in turn contradicts Brouwer's continuity principles. All this let us conclude that our genuine quasi-toposes provide two different universes where to reconcile effectiveness à la Markov with continuity à la Brouwer.

As future work, we intend to employ such quasi-toposes to show the consistency of the Minimalist Foundation in Maietti (2009) and Coquand-Huet's Calculus of Constructions with TCT and all Brouwer's intuitionistic principles.

The paper is organized as follows.

In Section 2, we lay out the categorical preliminaries on quasi-toposes and on various completions of suitable triposes. In particular, we focus on what we call *arithmetic CC-strong triposes*, of which we describe the internal language in the style of that for toposes in Lambek and Scott (1986), and we use it to formulate TCT. In Section 3, we introduce Brouwer's continuity principles. In Section 4, we specify our meta-theory. In Section 5, we build the intuitionistic quasi-topos of assemblies and that defined as the elementary quotient completion of strong subobjects over partitioned assemblies; we show that they can not be equivalent, and that they validate both TCT and the mentioned Brouwer's continuity principles.

2. Categorical preliminaries on quasi-toposes and triposes

Loosely, a quasi-topos is a generalization of the notion of topos that requires a classifier just for strong monomorphisms, which in general do not coincide with all monomorphisms.

Definition 1. *In any category, a* strong monomorphism *is a monomorphism $m : A \to B$ such that, for each epimorphism $e : C \to D$ and each pair of arrows $f : C \to A$ and $g : D \to B$ with $g \circ e = m \circ f$, there exists a unique arrow $h : D \to A$ making the following diagram commute.*

$$\begin{array}{ccc} C & \xrightarrow{e} & D \\ {\scriptstyle f}\downarrow & \overset{h}{\swarrow} & \downarrow{\scriptstyle g} \\ A & \xrightarrow[m]{} & B \end{array}$$

Definition 2. *In any finitely complete category, a* strong-subobject classifier *is an object $\mathscr{P}(1)$ together with an arrow $t : 1 \to \mathscr{P}(1)$ from the terminal object to it such that for each strong monomorphism $m : A \to B$ there exists a unique arrow $\chi_m : B \to \mathscr{P}(1)$ making the following a pullback square.*

$$\begin{array}{ccc} A & \xrightarrow{!_A} & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle t} \\ B & \underset{\chi_m}{\dashrightarrow} & \mathscr{P}(1) \end{array}$$

Definition 3. *A* quasi-topos *is a finitely complete, finitely cocomplete, locally cartesian closed category with a strong-subobject classifier. A quasi-topos is said to be* solid *if the unique map $0 \to 1$ from an initial object 0 to a terminal one 1 is a strong-monomorphism (or equivalently, it has disjoint coproducts, see Johnstone (2002)).*

We also recall the notion of weak dependent product from Birkedal et al. (1998); Carboni and Rosolini (2000), which will be useful in the sequel. To give such a definition, we make use of the following notation: for any finitely complete category $\mathscr{C}$ and any arrow $g : J \to I$ we denote the pullback functor

$$g^*(-) : \mathscr{C}/I \to \mathscr{C}/J$$

Definition 4. *In a finitely complete category $\mathscr{C}$ a* weak dependent product *of an arrow $f\colon X \to J$ along a map $g\colon J \to I$ consists of an arrow $\Pi_g(f) : Z \to I$ and an arrow $ev : J \times_I Z \to X$ such that the following diagram commutes*

$$\begin{array}{ccccc} X & \xleftarrow{ev} & J \times_I Z & \longrightarrow & Z \\ & \searrow^{f} & \downarrow & \lrcorner & \downarrow{\scriptstyle \Pi_g(f)} \\ & & J & \xrightarrow[g]{} & I \end{array}$$

where the right square is a pullback; moreover, we require the above diagram to be weakly terminal in the category of diagrams with such a shape, namely: for any other diagram of the form

$$\begin{array}{ccccc} X & \xleftarrow{m} & J \times_I H & \longrightarrow & H \\ & \searrow^{f} & \downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ & & J & \xrightarrow[g]{} & I \end{array}$$

where the right square is a pullback, there exists an arrow $n : H \to Z$ such that $h = \Pi_g(f) \circ n$ and $m = ev \circ g^*(n)$.

Whenever the arrow e exists and it is unique, then we say that $\Pi_g(f) : Z \to I$ and $ev : J \times_I Z \to X$ gives a dependent product *of $f\colon X \to J$ along $g\colon J \to I$.*

Finally, recall that if $\mathscr{C}$ has dependent products for all map $f\colon X \to J$ along any other $g\colon J \to I$ then it is locally cartesian closed.

In our treatment, we will rely on suitable notions of triposes, as introduced by Hyland et al. (1980); Pitts (2002), to organize the internal logic of a quasi-topos. We start by recalling the notion of *first-order doctrine* validating first-order intuitionistic logic with equality; then, we recall a strengthening of the notion of tripos, called *strong tripos*, introduced in (Maietti et al., 2023, Def. 2.21).

Let **Heyt** be the category of Heyting algebras and Heyting-algebra homomorphisms.

Definition 5. *A* first-order (intuitionistic) doctrine *on a finite product category $\mathscr{C}$ is a contravariant functor $P : \mathscr{C}^{\mathrm{op}} \to$* **Heyt** *equipped with the following structure:*

- *for each projection $\pi : A \times B \to A$ in $\mathscr{C}$, the functor $P_\pi : P(A) \to P(A \times B)$ has a left adjoint functor $\exists_\pi$ and a right adjoint functor $\forall_\pi$ (not required to be morphisms in* **Heyt***); moreover, these adjoints satisfy the following Beck-Chevalley condition: for any projection $\pi : A \times B \to A$ and any map $f : A' \to A$, the following diagrams commute;*

$$\begin{array}{ccc} P(A' \times B) & \xrightarrow{\exists_{\pi'}} & P(A') \\ {\scriptstyle P_{f\times id}}\uparrow & & \uparrow{\scriptstyle P_f} \\ P(A \times B) & \xrightarrow[\exists_\pi]{} & P(A) \end{array} \qquad \begin{array}{ccc} P(A' \times B) & \xrightarrow{\forall_{\pi'}} & P(A') \\ {\scriptstyle P_{f\times id}}\uparrow & & \uparrow{\scriptstyle P_f} \\ P(A \times B) & \xrightarrow[\forall_\pi]{} & P(A) \end{array}$$

- *P is* elementary*, i.e. for every object A in $\mathscr{C}$, there is an element δ_A in the fiber $P(A \times A)$ such that for every arrow e of the form $\langle \pi_1, \pi_2, \pi_2 \rangle : X \times A \to X \times A \times A$ in $\mathscr{C}$, the assignment*

$$\exists_e(\alpha) := P_{\langle \pi_1,\pi_2 \rangle}(\alpha) \wedge_{X\times A\times A} P_{\langle \pi_2,\pi_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint functor to

$$P_e : P(X \times A \times A) \to P(X \times A).$$

Definition 6. *A* strong tripos $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *is a first-order doctrine with* power-objects*; namely, for any object A of $\mathscr{C}$ there is another object $\mathscr{P}(A)$ and an element $\in_A$ in the fiber $P(A \times \mathscr{P}(A))$ such that for every object B in $\mathscr{C}$ and ρ in $P(A \times B)$ there exists a unique morphism $\chi_\rho : B \to \mathscr{P}(A)$ satisfying $P_{id \times \chi_\rho}(\in_A) = \rho$.*

We can strengthen the notion of strong tripos to that of a CC-strong tripos by requiring function spaces in its base category. This stronger notion has great conceptual relevance. In fact, it is sufficiently expressive to formulate, on the one hand, the effective axiom TCT in Section 2.4 by defining lawlike sequences as elements of some function space; and, on the other, Brouwer's continuity principles in Section 3 by defining choice sequences, as elements of some power-object. This double treatment diverges essentially from that of Troelstra and van Dalen (1988), where both lawlike and choice sequences are defined using the same arrow type, and it is the key to guarantee consistency. Finally, CC-strong triposes will help us in Section 5 to identify by contrast the amount of common structure shared between the quasi-topos of assemblies and that of sets of its meta-theory.

Definition 7. *A* CC-strong tripos $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *is a strong tripos whose base category $\mathscr{C}$ is cartesian closed. The exponential object between two objects A and B of $\mathscr{C}$ will be denoted by $A \to B$, and its evaluation map as* $\mathrm{ev} : A \times (A \to B) \to B$.

Remark 8. Let $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ be a first-order doctrine. Let us denote *strong predicate classifier* for P an object $\mathscr{P}(1)$ in $\mathscr{C}$ and an element $\in_1$ in the fiber $P(\mathscr{P}(1))$ such that for every object A in $\mathscr{C}$ and α in $P(A)$ there is a unique morphism $\chi_\alpha : A \to \mathscr{P}(1)$ satisfying $P_{\chi_\alpha}(\in_1) = \alpha$.

Observe that, if the base category $\mathscr{C}$ is cartesian closed, then for P to be a (CC-)strong tripos, it is sufficient to require the existence of such a predicate classifier; indeed, for each object A of $\mathscr{C}$ we can then define $\mathscr{P}(A)$ as the exponential $\mathscr{P}(1)^A$ and the predicate $\in_A$ in the fiber $P(\mathscr{P}(A) \times A)$ as $P_{\mathrm{ev}}(\in_1)$. Conversely, if P is a strong tripos, a predicate classifier is easily obtained as the power object $\mathscr{P}(1)$ of the terminal object.

For example, in a quasi-topos $\mathscr{C}$ the *strong-subobject classifier* coincides with a *predicate classifier* of the doctrine of the strong subobjects of $\mathscr{C}$.

As we already mentioned, the main example of such triposes is the functor of strong subobjects of a quasi-topos.

Proposition 9. *Strong subobjects of a quasi-topos $\mathscr{C}$ define a CC-strong tripos*

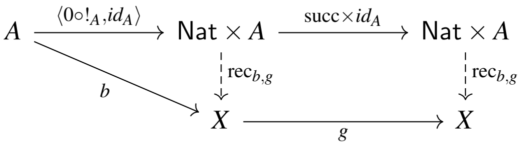
$$sSub_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$$

associating to each object A of $\mathscr{C}$ the Heyting algebra of isomorphism classes of strong monomorphisms over A.

Proof. This is a particular case of the more general result presented in (Frey, 2015, Lemma 4.13), stating that the strong-subobjects functor associated with a q-topos, a generalization of the notion of quasi-topos, is a tripos. Notice that in the specific case of a quasi-topos, we obtain a CC-strong tripos because, by definition, a quasi-topos is cartesian closed (while q-toposes are not required to be cartesian closed in general). □

We now define additional structure on a strong tripos sufficient to validate Peano's axioms for arithmetic. In particular, we require the existence of a natural numbers object in the base category, the definition of which we recall from Lambek and Scott (1986).

Definition 10. *In any category* $\mathscr{C}$ *with binary products, a* parameterized natural numbers object *(for short* NNO*) is an object* Nat *together with two arrows* $0 : 1 \to \mathsf{Nat}$ *and* $\mathrm{succ} : \mathsf{Nat} \to \mathsf{Nat}$ *such that for every objects* A *and* X *in* $\mathscr{C}$ *and every pair of arrows* $b : A \to X$ *and* $g : X \to X$*, there exists a unique arrow* $\mathrm{rec}_{b,g} : \mathsf{Nat} \times A \to X$ *such that the following diagram commutes.*



Definition 11. *An* arithmetic (CC-)strong tripos *is a (CC-)strong tripos* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *such that:*

(1) $\mathscr{C}$ *has a parameterized natural numbers object* $(\mathsf{Nat}, 0, \mathrm{succ})$ *satisfying* P-induction*; namely, for every* A *in* $\mathscr{C}$ *and* φ *in* $P(A \times \mathsf{Nat})$ *the following holds in* $P(A)$*.*

$$P_{\langle id_A, 0\rangle}(\varphi) \wedge \forall_{\pi_A}(\varphi \Rightarrow P_{id_A \times \mathrm{succ}}(\varphi)) \leq \forall_{\pi_A} \varphi$$

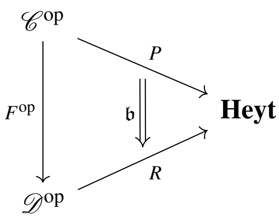
(2) P *is* extensional *(Maietti et al., 2023, Def. 2.6); namely, for every pair of parallel arrows* $f, g : X \to A$ *in* $\mathscr{C}$ *it holds that* f *is equal to* g *as arrows of* $\mathscr{C}$ *if and only if* $\top = P_{\langle f,g\rangle}(\delta_A)$ *in the fiber* $P(X)$*.*

Remark 12. Proposition 9 lifts to the case *arithmetic quasi-toposes*, namely quasi-toposes with an NNO.

We conclude this section by collecting both first-order doctrines and arithmetic strong triposes into a corresponding 2-category.

Definition 13. *Let* **FoD** *denote the 2-category having first-order doctrines as objects and the following 1-arrows and 2-arrows:*

- *a 1-arrow from any first-order doctrine* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *to any other* $R : \mathscr{D}^{\mathrm{op}} \to \mathbf{Heyt}$ *is given by a pair* $(F, \mathfrak{b})$



  where
  - $F : \mathscr{C} \to \mathscr{D}$ *is a functor preserving finite products;*
  - $\mathfrak{b} : P \Rightarrow R \circ F^{\mathrm{op}}$ *is a natural transformation preserving right and left adjoints, and such that for every object* A *of* $\mathscr{C}$

$$\mathfrak{b}_{A\times A}(\delta_A) = R_{\langle F\pi_1, F\pi_2\rangle}(\delta_{FA})$$

- *a 2-arrow from a 1-arrow* $(F, \mathfrak{b}) : P \to R$ *to any other* $(G, \mathfrak{c}) : P \to R$ *is a natural transformation* $\theta : F \Rightarrow G$ *such that for every* A *in* $\mathscr{C}$ *and every* α *in* $P(A)$*, we have*

$$\mathfrak{b}_A(\alpha) \leq R_{\theta_A}(\mathfrak{c}_A(\alpha)).$$

Definition 14. *Let* **asTr** *denote the locally full 2-subcategory of* **FoD** *whose objects are arithmetic strong triposes. The 1-arrows are those pairs* $(F, \mathfrak{b})$ *in* **FoD** *such that* F *preserves power-objects and the NNO, and such that for each object* A *of* $\mathscr{C}$

$$\mathfrak{b}_{A\times\mathscr{P}(A)}(\in_A) = R_{\langle F\pi_1, F\pi_2\rangle}(\in_{FA}).$$

2.1 Logical notation and the initial arithmetic strong tripos

In this section, we first introduce some logical notation *to reason within* a first-order doctrine as a syntactic calculus as done in (Maietti et al., 2019, Section 3). Then, we present a finitely axiomatized calculus suitable to define the free arithmetic strong tripos. We won't introduce an analogous calculus for arithmetic CC-strong triposes since, for our purposes, the logical notation introduced for them will suffice to our purposes.

Let $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ be a first-order doctrine. We denote any element α in the fiber $P(A_1 \times \ldots \times A_k)$ through the formal expression

$$x_1 : A_1, \ldots, x_k : A_k \mid \alpha(x_1, \ldots, x_k)$$

to emphasize its dependencies. In particular, an element of $P(1)$ will be called a *sentence*. Analogously, we write

$$x_1 : A_1, \ldots, x_k : A_k \mid \alpha_1(x_1, \ldots, x_k), \ldots, \alpha_n(x_1, \ldots, x_k) \vdash \beta(x_1, \ldots, x_k)$$

to mean that $\alpha_1 \wedge \ldots \wedge \alpha_n \leq \beta$ holds in the fiber $P(A_1 \times \ldots \times A_k)$, and we call *sequent* such an expression. We say that an element α in $P(A)$ is *true* if $\top_A \leq \alpha$, and we will write $P \models \alpha(x)$ to denote that a sentence α is true.

If α is an element in the fiber $P(A_1 \times \cdots \times A_k)$ and

$$f_i : B_1 \times \cdots \times B_h \to A_i \qquad i = 1, \ldots, k$$

are arrows of $\mathscr{C}$, then the element $P_{\langle f_1,\ldots,f_k\rangle}(\alpha)$ will be written as

$$y_1 : B_1, \ldots, y_h : B_h \mid \alpha(f(y_1, \ldots, y_h), \ldots, f_k(y_1, \ldots, y_h))$$

Moreover, in this situation composition of functions $(g \circ f)(y_1, \ldots, y_h)$ will be written as $g(f(y_1, \ldots, y_h))$, and we will simply write y_i in place of the i-th projection.

The element δ_A in the fiber $P(A \times A)$ will be written as $x : A, y : A \mid x =_A y$.

Given $x : A, y : B \mid \phi(x, y)$, we write $x : A \mid \exists_{y:B}\phi(x, y)$ and $x : A \mid \forall_{y:B}\phi(x, y)$ to denote the elements $\exists_{\pi_1}\phi$ and $\forall_{\pi_1}\phi$ of $P(A)$, respectively. As an additional shorthands on quantifier, suppose to have elements $x : A, y : B \mid \phi(x, y)$ and $y : B \mid \beta(y)$, then we set

$$x : A \mid \exists!_{y:B}\phi(x, y) :\equiv \exists_{y:B}\phi(x, y) \wedge \forall_{y:B}\forall_{y':B}(\phi(x, y) \wedge \phi(x, y') \Rightarrow y =_B y')$$
$$x : A \mid \forall_{y:\beta}\phi(x, y) :\equiv \forall_{y:B}(\beta(y) \Rightarrow \phi(x, y))$$
$$x : A \mid \exists_{y:\beta}\phi(x, y) :\equiv \exists_{y:B}(\beta(y) \wedge \phi(x, y))$$

and, similarly, we sometimes write $x : A, y : \beta \mid \phi(x, y)$ for $x : A, y : B \mid \beta(y) \Rightarrow \phi(x, y)$.

Finally, if P is a strong tripos, the element $P_{\langle\pi_2,\pi_1\rangle}(\in_A)$ in the fiber $P(A \times \mathscr{P}(A))$ will be written as $a : A, U : \mathscr{P}(A) \mid a \in_A U$. Moreover, given two elements $V : \mathscr{P}(A), y : B \mid \phi(V, y)$ and $x : A \mid \beta(x)$, we write

$$y : B \mid \phi(\{x : A \mid \beta(x)\}, y)$$

as a shorthand for $y : B \mid \forall_{V:\mathscr{P}(A)}(\,\forall_{x:A}(x \in_A V \Leftrightarrow \beta(x)) \;\Rightarrow\; \phi(V, y)\,)$.

We now define the internal language of arithmetic strong triposes as a slight extension of the *pure intuitionistic type theory* calculus $\mathscr{L}_0$ introduced in (Lambek and Scott, 1986, Section 1, Part II). Recall that $\mathscr{L}_0$ is a simply typed, higher-order logic whose basic types include the unit type 1, the type of natural numbers Nat, and the sentences classifier $\mathscr{P}(1)$ (called Ω in (Lambek and Scott, 1986, Section 1, Part II)); and whose type formers are the product type $A \times B$ and the powerset $\mathscr{P}(A)$. The propositions and the sequents of $\mathscr{L}_0$ are written as in the logical notation above.

Definition 15. *Let us call* Higher-order Arithmetic Type Theory, *for short* **HaTT**, *the extension of* $\mathscr{L}_0$ *obtained by introducing eliminator terms for the natural numbers type and the product type:*

(1) for any types B and X, and any terms $b(x) : B$ and $g(x,y) : B$ with $x : X$ and $y : B$, we introduce a new term

$$\mathsf{rec}_{b,g}(x,n) : B$$

where $y : X$ and $n : N$, together with the following axioms

$$x : X \mid \top \vdash \mathsf{rec}_{b,g}(x,0) =_B b(x) \qquad x : X, n : \mathsf{Nat} \mid \top \vdash \mathsf{rec}_{b,g}(x, \mathsf{succ}(n)) =_B g(x, \mathsf{rec}_{b,g}(x,n))$$

(2) for any product type $A \times B$, we introduce new terms

$$\pi_1(z) : A \qquad \pi_2(z) : B$$

with $z : A \times B$, together with the following axioms

$$x : A, y : B \mid \top \vdash \pi_1(\langle x, y \rangle) =_A x \qquad x : A, y : B \mid \top \vdash \pi_2(\langle x, y \rangle) =_B y$$

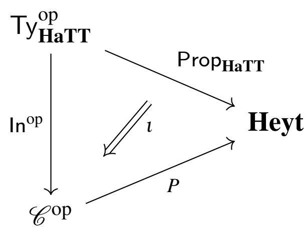
The theory **HaTT** induces the *syntactic* arithmetic strong tripos.

$$\mathsf{Prop}_{\mathbf{HaTT}} : \mathsf{Ty}_{\mathbf{HaTT}}^{\mathrm{op}} \to \mathbf{Heyt}$$

The base category $\mathsf{Ty}_{\mathbf{HaTT}}$ has the types of **HaTT** as objects; an arrow between two types A and B is a term $t(x) : B$ with $x : A$ up to propositional equality, i.e. two terms $t(x)$ and $s(x)$ are considered equal if $x : A \mid \top \vdash t(x) =_B s(x)$. The functor $\mathsf{Prop}_{\mathbf{HaTT}}$ maps a type A into the Heyting algebra $\mathsf{Prop}_{\mathbf{HaTT}}(A)$ of terms of type $\mathscr{P}(1)$ with one free variable in A ordered by $\vdash$.

Proposition 16. *The syntactic arithmetic strong tripos is a bicategorical initial object in the 2-category of arithmetic strong triposes; namely, for any other arithmetic strong tripos $P : \mathscr{C}^{\mathrm{op}} \to$ **Heyt**, the category* $\mathbf{asTr}(\mathsf{Prop}_{\mathbf{HaTT}}, P)$ *is equivalent to the terminal one.*

Proof. Given any arithmetic strong tripos $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$, there exists an arrow in **asTr**



which is defined by employing the following interpretation function of **HaTT** into P. Firstly, for any type A of **HaTT**, we define an object $[\![A]\!]$ of $\mathscr{C}$ recursively on the type structure in the obvious way. Then, by recursion on the term structure, we simultaneously define:

- for any term $x_1 : A_1, \ldots, x_n : A_n \mid t(x_1, \ldots, x_n) : B$ an arrow
$$[\![t]\!] : [\![A_1]\!] \times \cdots \times [\![A_n]\!] \to [\![B]\!] \text{ of } \mathscr{C}$$
- for any proposition $x_1 : A_1, \ldots, x_n : A_n \mid \varphi(x_1, \ldots, x_n)$ an element
$$[\![\varphi]\!]_{\mathscr{P}(1)} \text{ in the fiber } P([\![A_1]\!] \times \cdots \times [\![A_n]\!]).$$

In particular, the interpretation of a variable $[\![x_i]\!]$ is defined as the i-th projection out of the product. Substitution is interpreted using the action of P on arrows; for example, given two terms $a(x) : A$ and $u(x) : \mathscr{P}(A)$ with $x : X$, the interpretation of the proposition $x : X \mid a(x)\ \varepsilon\ u(x)$ is defined as $P_{\langle [\![a]\!], [\![u]\!] \rangle}(\in_{[\![A]\!]})$.

By induction, it is possible to check that the interpretation function is sound; namely that, for any two propositions φ and ψ, one has that $\varphi \vdash \psi$ implies $[\![\varphi]\!]_{\mathscr{P}(1)} \leq [\![\psi]\!]_{\mathscr{P}(1)}$. Therefore, $[\![-]\!]_{\mathscr{P}(1)}$ is a Heyting-algebra morphism and, together with the interpretation $[\![-]\!]$ on types and terms, they determine a morphism of **asTr** as follows:

$$\begin{aligned} \mathsf{In}(A) &:\equiv [\![A]\!] \\ \mathsf{In}(t : A \to B) &:\equiv [\![t]\!] \\ \iota_A(\varphi) &:\equiv [\![\varphi]\!]_{\mathscr{P}(1)} \end{aligned}$$

Given any two other 1-arrows (F, γ) and (G, δ) from $\mathsf{Prop}_{\mathbf{HaTT}}$ to P, we can show by induction that there is a unique 2-arrow η between them, and that such η is an isomorphism. □

A simple observation relating the previous initiality result with the logical notation of an arithmetic strong tripos P is the following. Given a proposition φ of **HaTT**, its interpretation $[\![\varphi]\!]_{\mathscr{P}(1)}$ in P *coincides* with φ read as a proposition in the logical notation of P.

Definition 17. *Let* λ**HaTT** *be the extension of* **HaTT** *with function types, denoted with the symbol* $\to$*, and governed by the same rules for function types in (Girard et al., 1989, Section 3.1) including the related equations on p.16 formulated through the equality predicate.*

The language of λ**HaTT** enjoys an initiality result analogous to that for **HaTT** but with respect to arithmetic CC-strong triposes. Since we will not use this result in this paper, we do not state it. In the following, we will employ the language λ**HaTT** to state an effective principle compatible with Brouwer's continuity principles which will be validated in the quasi-toposes we are going to build.

2.2 The elementary quotient completion

In this section, we recall the definition of *equivalence relation* relative to a doctrine, and the related notion of *elementary quotient completion* of an *elementary doctrine* introduced by Maietti and Rosolini (2013). For simplicity, here we restrict the exposition to the case of first-order doctrines. In the following, fix a first-order doctrine $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$.

Definition 18. *A* P-equivalence relation *over an object* A *of* $\mathscr{C}$ *is an element* ρ *of* $P(A \times A)$ *such that:*

(1) $x : A, y : A \mid x =_A y \vdash \rho(x, y)$
(2) $x : A, y : A \mid \rho(x, y) \vdash \rho(y, x)$
(3) $x : A, y : A, z : A \mid \rho(x, y), \rho(y, z) \vdash \rho(x, z)$

When no confusion arises, we shall refer at P-equivalence relations simply as equivalence relations*, without specifying the doctrine P.*

Note that, in any first-order doctrine, for any object A in the base the elements δ_A are equivalence relations.

Definition 19. *The* quotient *of a P-equivalence relation* ρ *on an object* A *is an arrow* $q : A \to C$ *in* $\mathscr{C}$ *such that* $x : A, y : A \mid \rho(x,y) \vdash q(x) =_C q(y)$*, and for every arrow* $g : A \to Z$ *with* $x : A, y : A \mid \rho(x,y) \vdash g(x) =_Z g(y)$*, there is a unique arrow* $h : C \to Z$ *such that* $g = h \circ q$*.*

A quotient as above is called stable *when, for every arrow* $f : C' \to C$ *in* $\mathscr{C}$*, there is a pullback*

$$\begin{array}{ccc} A' & \xrightarrow{q'} & C' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A & \xrightarrow{q} & C \end{array}$$

in $\mathscr{C}$ *and the arrow* $q' : A' \to C'$ *is a P-quotient.*

Let $f : A \to B$ *be an arrow in* $\mathscr{C}$*, its P*-kernel $\ker(f)$ *is the P-equivalence relation* $x : A, y : A \mid f(x) =_B f(y)$*. A quotient* $q : A \to C$ *as above is called* effective *if its P-kernel is* ρ*.*

Definition 20. *Given a first-order doctrine* $P : \mathscr{C}^{\text{op}} \to \textbf{Heyt}$ *and a P-equivalence relation* ρ *over* A *in* $\mathscr{C}$*, the Heyting algebra of* descent data $\mathscr{D}es(\rho)$ *is the sub-Heyting algebra of* $P(A)$ *consisting of those* α *such that*

$$x : A, y : A \mid \alpha(x), \rho(x,y) \vdash \alpha(y)$$

For $f : A \to B$ *in* $\mathscr{C}$*, the map* $P_f : P(B) \to P(A)$ *takes values in* $\mathscr{D}es(\ker(f))$*. We shall say that* f *is* of effective descent *if* $P_f : P(B) \to \mathscr{D}es(\ker(f))$ *is an isomorphism.*

In particular, this means that the functor P_f *is of* effective descent type *as defined in Barr and Wells (1984).*

Definition 21 (elementary quotient completion)**.** *Given a first-order doctrine* $P : \mathscr{C}^{\text{op}} \to \textbf{Heyt}$*, we call* $\mathscr{Q}_P$ *the category whose objects are pairs* (A, ρ) *with* A *in* $\mathscr{C}$ *and* ρ *a P-equivalence relation over A. An arrow* $[f] : (A,\rho) \to (B,\sigma)$ *is an equivalence class of arrows* $f : A \to B$ *in* $\mathscr{C}$ *such that* $x : A, y : A \mid \rho(x,y) \vdash \sigma(f(x), f(y))$*; two arrows* $f, g : A \to B$ *are in the same equivalence class if and only if* $x : A \mid \top \vdash \sigma(f(x), g(x))$ *holds.*

The elementary quotient completion *of P is defined as the functor* $\overline{P} : \mathscr{Q}_P^{\text{op}} \to \textbf{Heyt}$*, where*

$$\overline{P}(A,\rho) := \mathscr{D}es(\rho) \qquad \overline{P}_{[f]} := P_f$$

Maietti and Rosolini (2013) proved that the assignment on arrows of $\overline{P}$ in the definition above does not depend on the choice of representatives, and that $\overline{P}$ is a first-order doctrine. Moreover, it is immediate to see that $\overline{P}$ has stable, effective quotients: the quotient of an equivalence relation σ over an object (A,ρ) is given by the arrow $[id_A] : (A,\rho) \to (A,\sigma)$.

2.3 The regular and exact completions of arithmetic strong triposes

The regular and exact completions of a lex category presented in Carboni and Vitale (1998); Carboni (1995) were recognized to be instances of more general free constructions, namely the regular and exact completions of an existential elementary doctrine in Maietti et al. (2017); Maietti and Rosolini (2015). Here, we observe that the regular completion $\mathrm{Reg}(P)$ applied to a strong tripos P produces a topos, and this has an NNO when P is a arithmetic strong tripos. Furthermore, $\mathrm{Reg}(P)$ coincides also with the tripos-to-topos construction by Hyland et al. (1980) and Pitts (2002).

In the following, it is enough to recall the notions of regular and exact completions for first-order doctrines by using the logical notation of the involved doctrines, after reminding the definition of the 2-category of Heyting categories.

Definition 22 (Johnstone (2002))**.** *Let* **HeytCat** *denote the following 2-category:*

- *its objects are* Heyting categories, *namely regular categories* $\mathscr{C}$ *such that their subobject functor* $Sub_{\mathscr{C}} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Pos}$ *is a first-order doctrine.*
- *A 1-arrow between two Heyting categories* $\mathscr{C}$ *and* $\mathscr{D}$ *is a regular functor* $F : \mathscr{C} \to \mathscr{D}$ *such that* $(F, \mathfrak{f})$ *is a 1-arrow between* $Sub_{\mathscr{C}}$ *and* $Sub_{\mathscr{D}}$ *in* **FoD**, *where* $\mathfrak{f}_A([m : M \to A]) := [F(m) : F(M) \to F(A)]$.
- *a 2-arrow between two 1-arrows* $F, G : \mathscr{C} \to \mathscr{D}$ *is just a natural transformation between functors.*

Definition 23. *The* regular completion *of a first-order doctrine* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$, *is the category* $\mathrm{Reg}(P)$ *defined as follows:*

- *the objects of* $\mathrm{Reg}(P)$ *are pairs* (A, α), *where* A *is an object of* $\mathscr{C}$ *and* α *is an element of* $P(A)$;
- *a morphism of* $\mathrm{Reg}(P)$ *between two objects* (A, α) *and* (B, β) *is an element* ϕ *in* $P(A \times B)$ *such that* $x : A \mid \alpha(x) \vdash \exists!_{y:B}(\phi(x, y) \wedge \beta(y))$ *holds.*

Proposition 24. *For any first-order doctrine* P, *the category* $\mathrm{Reg}(P)$ *is a Heyting category and the assignment* $P \mapsto \mathrm{Reg}(P)$ *extends to a 2-functor*

$$\mathbf{FoD} \underset{Sub_{(-)}}{\overset{\mathrm{Reg}(-)}{\rightleftarrows}} \mathbf{HeytCat} \qquad (\bot)$$

which is left biadjoint to the inclusion of the 2-category **HeytCat** *of Heyting categories in the 2-category* **FoD** *of first-order doctrines acting as* $\mathscr{C} \mapsto Sub_{\mathscr{C}}$.

Proof. By definition, the subobjects functor over a Heyting category is a first-order doctrine. Then, the subobjects doctrine $Sub_{\mathrm{Reg}(P)} : \mathrm{Reg}(P)^{\mathrm{op}} \to \mathbf{Heyt}$ of the regular completion of first-order doctrine $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ is a first-order doctrine analogously to the proof of (Johnstone, 2002, Lemma 1.4.10, Section D 1.4) since the regular completion of a syntactic first-order doctrine coincides with Johnstone's notion of syntactic category (Johnstone, 2002, Section D 1.4, p. 841) as observed in (Maietti and Trotta, 2024, Ex. 4.14).

To prove the universal property, first recall that the $\mathrm{Reg}(-)$ construction is left biadjoint to the inclusion of the 2-category elementary and existential doctrines in the 2-category of regular categories (Maietti et al., 2017, Thm. 3.3). Then observe that the action of $\mathrm{Reg}(-)$ on the morphisms of elementary and existential doctrine sends morphisms of first-order doctrines into morphisms of Heyting categories. Finally, the unit of this adjunction, in the case of first-order doctrines, is a morphism of first-doctrines, while the counit is an iso. □

Definition 25 (Exact completion of a first-order doctrine)**.** *Given a first-order intuitionistic doctrine* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$, *its* exact completion *is the category whose objects are pairs* (A, ρ_A), *where* A *is an object of* $\mathscr{C}$ *and* ρ_A *is a symmetric and transitive* P*-relation, i.e. it is an element of* $P(A \times A)$ *satisfying the following conditions:*

(1) $x : A, y : A \mid \rho(x, y) \vdash \rho(y, x)$
(2) $x : A, y : A, z : A \mid \rho(x, y), \rho(y, z) \vdash \rho(x, z)$

Morphisms between two objects (A, ρ_A) *and* (B, ρ_B) *are elements* ϕ *of* $P(A \times B)$ *which preserve the* P*-equivalence relations* ρ_A *and* ρ_B, *i.e. they satisfy the following conditions.*

(1) (well-defined) $x : A, y : B \mid \phi(x, y) \vdash \rho_A(x, x) \wedge \rho_B(y, y)$
(2) (totality on reflexive elements) $x : A \mid \rho_A(x, x) \vdash \exists_{y:B} \, (\, \phi(x, y) \wedge \rho_B(y, y)\,)$
(3) (preservation of P-relations)
$x_1 : A, x_2 : A, y_1 : B, y_2 : B \mid \phi(x_1, y_1) \wedge \rho_A(x_1, x_2) \wedge \rho_B(y_1, y_2) \vdash \phi(x_2, y_2)$
(4) (functionality) $x_1 : A, y_1 : B, y_2 : B \mid \phi(x_1, y_1) \wedge \phi(x_1, y_2) \vdash \rho_B(y_1, y_2)$

Remark 26. The fact that the exact completion of an elementary existential doctrine in (Maietti and Rosolini, 2015, Def.2.2) restricts to a completion of first-order doctrines into exact categories whose subobjects functor is a first-order doctrine was originally proved in (Pitts, 2002, Theorem 3.6). Furthermore, τ_P is called *tripos-to-topos construction* when P is a tripos as recalled in the mentioned reference.

From Maietti and Rosolini (2015) and Maietti et al. (2017) we know that the exact completion of an elementary existential doctrine can be decomposed into the regular completion and the ex/reg-completion $\mathscr{C}_{ex/reg}$ of a regular category $\mathscr{C}$ to an exact category in the sense of Carboni (1995). Hence, we get the following lemma:

Lemma 27. *The exact completion* $\mathrm{Ex}(P)$ *of a strong tripos* P *is equivalent to* $(\mathrm{Reg}(P))_{ex/reg}$*, namely the ex/reg completion of the regular completion of* P*.*

Now, we are ready to observe that the regular completion of an arithmetic strong tripos is indeed a topos with an NNO which coincides with its tripos-to-topos construction:

Theorem 28. *If* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *is an arithmetic strong tripos, then* $\mathrm{Reg}(P)$ *is a topos with an NNO, and the embedding* $P \hookrightarrow Sub_{\mathrm{Reg}(P)}$ *is a morphism of arithmetic strong triposes. Moreover,* $\mathrm{Reg}(P)$ *coincides with the exact completion* $\mathrm{Ex}(P)$ *of* P*.*

Proof. The proof follows from (Lambek and Scott, 1986, Thm. 12.5). Indeed, it is enough to observe that the category denoted by $T(\mathfrak{L})$ associated to a type theory $\mathfrak{L}$ in (Lambek and Scott, 1986, Sec. 11, p. 186) is precisely the regular completion of the syntactic strong tripos associated with this (higher-order) type theory. This category is proved to be a topos (with a natural numbers object) (Lambek and Scott, 1986, Thm. 12.5). Hence, given an arbitrary arithmetic strong tripos P, we can mimic the mentioned proof by Lambek and Scott to conclude that $\mathrm{Reg}(P)$ is a topos with an NNO.

The fact that the canonical embedding $P \hookrightarrow Sub_{\mathrm{Reg}(P)}$ is a morphism of arithmetic strong triposes follows by definition of the power-objects in $\mathrm{Reg}(P)$ and by the fact that the regular completion preserves the structure of first-order doctrine.

Finally, the tripos-to-topos construction $\mathrm{Ex}(P)$ of P coincides with $\mathrm{Reg}(P)$ since by lemma 27 $\mathrm{Ex}(P)$ coincides with $(\mathrm{Reg}(P))_{ex/reg}$, which is $\mathrm{Reg}(P)$, because $\mathrm{Reg}(P)$ is an exact category, being a topos, and the exact completion of a regular category in Carboni (1995) is idempotent. $\square$

Corollary 29. *The regular completion establishes a biadjunction between arithmetic strong triposes and toposes with an NNO:*

$$\mathbf{asTr} \underset{Sub_{(-)}}{\overset{\mathrm{Reg}(-)}{\rightleftarrows}} \mathbf{TopCat}_{NNO} \quad (\perp)$$

where $\mathbf{TopCat}_{NNO}$ *denotes the 2-category of toposes and logical functors (preserving their NNO).*

Proof. As recalled in Proposition 24, the regular completion provides a biadjunction between the 2-category of first-order doctrines and the 2-category of Heyting categories.

By Theorem 28 we know that it sends an arithmetic strong tripos into a topos with an NNO, and that the unit of the adjunction is a morphism of arithmetic strong triposes. Then, observe that the subobjects functor of a topos with an NNO is an arithmetic strong tripos. Indeed, it is a strong tripos because the base category is a topos, and it is arithmetic by (Maietti et al., 2019, Prop. 3.3) because the base category has, in particular, a parameterized natural number object.

Moreover, the counit of the adjunction is, in general, an isomorphism. So, to conclude, it is enough to observe that the regular completion sends morphisms of strong triposes into morphisms of toposes, because, by definition, a morphism of arithmetic strong triposes preserves power-objects and the natural numbers object. □

Corollary 30. *The regular completion* $\mathrm{Reg}(\mathsf{Prop}_{\mathbf{HaTT}})$ *of the initial arithmetic strong tripos* $\mathsf{Prop}_{\mathbf{HaTT}}$ *is the free topos with an NNO.*

Proof. It follows from Proposition 16 together with Corollary 29. Indeed, for any topos $\mathscr{T}$ we have

$$\mathbf{TopCat}_{NNO}(\mathrm{Reg}(\mathsf{Prop}_{\mathbf{HaTT}}), \mathscr{T}) \simeq \mathbf{asTr}(\mathsf{Prop}_{\mathbf{HaTT}}, Sub_{\mathscr{T}}) \simeq 1. \qquad \square$$

Remark 31. Corollary 29 says that the tripos-to-topos construction is a free construction into the category of toposes with an NNO (and logical functors) when applied to arithmetic strong triposes. Observe that localic and realizability triposes are not arithmetic strong tripos. Our main relevant example of arithmetic strong tripos is the doctrine of strong subobjects of the quasi-topos of Assemblies within the Effective topos in Hyland (1982).

2.4 Two versions of Church's Thesis

Let $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ be an arithmetic CC-strong tripos, and let A and B be two objects of $\mathscr{C}$. An important distinction to notice is the one between functions from A to B as arrows $f : A \to B$ of the base category $\mathscr{C}$, which we will call *operations*, and functions as *functional relations*, that is total and singled-valued relations in the internal logic of P. We reserve the name *function* for the notion of functional relation, which can be defined internally as a proposition $x : A, y : B \mid \phi(x, y)$ such that the sentence $\mathsf{Fun}(\phi) :\equiv \forall_{x:A}\exists!_{y:B}\, \phi(x, y)$ is true.

This distinction is genuine. In fact, given an element $x : A, y : B \mid \phi(x, y)$, the following sentence, called *Axiom of Unique Choice*, is not necessarily true.

$$\mathsf{Fun}(\phi) \Rightarrow \exists_{f:A\to B}\forall_{a:A}\, \phi(a, \mathrm{ev}(f, a)) \tag{AC!}$$

With this distinction in mind, we define two versions of the formal Church Thesis, one for functions, simply called *(formal) Church Thesis* CT, and one for operations, called *Type-theoretic (formal) Church Thesis* TCT, as the following sentences

$$\begin{aligned} &\forall_{\rho:\mathscr{P}(\mathsf{Nat}\times\mathsf{Nat})}\big(\forall_{x:\mathsf{Nat}}\exists!_{y:\mathsf{Nat}}\,(\langle x, y\rangle \in_{\mathsf{Nat}\times\mathsf{Nat}} \rho) \\ &\quad \Rightarrow \exists_{e:\mathsf{Nat}}\forall_{x:\mathsf{Nat}}\exists_{y:\mathsf{Nat}}(\mathsf{T}(e, x, y) \wedge \langle x, \mathsf{U}(y)\rangle \in_{\mathsf{Nat}\times\mathsf{Nat}} \rho)\big) \end{aligned} \tag{CT}$$

$$\forall_{f:\mathsf{Nat}\to\mathsf{Nat}}\exists_{e:\mathsf{Nat}}\forall_{x:\mathsf{Nat}}\exists_{y:\mathsf{Nat}}(\mathsf{T}(e, x, y) \wedge \mathsf{U}(y) =_{\mathsf{Nat}} \mathrm{ev}(f, x)) \tag{TCT}$$

where, in both cases, $\mathsf{T}(e, x, y)$ is the Kleene predicate expressing that y is the encoding of the computation history of the computable function encoded by e on input x, and $\mathsf{U} : \mathsf{Nat} \to \mathsf{Nat}$ is the arrow encoding the output extraction of a computation.

Observe that in the regular completion $Reg(\mathscr{Q})$ in Section 2.3 applied to a quasi-topos $\mathscr{Q}$, the exponential objects of two objects embedded from $\mathscr{Q}$ is given by the set of functional relations in $\mathscr{Q}$ between these two objects.

3. Brouwer's continuity principles

In general, the unique choice axiom AC! does not hold in quasi-toposes, as it happens in foundations such as the Minimalist Foundation of Maietti and Sambin (2005); Maietti (2009). The non-validity of AC! in such contexts allows to employ the two concepts of number-theoretic functional relation and operation to identify Brouwer's notion of *choice sequence* with that of functional relation, as done in the context of axiomatic set theory in Rathjen (2005), and that of *lawlike sequence* with that of operation (see Section 2.4). We refer to Troelstra and van Dalen (1988) for all the explanations about Brouwer's use of these notions.

Here, we employ the notion of choice sequence to state Brouwer's principles in the language of doctrines in terms of the spatiality of suitable locales following Fourman and Grayson (1982). Such locales will be defined inductively, employing notions from the predicative and constructive approach to locale theory known as *formal topology*, following (Maietti and Sambin, 2013).

Intuitively, in formal topology, one starts with a set A of *formal basic opens*, and a *cover relation* $a \lhd V$ between elements $a \in A$ and subsets $V \in \mathscr{P}(A)$, meaning that the basic open a is covered by the family of basic opens V. Such a cover induces a *closure operator*

$$\lhd(-) : \mathscr{P}(A) \to \mathscr{P}(A)$$
$$\lhd(V) :\equiv \{x \in A \mid x \lhd V\}$$

which always forms a *complete* suplattice with respect to the inclusion order. When the cover satisfies a convergence property (Coquand et al., 2003; Maietti and Valentini, 2004; Ciraulo et al., 2013), the fixed points of $\lhd(-)$ define a locale.

In Coquand et al. (2003), the authors introduced a powerful method to generate constructive and predicative examples of formal topologies by defining covers inductively generated from a given set of axioms. Relevant examples of inductively-generated formal topologies include *Cantor* and *Baire formal topologies* since Brouwer's principle of Fan theorem and Monotone Bar Induction can be equivalently formalized as the spatiality of Cantor formal topology and that of Baire formal topology, respectively, as originally stated by Fourman and Grayson (1982).

In the following, fix an arithmetic strong tripos $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$; all the definitions in this section will be given relatively to it, and in particular within the many-sorted logic **HaTT** describing the internal language of the initial arithmetic strong tripos of Section 2.1. In the following by a *P-set*, we mean an element α of $P(A)$ for some object A, denoted by the formal expression $\{x : A \mid \alpha(x)\}$. It is easy to define the following constructors on P-sets: cartesian product $\alpha \times \beta$; powerset $\mathscr{P}(\alpha)$; comprehension $\{x : \alpha \mid \varphi(x)\}$ by another element $\varphi \in P(A)$; and lists $\mathsf{List}(\alpha)$. The latter is defined by expressing a functional relation with finite support. It is precisely the list constructor that motivates us to switch from objects in the base category to (formal) comprehensions to define the intended notion of set; in fact, notice that we do not have list objects in the base. Finally, an object A of $\mathscr{C}$ will be identified with the P-set $\{x : A \mid \top\}$.

The following definitions relativize the definitions of function and operation introduced in Section 2.4 to P-sets and formalize the corresponding notions of sequence. Recall from Section 2.1, that by $\forall_{x:\alpha}\varphi(x)$ we mean $\forall_{x:A}(\alpha(x) \Rightarrow \varphi(x))$, and similarly for the existential quantifier.

Definition 32. *Let* $\{x : A \mid \alpha(x)\}$ *and* $\{x : B \mid \beta(x)\}$ *be P-sets. The* P-set of functions *between the two is defined as follows.*

$$\mathsf{Fun}(\alpha, \beta) :\equiv \{\, \rho \;: \mathscr{P}(\alpha \times \beta) \mid \forall_{x:\alpha}\exists!_{y:\beta}\langle x, y\rangle \in_{A\times B} \rho\,\}.$$

Moreover, if P is an arithmetic CC-strong tripos, the P-set of operations *between the two is defined as*

$$\mathsf{Op}(\alpha,\beta) :\equiv \{f : A \to B \mid \forall_{x:A}(\alpha(x) \Rightarrow \beta(ev(f,x)))\}$$

Definition 33. *A* choice sequence *on a P-set α is an element of* $\mathsf{Fun}(\mathsf{Nat},\alpha)$ *(recall that we identify* Nat *with the P-set* $\{x : \mathsf{Nat} \mid \top\}$*).*

Remark 34. In case P is an arithmetic CC-strong tripos satisfying TCT, a *lawlike sequence* on a P-set α is an element of $\mathsf{Op}(\mathsf{Nat},\alpha)$.

Remark 35. Note that in Troelstra and van Dalen (1988) the notion of choice sequence on Nat is identified with that of a type-theoretic function $f : \mathsf{Nat} \to \mathsf{Nat}$ while in Rathjen (2005) with that of a functional relation.

We now formalize the definitions of Baire and Cantor formal topologies in our impredicative setting, following the notation of Maietti et al. (2021). Both topologies are instances of the more general notion of *tree formal topology*. Given a P-set A, arising from an object of the base category $\mathscr{C}$, the tree formal topology relative to it is represented by the following cover on lists of A.

$$l : \mathrm{List}(A), V : \mathscr{P}(\mathrm{List}(A)) \mid l \lhd_{tr(A)} V$$

Such a cover is impredicatively defined as the smallest one satisfying the following sentences

$$\forall_{l:\mathrm{List}(A)}\ \ (\, l \lhd_{tr(A)} \{s : \mathrm{List}(A) \mid \exists_{a:A}\, s =_{\mathrm{List}(A)} \mathsf{cons}(l,a)\}\,)$$
$$\forall_{l:\mathrm{List}(A)}\forall_{s:\mathrm{List}(A)}\ \ (l \sqsubseteq s \Rightarrow l \lhd_{tr(A)} \{s\})$$

where $l \sqsubseteq s$ means that the list s is an initial segment of the list l, formally defined as

$$l \sqsubseteq s \ :\equiv\ \exists_{t:\mathrm{List}(A)}(l =_{\mathrm{List}(A)} [s,t])$$

and $[-,-] : \mathrm{List}(A) \times \mathrm{List}(A) \to \mathrm{List}(A)$ is the concatenation operation of two lists, and $\mathsf{cons}(l,a)$ is the list constructor adding an element a to the list l.

Definition 36. *The* (formal topology of the) Baire space $\lhd_{Baire} :\equiv \lhd_{tr(\mathsf{Nat})}$ *is the tree topology on* Nat.

The (formal topology of the) Cantor space $\lhd_{Cantor} :\equiv \lhd_{tr(2)}$ *is the tree topology on the boolean P-set defined as* $2 :\equiv \{x : \mathsf{Nat} \mid x = 0 \vee x = 1\}$.

We then recall the notion of a formal point for the tree formal topologies of the form $\lhd_{tr(A)}$ since each formal point of such a topology describes the graph of a sequence toward the set A defined as a functional relation, namely a choice sequence on A.

Definition 37 (Formal point of tree topology)**.** Let A be a P-set. We define what it means for an element $\Psi : \mathscr{P}(\mathrm{List}(A))$ to be a *formal point* of the formal topology $\lhd_{tr(A)}$, and we collect them in a P-set denoted $Pt(\lhd_{tr(A)})$ and defined as follows.

$$\begin{aligned}
&\Psi : \mathscr{P}(\mathrm{List}(A)) \mid Pt_{tr(A)}(\Psi) :\equiv \exists_{l:\mathrm{List}(A)}\, l \in_{\mathrm{List}(A)} \Psi \\
&\quad \wedge \forall_{l_1,l_2:\mathrm{List}(A)}\, (\, l_1 \in_{\mathrm{List}(A)} \Psi \wedge l_2 \in_{\mathrm{List}(A)} \Psi \Rightarrow \exists_{s:\mathrm{List}(A)}\, (\, s \in_{\mathrm{List}(A)} \Psi \wedge s \sqsubseteq l_1 \wedge s \sqsubseteq l_2\,)\,) \\
&\quad \wedge \forall_{l:\mathrm{List}(A)}\, (\, l \in_{\mathrm{List}(A)} \Psi \Rightarrow (\forall_{s:\mathrm{List}(A)}\, l \sqsubseteq s \Rightarrow s \in_{\mathrm{List}(A)} \Psi\,)\,) \\
&\quad \wedge \forall_{l:\mathrm{List}(A)}\, (\, l \in_{\mathrm{List}(A)} \Psi \Rightarrow \exists_{a:A}\, \mathsf{cons}(l,a) \in_{\mathrm{List}(A)} \Psi\,)
\end{aligned}$$

Now note that choice sequences on A are exactly the formal points of the tree formal topology over A.

Proposition 38. *Formal points $Pt_{tr(A)}$ of the tree formal topology relative to a P-set A are in bijection with the choice sequences on A.*

Proof. Given a formal point β, we can associate a functional relation ϕ_β to it using the following predicate

$$n : \mathsf{Nat}, a : A \mid \ \phi_\beta(n,a) :\equiv \exists_{l:\mathrm{List}(A)} \left(a =_A l_{n+1} \ \& \ l \in_{\mathrm{List}(A)} \beta \right)$$

where l_n is the n-th component of the list l; in fact, the above predicate satisfies the following

$$\beta : \mathscr{P}(\mathrm{List}(A)) \mid \beta \in_{\mathscr{P}(\mathrm{List}(A))} Pt(\lhd_{tr(A)}) \Rightarrow \mathsf{Fun}(\phi_\beta)$$

Conversely, given a relation $x : \mathsf{Nat}, y : A \mid \phi(x,y)$, consider the P-set

$$\beta :\equiv \{ l : \mathrm{List}(A) \mid \forall_{n:\mathsf{Nat}} \left(n+1 \leq \mathsf{lh}(l) \Rightarrow \phi(n, l_n) \right) \}$$

where $\mathsf{lh}(l)$ represents the length of the list l. It associates with each functional relation a formal point, in the sense that it satisfies the following

$$\mathsf{Fun}(\phi) \Rightarrow \{ l : \mathrm{List}(A) \mid \beta(l) \} \in_{\mathscr{P}(\mathrm{List}(A))} Pt(\lhd_{tr(A)})$$

□

Remark 39. An alternative proof follows after noting, as observed in Sigstam (1995), that any tree formal topology on A is the exponential formal topology of the discrete formal topology of natural numbers on the discrete topology on A (see Maietti (2005) for a constructive and predicative construction of exponentiation). Therefore, its formal points are in bijection with functional relations, being all these continuous. This explains why formal points of Cantor and Baire topologies are related to Brouwer's choice sequences as defined in Definition 32.

Now, we are ready to formulate Bar Induction as *spatiality of the tree formal topology on a given P-set A*, as originally pointed out in (Fourman and Grayson, 1982, Theorem 3.4). We underline that this formulation of Bar Induction was one of the motivations behind the birth of formal topology in Sambin (1987, 2008) within a predicative foundation.

Definition 40. *The* principle $\mathsf{BI}(A)$ of Bar Induction in topological form *relative to a tree formal topology $tr(A)$ is the following sentence:*

$$\forall_{l:\mathrm{List}(A)} \forall_{V:\mathscr{P}(List(A))} \left(\forall_{\Psi:Pt_{tr(A)}} (l \in_{\mathrm{List}(A)} \Psi \Rightarrow \exists_{s:\mathrm{List}(A)} (s \in_{\mathrm{List}(A)} V \wedge s \in_{\mathrm{List}(A)} \Psi)) \quad \Rightarrow \quad l \lhd_{tr(A)} V \right)$$

This formulation of $\mathsf{BI}(A)$ essentially means the spatiality of the tree formal topology since the topology put on the formal points of the tree $\mathrm{List}(A)$, that are its choice sequences, coincides with the point-free one. The main consequence is that *we can reason on the topology of choice sequences by induction on finite sequences*, being the point-free one inductively generated.

As shown in full details in Gambino and Schuster (2007) and in Maietti (2012) within the Minimalist Foundation of Maietti (2009), Bar Induction on the Baire formal topology $\mathsf{BI}(\mathsf{Nat})$ corresponds to Brouwer's Monotone Bar Induction, as well as Bar Induction on the Cantor formal topology $\mathsf{BI}(2)$ corresponds to the Fan theorem. Hence, we use the following names:

Definition 41. *We call* Bar Induction *the principle* $\mathsf{BI}(\mathsf{Nat})$ *stating the spatiality of the Baire space and* Fan *the principle* $\mathsf{BI}(2)$ *stating the spatiality of the Cantor space.*

As shown in Dummett (2000):

Lemma 42. *In any arithmetic strong tripos P it holds that* $P \models \mathsf{Bar\ Induction} \Rightarrow \mathsf{Fan}$.

Proof. Apply the argument in (Maietti, 2012, Prop.3.19). □

Proposition 43. *In any arithmetic strong tripos P, the formal Church Thesis* CT *is inconsistent with the Fan theorem* Fan*, i.e. it holds that* $P \models \mathsf{CT} \wedge \mathsf{Fan} \Rightarrow \bot$.

Proof. Apply Kleene's argument in (Troelstra and van Dalen, 1988, Section 7.6, Chapter 4, p. 220). □

In addition to Bar Induction, we will consider two other principles that characterize Brouwer's intuitionism, namely the local continuity principle and a choice principle for the Baire space.

Definition 44. Brouwer's local continuity principle*, for short* LCP*, states that any total relation from the formal points of the Baire space to natural numbers is continuous.*

$$\forall_{\Psi:Pt(\lhd_{Baire})}\ \exists_{n:\mathsf{Nat}}\ \rho(\Psi,n) \Rightarrow \forall_{\Psi:Pt(\lhd_{Baire})}\ \exists_{n,m:\mathsf{Nat}} \forall_{\xi:Pt(\lhd_{Baire})}$$
$$(\ \exists_{l:\mathrm{List}(\mathsf{Nat})}(\mathsf{lh}(l) =_{\mathsf{Nat}} m \wedge l \in_{\mathrm{List}(\mathsf{Nat})} \Psi \wedge l \in_{\mathrm{List}(\mathsf{Nat})} \xi) \ \Rightarrow\ \rho(\xi,n)\ )$$

The above principle is called WC-N on p. 209 of Troelstra and van Dalen (1988).

Definition 45. Choice for Baire space*, for short* AC_{Baire}*, states that any total relation* $\Psi : Pt(\lhd_{Baire}), n : \mathsf{Nat} \,|\, R(\Psi,n)$ *between the Baire space and natural numbers contains the graph of a choice function.*

$$\forall_{\Psi:Pt(\lhd_{Baire})}\exists_{n:\mathsf{Nat}}\ \rho(\Psi,n) \Rightarrow \exists_{\phi:\mathsf{Fun}(Pt(\lhd_{Baire}),\mathsf{Nat})}\ \forall_{\Psi:Pt(\lhd_{Baire})}\forall_{n:\mathsf{Nat}}\ (\ \langle \Psi,n\rangle \in\ \phi \Rightarrow\ \ \rho(\Psi,n)\ )$$

Finally, we recall the following relevant fact about functions between real numbers defined as Cauchy sequences.

Theorem 46 (Brouwer's continuity)**.** *In any arithmetic strong tripos P, it holds that* $\mathsf{Bar\ Induction} + \mathsf{LCP}$ *implies that all functional relations between real numbers are uniformly continuous.*

Proof. See (Troelstra and van Dalen, 1988, Theorem 3.6, Chapter 6, p. 306). □

4. Our intuitionistic meta-theory and the category $\mathbf{Set}_i$

Let $\mathbf{IZF}_{\mathsf{BP}}$ be the extension of Intuitionistic Zermelo-Fraenkel Set Theory (Friedman, 1977) with Brouwer's principles $\mathsf{Bar\ Induction}$, LCP, and AC_{Baire}. Realizability models of **IZF** extended with these principles can be constructed using Kleene's second algebra by adopting the techniques in (Rathjen, 2005; Rosolini, 1982; Friedman, 1973; McCarty, 1984). We are going to use $\mathbf{IZF}_{\mathsf{BP}}$ as the meta-theory in which to formalize our quasi-toposes.

Define the large category $\mathbf{Set}_i$ whose objects are sets in $\mathbf{IZF}_{\mathsf{BP}}$ and the morphisms are functional relations in $\mathbf{IZF}_{\mathsf{BP}}$, and denote by

$$\mathbb{P} : \mathbf{Set}_i{}^{\mathrm{op}} \to \mathbf{Heyt}$$

the *powerset functor* associating to any set A its powerset $\mathbb{P}(A)$, and to any set-theoretic function $f : A \to B$, the inverse image function $\mathbb{P}_f \equiv f^{-1} : \mathbb{P}(B) \to \mathbb{P}(A)$.

Such a functor is, of course, an arithmetic CC-strong tripos. Two essential, although trivial, observations can be made about it. Firstly, truth in its logical notation (as specified in Section 2.1) coincides with truth in the meta-theory; namely, $\mathbb{P} \models \varphi(x)$ holds if and only if the obvious externalization of φ is true in $\mathbf{IZF}_{\mathsf{BP}}$. Secondly, notice that $\mathbb{P}$ coincides with the functor of subobjects of the topos $\mathbf{Set}_i$.

5. The intuitionistic quasi-toposes of assemblies

Here, we build two intuitionistic quasi-toposes validating both the effective principle TCT, and Brouwer's intuitionistic principles Bar Induction, LCP and AC_{Baire}. One quasi-topos is built by formalizing in $\mathbf{IZF}_{\mathsf{BP}}$ the quasi-topos of assemblies **Asm** within the Effective topos introduced in Hyland (1982). The other quasi-topos is built as the elementary quotient completion of the subdoctrine of strong subobjects on the full subcategory of **Asm** with *partitioned assemblies*, denoted **PAsm** and also formalized within $\mathbf{IZF}_{\mathsf{BP}}$. It is worth recalling that both the notions of *assembly* and *partitioned assembly* were originally introduced in Carboni et al. (1988), while partitioned assemblies have been used in Robinson and Rosolini (1990) to show that Hyland's effective topos is an exact completion of a lex category.

We then show that our two quasi-toposes can not be proved to be equivalent in $\mathbf{IZF}_{\mathsf{BP}}$. This in contrast with what happen when they are formalized in the classical meta-theory **ZFC** (Zermelo-Fraenkel Set Theory with the Axiom of Choice), where they are equivalent as shown in Maietti et al. (2019). In particular, it turns out that the intuitionistic rendering of **Asm** loses some projectivity properties.

5.1 Categories and doctrines of assemblies

We begin by formalizing the category of assemblies and partitioned assemblies over $\mathbf{IZF}_{\mathsf{BP}}$, and by deriving some fundamental properties regarding their structure.

Therefore, all the constructions in this section are assumed to be formalized in $\mathbf{IZF}_{\mathsf{BP}}$ unless explicitly stated otherwise.

Definition 47. *An* assembly *over* $\mathbf{IZF}_{\mathsf{BP}}$ *is a pair* $(A, \Vdash_A)$*, where* A *is a set, called* support*, and* $n \Vdash_A a$ *is a total relation, called* realizability*, between the set of natural numbers* $\mathbb{N}$ *and* A*, i.e. a relation such that for all* $a \in A$ *there exists an* $n \in \mathbb{N}$ *such that* $n \Vdash_A a$*, and in this case we say that* n realizes a.

The category $\mathbf{Asm}_i$ *of* intuitionistic assemblies *has assemblies over* $\mathbf{IZF}_{\mathsf{BP}}$ *as objects; a morphism of assemblies* $(A, \Vdash_A) \to (B, \Vdash_B)$ *consists of a function* $f : A \to B$ *such that there exists a natural number* $e \in \mathbb{N}$ *satisfying the following condition. Whenever* $n \Vdash_A a$ *holds, if follows that* $\{e\}(n)$ *is defined and* $\{e\}(n) \Vdash_B f(a)$ *holds too, where we denoted by* $\{e\}(n)$ *the value (if any) of the recursive function coded by* e *on argument* n*. In such a case, we will say that the number* e tracks f*, in symbols* $e \Vdash_{fun} f$.

Identity and composition are inherited from the usual ones in $\mathbf{Set}_i$.

We call $\mathbf{PAsm}_i$ *the full subcategory of* $\mathbf{Asm}_i$ *of* partitioned assemblies*, whose objects are those assemblies* $(A, \Vdash_A)$ *such that for each* $a \in A$ *there exists a unique* $n \in \mathbb{N}$ *satisfying* $n \Vdash_A a$.

Lemma 48. *Both* $\mathbf{Asm}_i$ *and* $\mathbf{PAsm}_i$ *are lextensive, i.e. they have finite limits and stable disjoint finite coproducts, and they both have an NNO. Furthermore,* $\mathbf{Asm}_i$ *has all finite colimits and it is locally cartesian closed, while* $\mathbf{PAsm}_i$ *has weak dependent products in the sense of Def. 4.*

Proof. We refer to (van Oosten, 2008, Theorem 1.5.2) for a detailed proof regarding the structure of $\mathbf{Asm}_i$, which also works in a constructive meta-theory as $\mathbf{IZF}_{\mathsf{BP}}$.

Firstly, we just recall the definitions of binary products and binary coproducts both in $\mathbf{Asm}_i$ and in $\mathbf{PAsm}_i$. The binary product of two (partitioned) assemblies $(A, \Vdash_A)$ and $(B, \Vdash_B)$ is given by

$(A \times B, \Vdash_{A\times B})$ with $\mathbf{p}(n_1, n_2) \Vdash_{A\times B} (x, y)$ whenever both $n_1 \Vdash_A x$ and $n_2 \Vdash_B y$ hold, where $\mathbf{p}$ is the recursive encoding of a pair of natural numbers. The coproduct of two (partitioned) assemblies $(A, \Vdash_A)$ and $(B, \Vdash_B)$ is given by the assembly $(A+B, \Vdash_{A+B})$, with $\mathbf{p}(0,n) \Vdash_{A+B} \iota_A(x)$ whenever $n \Vdash_A x$, and $\mathbf{p}(1,n) \Vdash_{A+B} \iota_B(y)$ whenever $n \Vdash_B y$.

It is easy to check that $\mathbf{Asm}_i$ inherits coequalizers from those of $\mathbf{Set}_i$, where the realizers of an equivalence class of a quotient are the union of the realizers of its representatives.

Furthermore, $\mathbf{Asm}_i$ is locally cartesian closed, since the dependent product of a morphism $g\colon (Y', \Vdash_{Y'}) \to (Y, \Vdash_Y)$ along $f\colon (Y, \Vdash_Y) \to (X, \Vdash_X)$ is given by the first projection map $\pi_1 : (E, \Vdash_E) \to (X, \Vdash_X)$ where

$$E :\equiv \{(x,h) | \, x \in X, h : (f^{-1}(x), \Vdash_{Y|f^{-1}(x)}) \to (Y', \Vdash_{Y'}) \text{ in } \mathbf{Asm}_i \text{ s.t. } g \circ h = id_{f^{-1}(x)}\}$$

and $n \Vdash_E (x,h)$ iff there exists an $m \in \mathbb{N}$ such that $n = \mathbf{p}(m,e)$, $m \Vdash_X x$, and $e \Vdash_{fun} h$ hold.

Instead, in $\mathbf{PAsm}_i$ only weak dependent products in the sense of Def. 4. exist: given two morphisms of partitioned assemblies $f\colon (Y, \Vdash_Y) \to (X, \Vdash_X)$ and $g\colon (Y', \Vdash_{Y'}) \to (Y, \Vdash_Y)$ their weak dependent product is given by the first projection map $\pi_1 : (E, \Vdash_E) \to (X, \Vdash_X)$ where

$$E :\equiv \{(x,h,e) \,|\, x : X, h : (f^{-1}(x), \Vdash_{Y|f^{-1}(x)}) \to (Y', \Vdash_{Y'}) \text{ in } \mathbf{PAsm}_i \text{ s.t. } g \circ h = id_{f^{-1}(x)}, e \Vdash_{fun} h\}$$

and $n \Vdash_E (x,h,e)$ iff there exists an $m \in \mathbb{N}$ such that $n = \mathbf{p}(m,e)$ and $m \Vdash_X x$ hold. Indeed, $\mathbf{PAsm}_i$ can not have function spaces because of the undecidability of function extensionality, see (Maietti et al., 2019, Remark 4.10).

Finally, observe that $(\mathbb{N}, \Vdash_{\mathbb{N}})$ where $m \Vdash_{\mathbb{N}} n :\equiv m = n$ is an NNO both in $\mathbf{Asm}_i$ and in $\mathbf{PAsm}_i$. □

The interplay between assemblies and sets is key to deriving our main results. In the following, we start the investigation of such a relationship, which will culminate in Theorem 60 in the next section.

Definition 49. *Let the* global section functor *be the functor*

$$\Gamma :\equiv \mathbf{Asm}_i(\mathbf{1}, -) : \mathbf{Asm}_i \to \mathbf{Set}_i$$

which is the forgetful functor assigning the set A to each assembly $(A, \Vdash_A)$.

Similarly, we consider its restriction to partitioned assemblies.

$$\Gamma :\equiv \mathbf{PAsm}_i(\mathbf{1}, -) : \mathbf{PAsm}_i \to \mathbf{Set}_i$$

Definition 50. *Let $\nabla : \mathbf{Set}_i \to \mathbf{PAsm}_i$ be the functor, called* canonical embedding*, which sends a set A to the assembly $(A, \Vdash_\nabla)$, where $n \Vdash_\nabla a$ iff $n = 0$, and sends any map $f : A \to B$ to itself (which is tracked by the code of the identity recursive function).*

We also call $\nabla : \mathbf{Set}_i \to \mathbf{Asm}_i$ the composition of ∇ above with the inclusion of $\mathbf{PAsm}_i$ within $\mathbf{Asm}_i$.

Proposition 51. *The forgetful functor $\Gamma : \mathbf{PAsm}_i \to \mathbf{Set}_i$ is left adjoint to the canonical inclusion functor $\nabla\colon \mathbf{Set}_i \hookrightarrow \mathbf{PAsm}_i$, as well as $\Gamma : \mathbf{Asm}_i \to \mathbf{Set}_i$ is left adjoint to $\nabla\colon \mathbf{Set}_i \hookrightarrow \mathbf{Asm}_i$. Each of these adjunctions is a mono-localization, namely a reflection such that the left adjoint preserves finite limits and the unit is monic.*

Proof. The proof follows the same line used by (Menni, 2000, Sec. 2.3.2 and Sec. 7.3). □

Proposition 52. *Both the doctrine $sSub_{\mathbf{Asm}_i} : \mathbf{Asm}_i^{op} \to \mathbf{Heyt}$ of strong subobjects on assemblies and the doctrine $sSub_{\mathbf{PAsm}_i} : \mathbf{PAsm}_i^{op} \to \mathbf{Heyt}$ of strong subobjects on partitioned assemblies are* *equivalent to the power-object functor $\mathbb{P}$ of* $\mathbf{Set}_i$ *composed with the corresponding forgetful functor* Γ. *Furthermore, they both admit a predicate classifier, or equivalently, both* $\mathbf{PAsm}_i$ *and* $\mathbf{Asm}_i$ *have a strong-subobject classifier.*

Proof. First, observe that from Proposition 51 we can deduce from (Menni, 2000, Lemma 7.4.3) that both $\mathbf{PAsm}_i$ and $\mathbf{Asm}_i$ have a stable epi/regular-mono factorization system induced by that in $\mathbf{Set}_i$, where epis/regular monos of this factorization system in $\mathbf{PAsm}_i$ are sent to epis/regular monos in $\mathbf{Set}_i$ by Γ and the poset of monos of this factorization system on a (partitioned) assembly $(A, \Vdash_A)$ is in bijection with $\mathbb{P}\Gamma^{op}(A, \Vdash_A) = \mathbb{P}(A)$. Since regular monos are strong, we have a factorization system with strong monos and epis. Hence, we conclude that the strong monos of the factorization system include all strong monos.

Finally, a strong-subobject classifier both in $\mathbf{PAsm}_i$ and $\mathbf{Asm}_i$ is given by $\nabla(\mathbb{P}(1))$ in analogy to what happens when adopting a classical meta-theory as pointed out in Menni (2000) after Def. 2.3.10. $\square$

Now, as it happens in (Hyland, 1982; van Oosten, 2008), where the category of assemblies is formalized in a classical meta-theory, we are ready to conclude the following.

Corollary 53. *The category* $\mathbf{Asm}_i$ *is an arithmetic solid quasi-topos.*

Proof. From Lemma 48 and Proposition 52. $\square$

Now, we build another quasi-topos by taking the elementary quotient completion $\overline{(\mathbb{P}\Gamma^{op})}$: $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}^{op} \to \mathbf{Heyt}$ of the doctrine $\mathbb{P}\Gamma^{op} : \mathbf{PAsm}_i^{op} \to \mathbf{Heyt}$ on partitioned assemblies, which is the doctrine of strong subobjects on $\mathbf{PAsm}_i$.

To show that the base category $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ is an arithmetic quasi-topos we need to rely heavily on the results of Maietti et al. (2023).

Theorem 54. *The category* $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ *is an arithmetic solid quasi-topos and the doctrine* $\overline{\mathbb{P}\Gamma^{op}}$: $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}^{op} \to \mathbf{Heyt}$ *is equivalent to its strong subobjects doctrine denoted* $sSub_{\mathscr{Q}_{\mathbb{P}\Gamma^{op}}}$.

Proof. From Theorem 7.10 and 7.14 of Maietti et al. (2023) we know that the quotient completion of a doctrine is the strong subobject doctrine of an arithmetic quasi-topos if and only if the starting doctrine is an arithmetic hyper-tripos; namely, an elementary existential doctrine with a weak predicate classifier and full weak comprehensions and comprehensive diagonals, such that its base category has a natural numbers object, weak pullbacks, it is slice-wise weakly cartesian closed and has finite distributive coproducts. Moreover, we recall from (Maietti et al., 2023, Prop. 6.44 and 7.4), that a hyper-tripos P has always P-disjoint coproducts and hence its quotient completion has $\overline{P}$-disjoint coproducts (in the case P is the strong subobject we obtain disjoint coproducts).

Hence, to prove that $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ is an arithmetic quasi-topos, it is sufficient to prove that the doctrine $\mathbb{P}\Gamma^{op}$ has the listed structure.

First, recall that $\mathbb{P}\Gamma^{op}$ is a first-order doctrine with full comprehensions and comprehensive diagonals. From Lemma 48 we know that $\mathbf{PAsm}_i$ is lextensive, with an NNO and weak dependent products, which implies its slice-wise weakly cartesian closure in the sense of Def. 6.9 in Maietti et al. (2023) as follows. Since $\mathbf{PAsm}_i$ has pullbacks, any slice category $\mathbf{PAsm}_i/(A, \Vdash_A)$ has finite products. Hence, to show that each dependent projective $f : (B, \Vdash_B) \to (A, \Vdash_A)$ is weakly exponentiable in $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}/(A, \Vdash_A)$ for partitioned objects $(B, \Vdash_B)$ and $(A, \Vdash_A)$, it is enough to show that any $f : (B, \Vdash_B) \to (A, \Vdash_A)$ is weakly exponentiable in $\mathbf{PAsm}_i/(A, \Vdash_A)$.

More in detail, a weak exponential of $g\colon (Y, \Vdash_Y) \to (A, \Vdash_A)$ over $f\colon (X, \Vdash_X) \to (A, \Vdash_A)$ is given by the first projections $\pi_1 : E \to X$ where

$$E := \{(x, h, e) \mid x : X \ , \ h\colon (f^{-1}(x), \Vdash_{A|f^{-1}(x)}) \ \to \ (g^{-1}(x), \Vdash_{A|g^{-1}(x)}) \text{ in } \mathbf{PAsm}_i \ , \ e \Vdash_{fun} h\}$$

and $n \Vdash (x,h,e) := n = \mathsf{pair}(m,e) \;\&\; m \Vdash_X x$.

Lastly, by Proposition 52 we know that $\mathbf{PAsm}_i$ has a strong-subobject classifier and hence $\mathbb{P}\Gamma^{\mathrm{op}}$ has a predicate classifier as pointed out in Remark 8. This concludes the proof that $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ is an arithmetic quasi-topos.

Finally, the fact that $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ is solid, namely that the unique map $0 \to 1$ is a strong monomorphism, follows easily from Proposition 52 and the definition of $\overline{\mathbb{P}\Gamma^{\mathrm{op}}}$. □

In Maietti et al. (2019), it was proved that the two quasi-toposes $\mathbf{Asm}_i$ and $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ are equivalent when they are formalized in a classical meta-theory as **ZFC**, with a crucial use of the axiom of choice. Now, we show that such an equivalence fails to be proved when working within our intuitionistic meta-theory.

Firstly, let $\overline{\nabla}\colon \mathbf{Set}_i \to \mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ denote the functor sending an object A to $((A, \Vdash_\nabla), \delta_A)$, where δ_A is the equality relation on A, namely the functor obtained by composing the canonical embedding $\nabla\colon \mathbf{Set}_i \to \mathbf{PAsm}_i$ with the canonical embedding of $\mathbf{PAsm}_i$ into $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$.

Lemma 55. *The functor $\overline{\nabla}$ preserves regular epimorphisms and finite coproducts.*

Proof. Both these facts are straightforward due to the nature of objects and arrows of $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$. Indeed, recall that the coproduct $((A, \Vdash_A), \rho) + ((B, \Vdash_B), \sigma)$ in $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ is defined as the object $((A+B, \Vdash_{A+B}), \rho \boxplus \sigma)$, where $\rho \boxplus \sigma :\equiv \exists_{\iota_A \times \iota_A}(\rho) \vee \exists_{\iota_B \times \iota_B}(\sigma)$, see e.g. (Maietti et al., 2023, Theorem 6.42). Hence, we have that

$$\overline{\nabla}(A+B) = ((A+B, \Vdash_\nabla), \delta_{A+B}) \cong ((A, \Vdash_\nabla), \delta_A) + ((B, \Vdash_\nabla), \delta_B) = \overline{\nabla}(A) + \overline{\nabla}(B).$$

Similarly, one can check that the functor $\overline{\nabla}\colon \mathbf{Set}_i \to \mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ preserves regular epis. □

Theorem 56. *The categories $\mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ and $\mathbf{Asm}_i$ can not be proven to be equivalent by $\mathbf{IZF}_{\mathsf{BP}}$ (and hence by* **IZF***).*

Proof. By contradiction, assume that there is an equivalence of categories $F\colon \mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}} \to \mathbf{Asm}_i$. Recall that, by the characterization of the quotient completions in (Maietti et al., 2023, Thm. 5.5), the subcategory of $sSub_{\mathbf{Asm}_i}$-projectives objects of $\mathbf{Asm}_i$ is equivalent to that of $\overline{(\mathbb{P}\Gamma^{\mathrm{op}})}$-projectives, i.e. to that of partitioned assemblies.

Now fix an arbitrary proposition φ, and consider the partitioned assembly $(Bool, \Vdash_{Bool})$ with equivalence relation $\rho_\varphi(x,y) :\equiv x = y \vee \varphi$, where $Bool$ is the set $\{\mathsf{true}, \mathsf{false}\}$ and the unique realizers for its elements are $0 \Vdash_{Bool} \mathsf{false}$ and $1 \Vdash_{Bool} \mathsf{true}$. Now we can consider in $\mathbf{Set}_i$ the quotient $Bool/\rho_\varphi$ and its regular epi $q\colon Bool \to Bool/\rho_\varphi$. By Lemma 55, the functor $\overline{\nabla}\colon \mathbf{Set}_i \to \mathcal{Q}_{\mathbb{P}\Gamma^{\mathrm{op}}}$ preserves coproducts and regular epis, and hence $(F \circ \overline{\nabla})(q)$ is a regular epi in $\mathbf{Asm}_i$ with a $sSub_{\mathbf{Asm}_i}$-projective codomain equivalent to $(Bool/\rho_\varphi, \Vdash')$ for some relation $\Vdash'$ given by the equivalence F, after observing that the strong subobject determined by φ is preserved by the equivalence with its logical structure.

We can consider now the identity map id (with the track of the constant computable function) $id : (Bool/\rho_\varphi, \Vdash') \to (Bool/\rho_\varphi, \Vdash_{all})$ where $\Vdash_{all}$ is the relation $n \Vdash_{all} x$ for every natural number n and every x of $Bool/\rho_\varphi$. Furthermore, consider the following regular epi, sending an element of $Bool$ into its equivalence class.

$$p : (Bool, \Vdash_{Bool}) \to (Bool/\rho_\varphi, \Vdash_{all})$$

Since $(Bool/\rho_\varphi, \Vdash')$ is projective, there exists a unique arrow f in $\mathbf{Asm}_i$ making the following diagram commute.

$$\begin{array}{ccc} & & (Bool, \Vdash_{Bool}) \\ & \overset{\exists! f}{\nearrow} & \big\downarrow p \\ (Bool/\rho_\varphi, \Vdash') & \xrightarrow[id]{} & (Bool/\rho_\varphi, \Vdash_{all}) \end{array}$$

In particular, we have that $p \circ f = id$ in $\mathbf{Set}_i$. Now, observe that $f([\mathsf{true}]) = f([\mathsf{false}])$ is true if and only if $\rho_\varphi(\mathsf{true}, \mathsf{false})$ holds, if and only if φ holds; since the equality between booleans is decidable, we have that

$$f([\mathsf{true}]) = f([\mathsf{false}]) \vee f([\mathsf{true}]) \neq f([\mathsf{false}])$$

holds in **IZF**, and hence also $\varphi \vee \neg\varphi$. Since φ is arbitrary, we would conclude that the law of excluded middle is valid in **IZF**, which is a contradiction.

The same argument applies if working within $\mathbf{IZF}_{\mathsf{BP}}$. □

Although distinct, the two quasi-toposes of assemblies are crucially related in the following way

Definition 57. *Define $Q : \mathscr{Q}_{\mathbb{P}\Gamma^{op}} \to \mathbf{Asm}_i$ as the functor sending an object $((A, \Vdash_A), \rho)$ of $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ into the quotient $(A/\rho, \Vdash_\rho)$ with $n \Vdash_\rho [a] :\equiv n \Vdash_A a$, after recalling that ρ yields an equivalence relation on A in $\mathbf{Set}_i$.*

Proposition 58. *The doctrine of strong subobjects in $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ is equivalent to the composition of the doctrine of strong subobjects in $\mathbf{Asm}_i$ with the functor $Q^{op} : \mathscr{Q}^{op}_{\mathbb{P}\Gamma^{op}} \to \mathbf{Asm}_i^{op}$.*

Proof. It follows from the fact that $\mathbf{Set}_i$ is an exact category whose quotients are of effective descent type as defined in Definition 20. □

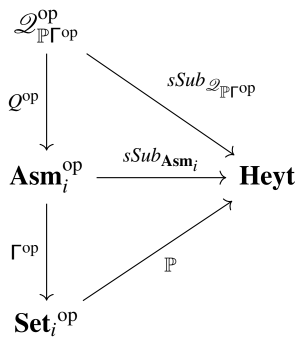
Remark 59. Notice that the functor $Q : \mathscr{Q}_{\mathbb{P}\Gamma^{op}} \to \mathbf{Asm}_i$ has a right adjoint $\Sigma : \mathbf{Asm}_i \longrightarrow \mathscr{Q}_{\mathbb{P}\Gamma^{op}}$ defined as follows: for any assembly $(A, \Vdash_A)$ we associate the object (Σ_A, ρ_Σ) where Σ_A is the partitioned assembly whose support is

$$\{(a, m) \mid a \in A, m \in \mathbb{N} \text{ s.t. } m \Vdash_A a\}$$

and $n \Vdash_{\Sigma_A} (a, m) :\equiv m = n$, while $\rho_\Sigma((a, m), (a', m')) :\equiv a = a'$. Indeed, we can define a natural bijection $\mathbf{Asm}_i(Q(A, \Vdash_A), \rho), (B, \Vdash_B)) \cong \mathscr{Q}_{\mathbb{P}\Gamma^{op}}(((A, \Vdash_A), \rho), \Sigma(B, \Vdash_B))$ by associating $g(a) = (f([a]), \{e\}(n_a))$ to a morphism $f : Q((A, \Vdash_A), \rho) \to (B, \Vdash_B)$ with track e. Conversely, it associates $f([a]) = \pi_1 \circ g(a)$ for any a in A to any $g : ((A, \Vdash_A), \rho) \to \Sigma(B, \Vdash_B)$, where $\pi_1 : \Sigma_B \to B$ is the function sending $(b, m) \mapsto b$. Of course, after Theorem 56 this adjunction can not be an equivalence.

5.2 Effectiveness and continuity principles validated

First, recall that Proposition 52, Theorem 54, and Proposition 58 together express the commutativity up to natural isomorphisms of the following diagram of functors.

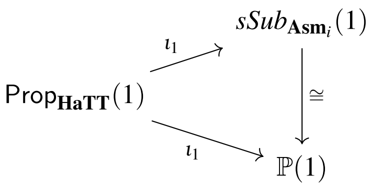


(1)

Moreover, observe that the diagram above can be read as a composition of arithmetic strong tripos 1-arrows. We can then show how the higher-order arithmetic of the meta-theory reflects into the quasi-toposes $\mathbf{Asm}_i$ and $\mathscr{Q}_{\mathbb{P}\Gamma^{op}}$.

Theorem 60 (transfer principle)**.** *If a sentence formulated in the language of* **HaTT** *is true in* $\mathbf{IZF}_{\mathsf{BP}}$*, then it is true in* $sSub_{\mathbf{Asm}_i}$ *and* $sSub_{\mathscr{Q}_{\mathbb{P}\Gamma^{op}}}$.

Proof. By the initiality of the arithmetic strong tripos $\mathsf{Prop}_{\mathbf{HaTT}}$ (Proposition 16), we know that the following commutes, where the unlabeled vertical arrow is the isomorphism obtained by specializing the lower triangle of Diagram 1.



It is therefore immediate to conclude that $\mathbb{P} \models \varphi(x)$ if and only if $sSub_{\mathbf{Asm}_i} \models \varphi(x)$ for any sentence $\varphi(x)$ expressed in **HaTT**. An analogous proof can be carried out for the arithmetic strong tripos $sSub_{\mathscr{Q}_{\mathbb{P}\Gamma^{op}}}$ using the outer triangle of Diagram 1. □

It is worth noting that Theorem 60 is very general but it does not hold if we replace the initial arithmetic strong tripos with the initial arithmetic CC-strong tripos. This is because the forgetful functor Γ does not preserve exponentials. As an immediate corollary, we obtain:

Corollary 61. *In the arithmetic strong triposes* $sSub_{\mathbf{Asm}_i}$ *and* $sSub_{\mathscr{Q}_{\mathbb{P}\Gamma^{op}}}$ *the Brouwer's principles* Bar Induction, LCP*, and* AC_{Baire} *are true.*

Proof. From Theorem 60, since both Bar Induction, LCP, and AC_{Baire} can be written in the language of **HaTT**, and they are true (actually, they are axioms) in $\mathbf{IZF}_{\mathsf{BP}}$. □

Finally, we can show that assemblies model the Type-theoretic Church Thesis.

Theorem 62. TCT *is true in both the arithmetic strong triposes* $sSub_{\mathbf{Asm}_i}$ *and* $sSub_{\mathscr{Q}_{\mathbb{P}\Gamma^{op}}}$.

Proof. In the meta-theory, define the predicate *Comp* over the function space $\mathbb{N} \to \mathbb{N}$ as $Comp(f) :\equiv (\exists e \in \mathbb{N}) e \Vdash_{fun} f$. Then, TCT is interpreted in both $sSub_{\mathbf{Asm}_i}$ and $sSub_{\mathscr{Q}_{\mathbb{P}\mathsf{rop}}}$ as the following sentence of the meta-theory

$$(\forall f \in \{g \in \mathbb{N} \to \mathbb{N} \mid Comp(g)\})\, Comp(f)$$

which holds trivially because of Lemma 52 and Lemma 58. □

Remark 63. Observe that in Corollary 4.9 of Maietti et al. (2019) it is proved that the doctrine of strong subobjects in the classical category of assemblies **Asm** inherits the validity of TCT from the doctrine of strong subobjects on $\mathbf{PAsm}_i$.

After Theorem 56, such an explanation survives within an intuitionistic meta-theory, as **IZF**, only to explain why $sSub_{\mathscr{Q}_{\mathbb{P}\mathsf{rop}}}$ inherits the validity of TCT.

We end this section by proving negative results, showing on the one hand the necessity in our treatment of considering genuine quasi-toposes instead of arbitrary toposes, and, on the other, the poor behavior of the regular completion with respect to the structures we are interested in.

The following proposition is what motivates the search for genuine quasi-toposes validating these principles.

Proposition 64. *There exists no non-trivial arithmetic topos validating* $\mathsf{TCT} + \mathsf{Bar\ Induction}$*, and even more with also* $\mathsf{LCP} + \mathsf{AC}_{Baire}$.

Proof. Any topos validates the Axiom of Unique Choice AC! as shown by using its internal language in Maietti (2005). Therefore, if a topos would validate $\mathsf{TCT} + \mathsf{Bar\ Induction}$, it would also validate CT, and hence the falsum by Lemmas 42 and 43. □

Proposition 65. *The regular completions of the arithmetic strong triposes* $sSub_{\mathbf{Asm}_i}$ *and* $sSub_{\mathscr{Q}_{\mathbb{P}\mathsf{rop}}}$ *are both equivalent to* $\mathbf{Set}_i$ *and the free topos with a NNO* $\mathrm{Reg}(Init)$ *embeds into them, i.e. there is a faithful functor from* $\mathrm{Reg}(Init)$ *to* $\mathbf{Set}_i$

$$\mathrm{Reg}(Init) \hookrightarrow \mathrm{Reg}(sSub_{\mathbf{Asm}_i}) = \mathrm{Reg}(sSub_{\mathscr{Q}_{\mathbb{P}\mathsf{rop}}}) = \mathbf{Set}_i.$$

Proof. It follows from Definition 23 of regular completion, Proposition 52 and, Corollary 30. □

Theorem 66. *There exists a CC-strong tripos* $P : \mathscr{C}^{\mathrm{op}} \to \mathbf{Heyt}$ *for which the embedding into its regular completion* $\mathrm{Reg}(P)$ *does NOT preserve the cartesian closed structure.*

Proof. Thanks to Proposition 65 both $sSub_{\mathbf{Asm}_i} : \mathbf{Asm}_i^{\mathrm{op}} \to \mathbf{Heyt}$ and $sSub_{\mathscr{Q}_{\mathbb{P}\mathsf{rop}}} : \mathscr{Q}_{\mathbb{P}\mathsf{rop}}^{\mathrm{op}} \to \mathbf{Heyt}$ provide examples of CC-strong triposes whose regular completion is $\mathbf{Set}_i$ and whose embedding into it fails to preserve the cartesian closed structure of their bases. Indeed, if it did, then TCT would hold in $\mathbf{IZF}_{\mathsf{BP}}$. Moreover, CT would also hold, since exponentials are interpreted in $\mathbf{IZF}_{\mathsf{BP}}$ as sets of functional relations. However, this would lead to a contradiction, since Brouwer's principles of $\mathbf{IZF}_{\mathsf{BP}}$ are inconsistent with CT by Lemmas 42 and 43. □

Remark 67. From what we have shown here, we can conclude that the calculus λ**HaTT** is not complete with respect to toposes. Indeed, a calculus that extends λ**HaTT** and enjoys such completeness property must satisfy the Axiom of Unique Choice AC!. One possible extension is the dependently typed calculus $\mathscr{T}_{\mathbf{Topos}}$ by Maietti (2005), where the axiom of unique choice is not postulated, but rather derived from the identification of propositions with mono-types.

6. Conclusion

We have built two intuitionistic quasi-toposes $\mathbf{Asm}_i$ and $\mathcal{Q}_{\mathbb{P}\Gamma^{op}}$ validating Brouwer's continuity principles, including Bar Induction, the local continuity principle LCP, and an instance of choice AC_{Baire}. The striking property is that they also validate a restricted form of Church's Thesis TCT, expressing that all morphisms of the considered quasi-topos are computable. This is possible because *Brouwer's choice sequences* are interpreted as *number-theoretic functional relations*. Furthermore, within these quasi-toposes the validity of these continuity principles is inherited from the meta-theory $\mathbf{IZF}_{\mathsf{BP}}$ due to the particular nature of their doctrine of strong subobjects. Only the quasi-topos $\mathcal{Q}_{\mathbb{P}\Gamma^{op}}$ retains the projectivity originally proved for the classical version of $\mathbf{Asm}_i$ in Maietti et al. (2019).

These universes show that it is possible to reconcile Markov's constructivism with Brouwer's intuitionism without renouncing to all the effective and continuity principles stipulated in these approaches. In particular, they show that Heyting Arithmetic with finite types extended with power-objects, as those in the generic calculus of arithmetic CC-strong triposes is consistent both with Brouwer's continuity principles and TCT.

As future work, we want to show the consistency of both principles within predicative settings including the two-level Minimalist Foundation (Maietti, 2009), and even richer setting including Coquand-Huet's Calculus of Constructions (Coquand and Huet, 1988) by modeling them in either of two quasi-toposes depending of the need of projectivity we will have.

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Declarations

The authors declare that they have no competing interests.

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