Consistency of the minimalist foundation with Church thesis and Bar Induction

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Abstract

The well-known result by Kleene that Brouwer's principle of Bar Induction is inconsistent with the formal Church thesis for choice sequences can be decomposed in our minimalist foundation as follows: Brouwer's Bar Induction, where choice sequences are functional relations, is inconsistent with the formal Church thesis for type-theoretic functions from natural numbers to natural numbers and the axiom of unique choice transforming a functional relation between natural numbers into a type-theoretic function.

Here, we show that a version of our minimalist constructive foundation is consistent with Bar Induction together with Church thesis, and hence it does not validate the axiom of unique choice. To prove this we build a simple realizability model. In this model our sets are interpreted by following usual Kleene realizability. Instead propositions are interpreted in a proof-irrelevant way.

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1 Introduction

In joint work with G. Sambin started in [MS05] we argued about the necessity of building a foundation for constructive mathematics to be taken as a minimal core among the most relevant constructive ones. There we also defined what we mean by a constructive foundation by requiring that it should consist of two levels: *one*, called *intensional*, must be given by a *proofs-as-programs theory* acting as a programming language, and *the other*, called *extensional*, must be a set theory, where to formalize mathematical proofs. In addition the extensional level is required to be obtained by abstraction from the intensional one according to Sambin's forget-restore principle in [SV98] so to preserve the extraction of programs from proofs. Moreover, by a *proofs-as-programs theory* we meant one consistent with the formal Church thesis and the axiom of choice. This is indeed a very technical definition compared to the intuitive idea of a proofs-as-programs theory. However it turned out to be very useful in discriminating intensional and constructive theories versus extensional and classical ones.

In [Mai09] we built an example of our desired constructive foundation. Its two levels are both given by a type theory à la Martin-Löf: the first is an intensional type theory as [NPS90], called mTT, and the latter is an extensional one with quotients and proof-irrelevance of propositions. The extensional level is then interpreted in the intensional one via a quotient model based on total setoids à la Bishop [Bis67, Hof97, BCP03, Pal05]. This means that extensional concepts are obtained by just abstracting from equalities of intensional ones.

As advocated in [MS05], a main novelty of our foundation in [Mai09], which is also a major difference with respect to Martin-Löf's type theory, is that it should not validate the axiom of unique choice turning a functional relation into a type-theoretic function, even restricted to natural numbers. Formally, this distinction is possible because, at both levels of our foundation, we discharged the isomorphism "propositions-as-sets" of Martin-Löf's type theory. Indeed we build propositions via primitive constructors distinct from those for sets, as it happens in the Calculus of Constructions [Coq90] of which mTT is a predicative version.

Therefore in our foundation, contrary to most extensional constructive theories in the literature, such as Aczel's one [AR01] or the internal theory of a topos (for example in [Mai05a]), we have two distinct notions of function: the usual notion of functional relation and that of type-theoretic function.

The benefit of this is the possibility of revisiting the well known result by Kleene [Tv88a] that Brouwer's principle of Bar Induction, or better the Fan theorem derived from it, is inconsistent with the formal Church thesis for choice sequences [Tv88a, Rat05, Dum00]. Indeed, at the extensional level of our foundation emTT this result gives that Brouwer's Bar Induction ($\mathrm{BI}_{\mathrm{fr}}$) where choice sequences are functional relations is inconsistent with the formal Church thesis ($\mathrm{CT}_{\mathrm{tt}}$) for type-theoretic functions from natural numbers to natural numbers *in the presence of the axiom of unique choice* on natural numbers ($\mathrm{AC!}_{\mathrm{N,N}}$) turning a functional relation into a type-theoretic function. Therefore, in the absence of unique choice, it makes sense to investigate consistency of our foundation with $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$.

The importance of finding a consistent extension of emTT where $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$ are valid is that of providing a setting apt to develop constructive analysis, where $\mathrm{BI}_{\mathrm{fr}}$, or better the Fan theorem, has already shown to be very useful (see for example [Bri08, BR87]). In the same time we want to keep a computational interpretation of type-theoretic functions thanks to the presence of $\mathrm{CT}_{\mathrm{tt}}$ as advocated by Feferman in designing his theories [Fef79]. Then, in this extension we can well identify *choice sequences* with functional relations and *lawlike sequences* with type-theoretic functions, which happen to be also *recursive* by the presence of $\mathrm{CT}_{\mathrm{tt}}$. The absence of unique choice in the extension says that choice sequences and lawlike sequences do not collapse together. Hence there is a clear distinction between computational concepts and non-computational or ideal ones as explained in [Sam08]. Indeed, as we know from [FG82, Sam87, GS07], a choice sequence defined as a functional relation happens to be a formal point of a suitable formal topology. Then, Bar Induction amounts to spatiality of such a topology. Therefore, without unique choice we have that the collection of formal points happen to be an ideal concept that does not collapse with the set of lawlike sequences which is a computational one (see [Sam08]).

Here we prove consistency with $\mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}}$ of a slightly modified version of our two-level foundation in [Mai09], where we restricted the collection constructors to a minimum to represent the power collection of a set. Its intensional level is called mTT_0 and its extensional one emTT_0.

In particular we prove the consistency of emTT_0 with $\mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}}$ by reducing it to the consistency of the corresponding intensional level $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$, where $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ are the mTT_0-translations of the corresponding emTT_0-formulations. In turn the consistency of $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ is obtained by building a simple realizability interpretation in classical set theory ZFC.

In the realizability model, to validate $\mathrm{CT}^i_{\mathrm{tt}}$ we interpret mTT_0-sets as subsets of natural numbers like in the realizability interpretation à la Kleene built in [Tv88b] for a version of Martin-Löf's type theory. Then, we interpret mTT_0-propositions as their boolean value and mTT_0-set elements as suitable computable functions. Finally to validate $\mathrm{BI}^i_{\mathrm{fr}}$ we interpret mTT_0-collections and their elements as ZFC-sets and functions respectively, with no computational contents.

As a corollary of our results we get a proof that, as expected, mTT_0 as well as emTT_0 do not validate unique choice including $\mathrm{AC!}_{\mathrm{N,N}}$.

The model presented here provides a natural way to interpret our minimalist foundation in classical set theory, or in Feferman's theories [Fef79], by keeping the interpretation of operations as recursive functions.

In the future we hope to extend such a realizability model to model the intensional level of our original foundation in [Mai09]. Moreover, it would be interesting to investigate consistency with $\mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ of an impredicative version of our intensional level as the Calculus of Constructions [Coq90].

2 The two-level theory: mTT_0 and emTT_0

As described in the introduction in [Mai09] we built a two-level foundation meeting the requirements in [MS05].

Here we consider a slightly modified version of this foundation where we restrict collection constructors to a minimum to formalize Bar Induction. Its intensional level is called mTT_0 and its extensional one emTT_0.

mTT_0 is a fragment of mTT and, as mTT it has the following features: it is represented by an intensional type theory as [NPS90] with collections distinct from sets to represent the power collection of a set in a predicative way; propositions are defined in a primitive way to avoid the validity of choice principles; we distinguish small propositions as those propositions closed only under quantification over sets to define subsets of a set; we identify any proposition with the collection of its proofs, as well as any small proposition with the set of its proofs to implement useful operations on subsets advocated in [SV98, Sam10]; we replace usual equality rules in [NPS90] with *substitution rules* given explicitly, in order to avoid the presence of the ξ-rule for λ-terms.

We recall from [Mai09] that the mentioned change of equality rules enable us to show consistency of mTT_0 with the axiom of choice and formal Church thesis, as advocated in [MS05] via a realizability interpretation à la Kleene (see [Mar75]). Luckily, *this change of equality rules does not affect the interpretation of the extensional level* emTT_0 where the ξ-rule is present and it is equivalent to extensionality of type-theoretic functions.

Also the extensional level emTT_0 shares with the extensional level emTT in [Mai09] the fact that is an extensional type theory as [Mar84] which is closed under effective quotient sets. Moreover its propositions, defined primitively as in mTT_0, are proof-irrelevant, namely they are equipped with at most an unique canonical proof-term.

The only difference between mTT_0 and mTT as well as between emTT_0 and emTT is that indexed sum of collection families are restricted to indexed sums of propositional functions only. In other terms the rules F-Σ), I-Σ), E-Σ), C-Σ) of strong indexed sums in mTT and emTT in [Mai09] are simply replaced by the following ones

Strong Indexed Sum of a propositional function

$$\text{F-ip)}\ \frac{C(x)\ \ prop\ [x\in B]}{\Sigma_{x\in B}C(x)\ col} \qquad \text{I-ip)}\ \frac{b\in B\quad d\in C(b)\quad C(x)\ \ prop\ [x\in B]}{\langle b,d\rangle\in\Sigma_{x\in B}C(x)}$$

$$\text{E-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z\in\Sigma_{x\in B}C(x)] \\ d\in\Sigma_{x\in B}C(x)\quad m(x,y)\in M(\langle x,y\rangle)\ [x\in B, y\in C(x)]\end{array}}{El_{\Sigma}(d,m)\in M(d)}$$

$$\text{C-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z\in\Sigma_{x\in B}C(x)] \\ b\in B\quad c\in C(b)\quad m(x,y)\in M(\langle x,y\rangle)\ [x\in B, y\in C(x)]\end{array}}{El_{\Sigma}(\,\langle b,c\rangle, m\,)=m(b,c)\in M(\langle b,c\rangle)}$$

In emTT_0 we add also the equality rules of such indexed sums corresponding to eq-Σ), I-eq Σ), E-eq Σ) in emTT.
Moreover, in emTT_0 we have also the collection of subsets of the singleton as in emTT (in [Mai09] we forgot to add to emTT the rule sm-eq) saying that the propositional equality of subsets is small):

Power collection of the singleton

$$\text{F-P)}\ \ \mathcal{P}(1)\ col \qquad \text{I-P)}\ \frac{B\ prop_s}{[B]\in\mathcal{P}(1)} \qquad \text{eq-P)}\ \frac{\mathsf{true}\in B\leftrightarrow C}{[B]=[C]\in\mathcal{P}(1)} \qquad \text{eff-P)}\ \frac{[B]=[C]\in\mathcal{P}(1)}{\mathsf{true}\in B\leftrightarrow C}$$

$$\text{sm-eq)}\ \frac{U\in\mathcal{P}(1)\qquad V\in\mathcal{P}(1)}{\mathsf{Eq}(\mathcal{P}(1),\,U,\,V\,)\ prop_s} \qquad \eta\text{-P)}\ \frac{U\in\mathcal{P}(1)}{U=[\,\mathsf{Eq}(\mathcal{P}(1),\,U,\,[\mathsf{tt}]\,)\,]}$$

where $\mathsf{tt}\equiv\bot\to\bot$ represents the truth constant.

Then, we have also function collections from a set toward $\mathcal{P}(1)$ to represent the power collection of a set:

Function collection to $\mathcal{P}(1)$

$$\text{F-Fc)}\ \frac{B\ set}{B \to \mathcal{P}(1)\ col} \qquad\qquad \text{I-Fc)}\ \frac{c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\lambda x^B.c(x) \in B \to \mathcal{P}(1)}$$

$$\text{E-Fc)}\ \frac{b \in B \quad f \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f,b) \in \mathcal{P}(1)} \qquad\qquad \beta\text{C-Fc)}\ \frac{b \in B \quad c(x) \in \mathcal{P}(1)\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^B.c(x), b) = c(b) \in \mathcal{P}(1)}$$

$$\eta\text{C-Fc)}\ \frac{f \in B \to \mathcal{P}(1)}{\lambda x^B.\mathsf{Ap}(f,x) = f \in B \to \mathcal{P}(1)} (x\ not\ free\ in\ f)$$

The above restriction of indexed sums is enough to interpret these function collections in mTT_0 as in [Mai09]. Hence we define an interpretation of the extensional level emTT_0 into mTT_0 as that in [Mai09]:

Def. 2.1 We call

$$(-)^i : \mathrm{emTT}_0 \to \mathrm{mTT}_0$$

the restriction of the interpretation of emTT-dependent types and terms into mTT-extensional dependent types and terms in [Mai09], where, in particular, dependent sets are interpreted as total dependent setoids à la Bishop [Bis67, Pal05] [1].

This is well defined because emTT_0-propositions are interpreted as mTT_0-propositions from which we get that strong indexed sums of propositional functions in emTT_0 are defined via indexed sums of propositional functions in mTT_0 as follows (recall that $\sigma^{\overline{x}}_{\overline{x'}}$ are isomorphisms needed to interpret substitution):

Strong Indexed Sum :

$(\,\Sigma_{y \in B} C(y)\,)^I\ col\ [\Gamma^I] \equiv \Sigma_{y \in B^I} C^I(y)\ col\ [\Gamma^I]$

and $z =_{\Sigma_{y \in B} C(y)^I} z' \equiv \quad \exists_{d \in \pi_1(z) =_{B^I} \pi_1(z')}\ \sigma^{\pi_1(z')}_{\pi_1(z)}(\pi_2(z)) =_{C^I(\pi_1(z'))} \pi_2(z')$ for $z, z' \in (\,\Sigma_{y \in B} C(y)\,)^I$.

with terms constructors interpreted exactly as in [Mai09].

Moreover, the function collection towards $\mathcal{P}(1)$ is interpreted by using only indexed sums of propositional functions as follows:

Function collection toward $\mathcal{P}(1)$:

$(\,B \to \mathcal{P}(1)\ col\ [\Gamma]\,)^I \equiv \Sigma_{h \in B^I \to \mathsf{props}} \quad \forall_{y_1 \in B^I}\ \forall_{y_2 \in B^I}\ (\,y_1 =_{B^I} y_2 \to (\,\mathsf{Ap}(h, y_1) \leftrightarrow \mathsf{Ap}(h, y_2)\,)\,)$

with equality $z =_{\mathcal{P}} z' \equiv \forall_{y \in B^I}\ (\,\mathsf{Ap}(\,\pi_1(z)\,, y) \leftrightarrow \mathsf{Ap}(\,\pi_1(z')\,, y)\,)$ for $z, z' \in (\,B \to \mathcal{P}(1)\,)^I$

$(\,\lambda y^B.c\,)^I \equiv \langle\,\lambda y^{\widetilde{B}}.c^{\widetilde{I}}\,, p\,\rangle$ where $p \in \forall_{y_1 \in B^I}\ \forall_{y_2 \in B^I}\ (y_1 =_{B^I} y_2 \to (\,c^{\widetilde{I}}(y_1) \leftrightarrow c^{\widetilde{I}}(y_2)\,)\,)$

$(\,\mathsf{Ap}(f, b)\,)^I \equiv \mathsf{Ap}(\,\pi_1(f^{\widetilde{I}})\,, b^{\widetilde{I}})$

$\sigma^{\overline{x'}}_{\overline{x}}(w) \equiv \langle\,\lambda y'^{B^I(\overline{x'})}.\,\sigma^{\overline{x'}, y'}_{\overline{x}, \sigma^{\overline{x}}_{\overline{x'}}(y')}(\,\mathsf{Ap}(\,\pi_1(w)\,, \sigma^{\overline{x}}_{\overline{x'}}(y')\,)\,), p\,\rangle$ for $\overline{x}, \overline{x'} \in \Gamma^I$ and $w \in (\,B \to \mathcal{P}(1)\,)^I(\overline{x})$ where p is a proof-term witnessing the preservation of equalities obtained from $\pi_2(w)$.

3 Formulation of $\mathbf{CT}_{\mathrm{tt}}$, $\mathbf{AC!}_{\mathbf{N},\mathbf{N}}$ and $\mathbf{BI}_{\mathbf{fr}}$

The goal of our work is to prove that the extensional level emTT_0 of our minimalist foundation is consistent with Bar Induction where choice sequences are functional relations, for short $\mathrm{BI}_{\mathrm{fr}}$, together

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[1] Note that we can not turn such an interpretation $(-)^i$ into one that interprets emTT_0 in a categorical model of quotients built over mTT_0 as done over mTT in [Mai09], because there we interpreted contexts via generic indexed sums not available in mTT_0. Here we can only turn $(-)^i$ into an interpretation of emTT_0 in a syntactic indexed category built out of mTT_0, for example as in [Hof97], where emTT_0-contexts are interpreted as mTT_0-extensional contexts exactly as done by $(-)^i$.

with the formal Church thesis for type-theoretic functions, for short $\mathrm{CT}_{\mathrm{tt}}$.

Given that our extensional level emTT_0 can be interpreted in mTT_0 via quotients, to fulfil our purpose it is enough to prove that the intensional level mTT_0 is consistent with the translations $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ of the emTT_0-formulations $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$.

Here we start by presenting the formulation of $\mathrm{CT}_{\mathrm{tt}}$ and of unique choice on natural numbers $\mathrm{AC!}_{\mathrm{N,N}}$. Then we pass to formulate $\mathrm{BI}_{\mathrm{fr}}$ in topological terms as pioneered in [FG82] but in the context of formal topology [Sam87, GS07], namely of point-free topology developed in a predicative way (for a survey and related notation see [Sam03]). We then review its connection with the traditional formulation of Bar Induction and Fan theorem [Dum00, Tv88a, Rat05] (see [Sam10] for a survey and further developments about this). Then we use Kleene's result [Tv88a] about inconsistency of Fan theorem with Church thesis for choice sequences to deduce that Bar Induction is inconsistent with $\mathrm{CT}_{\mathrm{tt}}$ and unique choice. Finally we formulate such principles at the intensional level.

3.1 Formulation of $\mathrm{CT}_{\mathrm{tt}}$, $\mathrm{AC!}_{\mathrm{N,N}}$ and $\mathrm{BI}_{\mathrm{fr}}$ at the extensional level

One of the benefits of our foundation emTT_0 is that it allows to have two notions of function: one is that of *functional relation* and the other is that of *type-theoretic function* (or *operation* as called in [Sam10]). In emTT_0 a functional relation from a set A to a set B is identified with a small proposition $a\,R\,b\ prop_s\ [a \in A, b \in B]$ satisfying [2]

$$\forall x \in A\ \exists! y \in B\quad R(x,y)$$

Instead in emTT_0 a type-theoretic function from a set A to a set B is identified with an element of the set $A \to B$

$$f \in A \to B$$

We recall that canonical terms of the set $A \to B$ are of the form $\lambda x.t(x)$ obtained by λ-abstraction from terms of the form $t(x) \in B\ [x \in A]$.

Note here that while the collection of functional relations from a set A to a set B does not need to form a set, instead by definition the type-theoretic functions from a set A to a set B form a set!
Given that type-theoretic functions are meant to be computational as in Feferman's theories [Fef79], it makes sense to formulate the formal Church thesis only for them, namely to say that all type theoretic functions from natural numbers to natural numbers are internally recursive: [3]

$$(\mathrm{CT}_{\mathrm{tt}})\qquad \forall f \in \mathrm{N} \to \mathrm{N}\quad \exists e \in \mathrm{N}\quad (\,\forall x \in \mathrm{N}\ \exists y \in \mathrm{N}\ T(e,x,y)\ \wedge\ U(y) =_{\mathrm{N}} f(x)\,)$$

Note that the notion of functional relation is more general than that of operation from A to B, unless we can derive the *axiom of unique choice on natural numbers*, formulated in emTT_0 as follows

$$(\ \mathrm{AC!}_{\mathrm{N,N}}\ )\qquad \forall x \in \mathrm{N}\ \exists! y \in \mathrm{N}\ R(x,y)\quad \longrightarrow\quad \exists f \in \mathrm{N} \to \mathrm{N}\ \forall x \in \mathrm{N}\ R(x, f(x))$$

The absence of unique choice was exactly one of the key features desired for our minimalist foundation as explained in [MS05]. The main motivation was to be able to identify the notion of *choice sequence* with that of functional relation, as done in the context of axiomatic set theory, and that of *lawlike sequence* with that of type-theoretic function (see [Sam08] for the relevance of this distinction).

To express this we identify the tree with nodes labelled by lists of elements in a set A with $A^* \equiv List(A)$ itself:

Def. 3.1 (choice sequence) Given a set A, a *choice sequence* on the tree A^* is a functional relation from N to A given by a small proposition $\alpha(x,y)\ prop_s\ [x \in \mathrm{N}, y \in A]$ in emTT_0.
We write $\alpha \varepsilon \mathrm{CH}(A)$ to mean that α is a choice sequence.

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[2] As usual $\exists! y \in \mathrm{N}\ R(x,y) \equiv \exists y \in B\ R(x,y)\ \wedge\ \forall y_1, y_2 \in B\ (\ R(x,y_1)\ \wedge\ R(x,y_1) \to \mathsf{Eq}(B, y_1, y_2)\ )$.

[3] As usual here $T(e,x,y)$ is the Kleene predicate expressing that y is the computation executed by the program numbered e on the input x and $U(y)$ is output of the computation y. Of course N is the set of natural numbers.

Def. 3.2 (lawlike sequence) Given a set A, a *lawlike sequence* on the tree A^* is a type-theoretic function $f \in \mathrm{N} \to A$ in emTT_0 from natural numbers to A.

Remark 3.3 Note that in [Tv88a] the notion of choice sequence is identified with that of type-theoretic function $f : \mathrm{N} \to \mathrm{N}$ while in [Rat05] with that of functional relation.

Now we formulate the principle of Bar Induction in topological terms by employing an inductively generated formal topology put on the tree A^* (see [Sam10]). In the next in emTT_0 we use the notion of subset with its ε-relation as in [Mai09]: a subset V of A^* is a term $V \in \mathcal{P}(A^*)$, where $\mathcal{P}(A^*) \equiv A^* \to \mathcal{P}(1)$, and $l \varepsilon V \equiv \mathsf{Eq}(\,\mathcal{P}(1),\, V(l),\, [\mathsf{tt}]\,)$ for a list $l \in A^*$.

Def. 3.4 *The tree formal topology over A* is the formal topology $(A^*, \lhd_{A^N})$ where $\lhd_{A^N}$ is inductively generated by the following rules (see [CSSV03])

$$\text{rfl}\ \frac{a \varepsilon V}{a \lhd_{A^N} V} \qquad \leq\ \frac{s \sqsubseteq^{op} l \qquad l \lhd_{A^N} V}{s \lhd_{A^N} V} \qquad \text{tr}\ \frac{\forall x \in A\ \ \mathsf{cons}(l, x) \lhd_{A^N} V}{l \lhd_{A^N} V}$$

where $s \sqsubseteq^{op} l \equiv \exists_{t \in A^*}\, s =_{A^*} [l, t]$, i.e. l is an initial segment of s.

The above tree topology is called *Cantor formal topology* when A is the boolean set $\{0, 1\}$ and we indicate its cover with $\lhd_C \equiv \lhd_{\{0,1\}^N}$. Moreover, it is called *Baire formal topology* when A is the set of natural numbers N and we indicate its cover with $\lhd_B \equiv \lhd_{N^N}$.

We just recall that a subset V of A^* is called a formal open of the formal topology $\lhd_{A^N}$ if $V = \lhd_{A^N}(V) \in \mathcal{P}(A^*)$ holds, where in turn $\lhd_{A^N}(V) \equiv \{\ l \in A^* \ \mid\ l \lhd_{A^N} V\ \}$.

Then the frame associated to the formal topology $\lhd_{A^N}$ is represented by the formal opens of $\lhd_{A^N}$, with inclusion as order. The intersection of two opens V, W in the frame is given by $\lhd_{A^N}(V \downarrow W)$ where $V \downarrow W \equiv \{\ s \in A^* \ \mid\ \exists l \varepsilon V\ s \sqsubseteq^{op} l\ \wedge\ \exists w \varepsilon W\ s \sqsubseteq^{op} w\ \}$. The arbitrary supremum of a family of opens $V_{i \in I}$ in the frame is given by $\lhd_{A^N}(\,\bigcup_{i \in I} V_i\,)$ where $\bigcup_{i \in I} V_i \equiv \{\ s \in A^* \ \mid\ \exists i \in I\ s \varepsilon V_i\ \}$.

In [Val07] it is shown how to build the above tree formal topologies in an extension of Martin-Löf's type theory with the help of the *intensional axiom of choice*. This axiom of choice is valid in the quotient built over Martin-Löf's type theory as that over mTT in [Mai09], and hence it does not entail classical logic as the full extensional one (see [ML06, Car04]). Given that we do not have such a choice principle in emTT_0 we *simply postulate the existence of tree formal topologies* as added axioms to emTT_0. We indicate this extension with $\mathrm{emTT}_0 + \lhd_{A^N}$.

Before proceeding we define a useful notation introduced in [Sam03]:

Def. 3.5 ($\between$-relation) Given a set A and subsets V, W of A^* we define

$$V \between W \ \equiv\ \exists l \in A^*\ (\, l \varepsilon V\ \wedge\ l \varepsilon W\,)$$

We then recall the notion of a formal point for the tree formal topologies of the form $\lhd_{A^N}$:

Def. 3.6 (formal point of $\lhd_{A^N}$) A subset α of A^* for a given set A is a *formal point*, written $\alpha \varepsilon Pt(\lhd_{A^N})$, if it satisfies the following conditions:

$$\begin{array}{l}
\exists_{l \in A^*}\ l \varepsilon \alpha \\
\forall_{l_1, l_2 \in A^*}\ (\, l_1 \varepsilon \alpha \wedge l_2 \varepsilon \alpha\ \to\ \exists_{s \in A^*}\ (\, s \varepsilon \alpha \wedge s \sqsubseteq^{op} l_1\ \wedge\ s \sqsubseteq^{op} l_2\,)\ \,) \\
\forall_{s \in A^*}\ (\, s \varepsilon \alpha\ \to\ (\forall_{l \in A^*}\ s \sqsubseteq^{op} l\ \to\ l \varepsilon \alpha\,)\ \,) \\
\forall_{l \varepsilon A^*}\ (\, l \varepsilon \alpha\ \to\ \exists_{a \in A}\ \mathsf{cons}(l, a) \varepsilon \alpha\,)
\end{array}$$

In the next, we use the following abbreviations to quantify over formal points: for any formula $\phi(\alpha)$
$\forall\, \alpha \varepsilon Pt(\lhd_{A^N})\ \ \phi(\alpha)\ \equiv\ \forall\, \alpha \in A^* \to \mathcal{P}(1)\ (\, \alpha \varepsilon Pt(\lhd_{A^N})\ \to\ \phi(\alpha)\,)$
$\exists\, \alpha \varepsilon Pt(\lhd_{A^N})\ \ \phi(\alpha)\ \equiv\ \exists\, \alpha \in A^* \to \mathcal{P}(1)\ (\, \alpha \varepsilon Pt(\lhd_{A^N})\ \wedge\ \phi(\alpha)\,)$.

Now note that choice sequences on the tree A^* are exactly the formal points of the tree formal topology over A:

Proposition 3.7 *The collection of formal points $Pt(\triangleleft_{A^N})$ of the tree formal topology over a set A are in bijection with the choice sequences on the tree A^*.*

Proof Given a formal point α, we can define a functional relation as follows:

$$\alpha_{fr}(n,a) \;\equiv\; \exists\; l \varepsilon \alpha \;\; \mathsf{Eq}(A, l_{n+1}, a)$$

where l_n is the n-th component of l.

Conversely, given a functional relation $\alpha(x,y)$ *props* $[x \in \mathrm{N}, y \in A]$ the following subset

$$\alpha_{pt} \;\equiv\; \{\, l \in A^* \;\mid\; \forall n \in \mathrm{N}\; (\, 1 \leq n \leq \mathsf{lh}(l) \;\rightarrow\; \alpha(n, l_{n+1})\,)\}$$

where $\mathsf{lh}(l)$ is the length of l, turns out to be a formal point.

An alternative proof follows after noting, as observed in [Sig95], that any tree formal topology is the exponential formal topology of the discrete formal topology of natural numbers on itself (see [Mai05b] for a constructive and predicative construction of exponentiation). Therefore its formal points are in bijection with functional relations, being these all continuous. This explains why we label the cover $\triangleleft_{A^N}$ of the tree formal topology over A with A^N.

Then, we are ready to formulate Bar Induction as *spatiality of the tree formal topology on a given set* A similarly to [FG82]:

Def. 3.8 (Bar Induction in topological form) In $\mathrm{emTT}_0 + \triangleleft_{A^N}$ the principle of Bar Induction in topological form is the following statement: for any given set A in emTT_0

$$(\ \mathrm{BI}_{\mathrm{fr}}(\mathrm{A})\ ) \qquad \forall l \in A^*\; \forall V \in \mathcal{P}(A^*)\;\; (\, \forall \alpha \varepsilon Pt(\triangleleft_{A^N})\;\; (\, l \varepsilon \alpha \;\rightarrow\; \alpha \between V\;) \rightarrow l \triangleleft_{A^N} V\;)$$

This formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ essentially means that the topology put on the formal points of the tree A^*, that are its choice sequences, coincides with the point-free one and hence we can reason on it by induction being the point-free one inductively generated (see [Sam08, Sam10]).

We give specific names to Bar Induction on the Baire formal topology and on Cantor formal topology:

Def. 3.9 (Bar Induction on Baire and Cantor formal topologies) We call $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ the above formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ on Baire formal topology, namely when $A \equiv \mathrm{N}$.

We call $\mathrm{BI}_{\mathrm{fr}}(\{\,0,1\,\})$ the above formulation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ on Cantor formal topology, namely when $A \equiv \{\,0,1\,\}$.

Note that spatiality of Cantor formal topology allows to derive compactness of Cantor space [FG82]. In the rest of the paper we just say $\mathrm{BI}_{\mathrm{fr}}$ to mean $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ for any given set A and $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}}$ to mean $\mathrm{emTT}_0 + \triangleleft_{A^N} + \mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ for any given set A.

3.2 Connection of $\mathrm{BI}_{\mathrm{fr}}$ with traditional formulations

Here, we review the connection of our topological formulation of Bar Induction with more traditional formulations of it and with Fan theorem. We want to make this clear in order to derive an inconsistency of $\mathrm{CT}_{\mathrm{tt}}$ with $\mathrm{AC!}_{\mathrm{N,N}}$ and $\mathrm{BI}_{\mathrm{fr}}$ in emTT_0 from Kleene's proof in [Dum00, Tv88a, Rat05] about inconsistency of the Fan theorem with Church thesis for functional relations.

We start with defining the notion of bar of a list:

Def. 3.10 (bar of a list) Given a set A, a *bar of a list* l on the tree A^* is a subset V of A^* satisfying

$$\forall \alpha \varepsilon CH(A)\;\; (\, \alpha \between \{\, l\,\} \;\rightarrow\; \alpha \between V\,)$$

We then say that

- V is *monotone* if $\forall l \in A^* \ (\ l \varepsilon V \ \rightarrow \ \forall a \in A \ \mathsf{cons}(l,a) \varepsilon V \ )$ holds.

- V is *inductive* if $\forall l \in A^* \ (\ \forall a \in A \ \mathsf{cons}(l,a) \varepsilon V \ \rightarrow \ l \varepsilon V \ )$ holds.

Now we are ready to give the traditional formulation of Monotone Bar Induction as in [Dum00, Tv88a, Rat05] for $A \equiv \mathrm{N}$ and here extended also when $A \equiv \{0,1\}$:

Def. 3.11 (traditional Bar Induction) For $A \equiv \mathrm{N}$ or $A \equiv \{0,1\}$ the principle of *Bar induction* BI^{tr}_A says that every inductive subset Q of A^* containing a monotone bar V of the empty list contains the empty list:

$$
\begin{array}{lll}
(\mathrm{BI}^{tr}_A) \qquad & \forall V, Q \in \mathcal{P}(A^*) \quad & (\ \forall \alpha \varepsilon \mathrm{CH}(A) \ \ \alpha \between V \\
 & & \wedge \ \ \forall l \in A^* \ (\ \ l \varepsilon V \ \rightarrow \ \ \forall a \in A \ \mathsf{cons}(l,a) \varepsilon V \ ) \\
 & & \wedge \ \ \forall l \in A^* \ (\ \forall a \in A \ \mathsf{cons}(l,a) \varepsilon Q \ \ \rightarrow \ \ l \varepsilon Q \ ) \\
 & & \wedge \ \ \forall l \in A^* \ (\ l \varepsilon V \ \rightarrow \ l \varepsilon Q \ ) \ ) \\
 & & \qquad \qquad \rightarrow \ \mathsf{nil} \varepsilon Q
\end{array}
$$

holds, where nil is the empty list.

In order to see the connection between the traditional formulation of Bar Induction and our topological form it is convenient to note that monotone inductive subsets of lists over a set A are in bijection with formal opens of the tree formal topology over a set A:

Lemma 3.12 *A subset V of A^* is monotone and inductive if and only if V is a formal open in the tree formal topology over A, i.e. we can derive a proof of*

$$\mathsf{Eq}(\ \mathcal{P}(A^*)\,,\ V\,,\ \lhd_{A^N}(V)\ )$$

Proof. Given a monotone inductive subset, we can prove by induction that $\lhd_{A^N}(V) \subseteq V$ and the other inclusion is obvious. The converse is trivial.

This lemma suggests a reformulation of BI^{tr}_A in terms of monotone inductive bars:

Def. 3.13 The principle of *monotone bar induction* $\mathrm{MBI}^{\mathsf{nil}}_A$ says that, for a given set A, every monotone inductive bar of the empty list in A^* contains the empty list (and hence by monotonicity every list):

$$
\begin{array}{ll}
(\mathrm{MBI}^{\mathsf{nil}}_A) & \text{for all } \textit{monotone inductive} \text{ subset } V \text{ of } A^* \\
 & \qquad \qquad \qquad \forall \alpha \varepsilon \mathrm{CH}(A) \ \ \alpha \between V \ \ \rightarrow \ \mathsf{nil} \varepsilon V
\end{array}
$$

We also give the following more general definition of monotone bar induction:

Def. 3.14 The general principle of *monotone bar induction* MBI_A says that, for a given set A, every monotone inductive bar of a list l in A^* contains the list l:

$$
\begin{array}{ll}
(\mathrm{MBI}_A) & \text{for all } \textit{monotone inductive} \text{ subset } V \text{ of } A^* \\
 & \qquad \qquad \forall l \in A^* \quad (\ \forall \alpha \varepsilon \mathrm{CH}(A) \ \ (\ \alpha \between \{\, l \,\} \rightarrow \alpha \between V\ ) \ \rightarrow \ l \varepsilon V\ )
\end{array}
$$

Now we show that all the above formulations of Bar Induction are equivalent when applied to Baire and Cantor formal topologies:

Theorem 3.15 *In* $\mathrm{emTT}_0 + \lhd_{A^N}$, *for* $A \equiv N$ *or* $A \equiv \{0,1\}$ *the following are equivalent:*

1. $\mathrm{BI}_{\mathrm{fr}}(A)$
2. MBI_A
3. $\mathrm{MBI}^{\mathsf{nil}}_A$
4. BI^{tr}_A

Proof. $1 \leftrightarrow 2$ Clearly $\mathrm{BI}_{\mathrm{fr}}(A)$ is equivalent to MBI_A because of lemma 3.12.
$2 \leftrightarrow 3$ To prove that $\mathrm{MBI}_A^{\mathsf{nil}}$ entails MBI_A note that given a monotone inductive bar V for l then

$$W \equiv V \bigcup \{ s \in \mathrm{N}^* \ \mid \ l \neq s\,^4 \ \wedge \ \mathsf{Eq}(\mathrm{N}, \mathsf{lh}(s)\,,\, \mathsf{lh}(l)) \ \}$$

is a bar of the empty list. Therefore, from lemma 3.12 we get that $\lhd_{A^N}(W)$ is a monotonic inductive bar of the empty list and from $\mathrm{MBI}_A^{\mathsf{nil}}$ we obtain that $\mathsf{nil}\varepsilon\lhd_{A^N}(W)$ and by monotonicity also that $l\varepsilon\lhd_{A^N}(W)$, i.e. $l \lhd_{A^N} W$. Now, by intersecting the open $\lhd_{A^N}(W)$ with the open generated from $\{\, l\,\}$ we get the open $\lhd_{A^N}(\{\, l\,\} \downarrow W)$ with $l \lhd_{A^N} \{\, l\,\} \downarrow W$. Now observe that $\{\, l\,\} \downarrow W \subseteq V$ being V monotone. Hence, by transitivity of $\lhd_{A^N}$ as a formal cover we conclude $l \lhd_{A^N} V$. Being V a monotone inductive bar, by lemma 3.12 we conclude $l\varepsilon V$.
$3 \leftrightarrow 4$ $\mathrm{MBI}_A^{\mathsf{nil}}$ entails BI_A^{tr}, as shown in [Sam10], if we consider the minimum inductive subset containing a given monotone bar V. This is monotone and it is contained in Q, and hence it happens to coincide with $\lhd_{A^N}(V)$. By $\mathrm{MBI}_A^{\mathsf{nil}}$ we get that $\mathsf{nil}\varepsilon \lhd_{A^N}(V)$ and hence we conclude $\mathsf{nil}\varepsilon Q$.
Conversely, BI_A^{tr} implies $\mathrm{MBI}_A^{\mathsf{nil}}$ trivially by taking $Q \equiv V$ for a given monotone inductive bar V.

A consequence of Bar Induction on the tree N^*, namely of $\mathrm{BI}_{\mathrm{N}}^{tr}$, is the well known Fan theorem regarding choice sequences from N to the boolean set (see [Dum00, Tv88a, Rat05]):

Def. 3.16 (traditional Fan theorem) The traditional formulation of Fan theorem, called here $\mathrm{FT}^{\mathsf{nil}}$, says that every bar V of the empty list in $\{0,1\}^*$ is uniform, namely there exists a subset of V, which is still a bar of the empty list, with lists bounded by a fixed natural number:

$$(\mathrm{FT}^{\mathsf{nil}}) \ \forall V \in \mathcal{P}(\{\,0,1\,\}) \ \ (\ \forall \alpha\varepsilon\mathrm{CH}(\{\,0,1\,\}) \ \ \alpha \between V \ \ \rightarrow \ \ \ \exists n \in \mathrm{N} \ \forall \alpha\varepsilon\mathrm{CH}(\{\,0,1\,\}) \ \alpha \between V_n \ )$$

where $V_n \equiv \{ \ v \in \{\,0,1\,\}^* \ \mid \ v\varepsilon V \ \wedge \ \mathsf{lh}(v) \leq n \ \}$.

We can extend the formulation of the Fan theorem to bars of a generic list:

Def. 3.17 (Fan theorem with bars of a generic list) The more general formulation of Fan theorem, called here FT, says that every bar V of a list l in $\{0,1\}^*$ is uniform, namely there exists a subset of V, which is still a bar of the list l, with lists bounded by a fixed natural number:

$$(\mathrm{FT}) \begin{array}{l} \forall l \in A^* \ \forall V \in \mathcal{P}(\{\,0,1\,\}) \\ \qquad (\ \forall \alpha\varepsilon\mathrm{CH}(\{\,0,1\,\}) \ (\ \alpha \between \{\, l\,\} \rightarrow \alpha \between V \ ) \ \rightarrow \ \ \exists n \in \mathrm{N} \ \forall \alpha\varepsilon\mathrm{CH}(\{\,0,1\,\}) \ \ (\ \alpha \between \{\, l\,\} \rightarrow \alpha \between V_n \ ) \ ) \end{array}$$

But with this formulation of Fan theorem on generic lists we do not get a stronger statement than the traditional one and more importantly this is also equivalent to Bar Induction on the Cantor formal topology $\mathrm{BI}_{\mathrm{fr}}(\{\,0,1\,\})$:

Theorem 3.18 *In* $\mathrm{emTT}_0 + \lhd_{A^N}$ *the following are equivalent*

1. $\mathrm{BI}_{\mathrm{fr}}(\{\,0,1\,\})$
2. FT
3. $\mathrm{FT}^{\mathsf{nil}}$

Proof. $1 \leftrightarrow 2$ The proof is given in [GS07] and it can be easily carried out in emTT_0 being based on induction over the generation of Cantor formal topology.
$2 \leftrightarrow 3$ To prove that $\mathrm{FT}^{\mathsf{nil}}$ entail FT note that given a bar V for l then

$$W \equiv V \bigcup \{ s \in \{\,0,1\,\}^* \ \mid \ l \neq s \ \wedge \ \mathsf{Eq}(\mathrm{N}, \mathsf{lh}(s), \mathsf{lh}(l)) \ \}$$

---

[4] The equality on N^* or $\{\,0,1\,\}^*$ is decidable being decidable that on N and on $\{\,0,1\,\}$.

is a bar of the empty list. Hence, by $\mathrm{FT}^{\mathsf{nil}}$ there exists a natural numbers n such that W_n is a bar of the empty list, and hence V_n is a bar of l.

We can also show that Bar Induction on the Cantor formal topology, or equivalently the Fan theorem, is a consequence of Bar Induction on the Baire formal topology:

Proposition 3.19 $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ *entails* FT *in* emTT_0.

Proof. Thanks to theorem 3.18 we just show that $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ entails $\mathrm{BI}_{\mathrm{fr}}(\{0,1\})$. As suggested to us by T. Streicher this follows from the fact that Cantor formal topology is a retract of Baire formal topology, i.e. that there exist morphisms $\mathcal{E} : \lhd_C \to \lhd_B$, $\mathcal{R} : \lhd_B \to \lhd_C$ such that $\mathcal{R} \cdot \mathcal{E} = \mathsf{id}_{\lhd_C}$ in the category of inductively generated formal topologies (the definition of such a category can be found in [Mai05b] and in loc. cit.). In particular, the existence of $\mathcal{R}$ is in turn based on a retraction $\sigma : \mathrm{N}^* \to \{0,1\}^*$ of the embedding $i : \{0,1\}^* \to \mathrm{N}^*$ of boolean lists into lists of natural numbers, where $\sigma \equiv \mathit{List}(\widetilde{\sigma})$ is the lifting of the operation $\widetilde{\sigma}(x) \in \{0,1\}$ $[x \in \mathrm{N}]$ defined as follows:

$$\widetilde{\sigma}(x) \equiv \begin{cases} 0 & \text{if } x \equiv 0 \\ 1 & \text{otherwise} \end{cases}$$

Indeed, we can define $\mathcal{E} : \lhd_C \to \lhd_B$ and $\mathcal{R} : \lhd_B \to \lhd_C$ as follows: given $s \in \mathrm{N}^*$ and $l \in \{0,1\}^*$

$$l\,\mathcal{E}\,s \equiv l \lhd_C \{\, x \in \{0,1\}^* \mid \mathsf{Eq}(\mathrm{N}^*, x, s) \,\} \qquad\qquad s\,\mathcal{R}\,l \equiv \sigma(s) \lhd_C l$$

where for easiness we just consider a list $l \in \{0,1\}^*$ also as a list in N^*. (We just recall that, for any formal open V of $\lhd_B$, then $\mathcal{E}^-(V)$ is a formal open of $\lhd_C$ and this gives rise to a frame morphism from the Baire frame to the Cantor one. Similarly $\mathcal{R}^-$ gives rise to a frame morphism from the Cantor frame to the Baire one.)
Then, one derives $\mathrm{BI}_{\mathrm{fr}}(\{0,1\})$ from $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ by using $\mathcal{E}$ and $\mathcal{R}$. The essence is that any bar V of a list l in the Cantor formal topology yields to a bar $\mathcal{R}^-(V) \equiv \{\, s \in \mathrm{N}^* \mid \exists v \varepsilon V\, s\,\mathcal{R}\,v \,\}$ for the list l in the Baire formal topology. Then by $\mathrm{BI}_{\mathrm{fr}}(\mathrm{N})$ we get $l \lhd_B \mathcal{R}^-(V)$ and hence also that $\mathcal{E}^-(l) \lhd_C \mathcal{E}^-(\mathcal{R}^-(V))$. From this and from $l \varepsilon \mathcal{E}^-(l)$ and $\mathcal{R} \cdot \mathcal{E} = \mathsf{id}_{\lhd_C}$ we conclude $l \lhd_C V$. Hence $\mathrm{BI}_{\mathrm{fr}}(\{0,1\})$ holds, as claimed.

3.3 Inconsistency of $\mathbf{AC!}_{\mathbf{N},\mathbf{N}}$+ $\mathbf{CT}_{\mathrm{tt}}$+ $\mathbf{BI}_{\mathbf{fr}}$

Here we reread in our foundation Kleene's well known result that the Fan theorem is inconsistent with the formal Church thesis saying that all choice sequences are recursive. We found at least two ways on which choice sequences are defined in the literature. Some authors, like [Tv88a], identify choice sequences with type-theoretic functions and hence their intended formal Church thesis to get Kleene's result coincides with our $\mathrm{CT}_{\mathrm{tt}}$. Since in the presence of unique choice our notion of choice sequences coincide with that in [Tv88a] Kleene's result in our setting amounts to say that our Fan theorem together with $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ is inconsistent with $\mathrm{CT}_{\mathrm{tt}}$.

Others authors identify choice sequences with functional relations as in [Rat05] and their intended formal Church thesis to get Kleene's result is then a consequence of combining our Church thesis $\mathrm{CT}_{\mathrm{tt}}$ for type-theoretic functions with the axiom of unique choice. Also in this case Kleene's result amounts to inconsistency of the Fan theorem together with $\mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N},\mathrm{N}}$.

Therefore we deduce for our foundation:

Proposition 3.20 *There is no model of* $\mathrm{emTT}_0 + \mathrm{FT} + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N},\mathrm{N}}$.

Proof. Mimick Kleene's proof done in arithmetics on finite types in [Tv88a].

Thanks to propositions 3.18,3.19 we then conclude:

Corollary 3.21 $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ *is inconsistent.*
Hence, $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ *is inconsistent, too, where we recall that* $\mathrm{BI}_{\mathrm{fr}}$ *means* $\mathrm{BI}_{\mathrm{fr}}(A)$ *for all set* A.

These inconsistency statements provide a rereading of Kleene's result as follows: *there is no way to identify all choice sequences*, defined as functional relations between natural numbers, *with lawlike ones*, defined as terms of type N → N, *if these are also internally recursive* (as stated in our formal Church thesis).

Then, it comes natural to ask whether without unique choice $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ our emTT_0 turns out to be consistent with $\mathrm{BI}_{\mathrm{fr}}$ and $\mathrm{CT}_{\mathrm{tt}}$. This is what we are going to show in the next. As a byproduct we will conclude that unique choice on natural numbers does not generally holds in emTT_0.

3.4 Formulation of $\mathrm{CT}_{\mathrm{tt}}$, $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ and $\mathrm{BI}_{\mathrm{fr}}$ at the intensional level

Now, we describe the interpretation of $\mathrm{CT}_{\mathrm{tt}}$ and $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ and $\mathrm{BI}_{\mathrm{fr}}$ at the intensional level mTT_0. The translations of $\mathrm{CT}_{\mathrm{tt}}$ and $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ are essentially the identity while that of $\mathrm{BI}_{\mathrm{fr}}$ is not because we need to represent the power collection of subsets as a suitable quotient (see [Mai09]).
We start by giving the definition of tree formal topology on a *setoid* or *extensional set* $(A, =_A)$ in mTT_0, namely on an mTT_0-set A equipped with an equivalence relation

$$x =_A y \; prop_s \; [x \in A, y \in A]$$

as in [Mai09]. The following formulation is obtained by translating in mTT_0 the notion of tree formal topology of emTT_0 by using the interpretation $(-)^i$ in definition 2.1:

Def. 3.22 (tree formal topology in mTT_0) A tree formal topology in mTT_0 on a *setoid* $(A, =_A)$ in mTT_0 consists of a proposition

$$l \lhd^i_{A^N} V \;\mathsf{prop_s}\; [l \in A^*, V \in A^* \to \mathsf{prop_s}]$$

with a proof of the proposition

$$l_1 \lhd^i_{A^N} V \leftrightarrow l_2 \lhd^i_{A^N} W \;\mathsf{prop_s} \quad \begin{array}{r} [l_1, l_2 \in A^*,\; V, W \in A^* \to \mathsf{prop_s},\; u \in l_1 =_{A^*} l_2 \\ z \in \forall_{x \in A^*}\; V(x) \leftrightarrow W(x)] \end{array}$$

where the equality on lists $=_{A^*}$ is defined from $=_A$ as in [Mai09], and $\lhd^i_{A^N}$ is inductively generated (see [CSSV03]) from the rules rfl, ≤, tr in def. 3.4 written as axioms in the implicative form with corresponding proof-terms.

Before giving the interpretation of $\mathrm{BI}_{\mathrm{fr}}$, $\mathrm{CT}_{\mathrm{tt}}$ and $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ in mTT_0, we need to extend the interpretation $(-)^i$ of emTT_0 into mTT_0 to include the existence of tree formal topologies in both theories:

$$(-)^i : \mathrm{emTT}_0 \; + \lhd_{A^N} \; \to \mathrm{mTT}_0 \; + \lhd^i_{A^{iN}}$$

by simply interpreting each $\lhd_{A^N}$ as $\lhd^i_{A^{iN}}$ supposing $(A^i =_{A^i})$ the setoid interpretation of the emTT_0-set A. Hence, we are ready to prove

Lemma 3.23 *According to the interpretation* $(-)^i : \mathrm{emTT}_0 \; + \lhd_{A^N} \to \mathrm{mTT}_0 \; + \lhd^i_{A^N}$ *just mentioned, supposing* $(A^i, =_{A^i})$ *the setoid interpretation in* mTT_0 *of the* emTT_0*-set A, then*
- The translation of $\mathrm{BI}_{\mathrm{fr}}(\mathrm{A})$ *for an* emTT_0*-set A in* $\mathrm{mTT}_0 + \lhd^i_{A^{iN}}$ *is the following:*

$$(\mathrm{BI}_{\mathrm{fr}}(A^i)) \qquad \begin{array}{l} \forall\, l \in List(A^i) \;\; \forall\, V \in List(A^i) \to \mathsf{prop_s} \\ \quad (\; \forall \alpha \in List(A^i) \to \mathsf{prop_s} \;\; (\; \alpha \varepsilon Pt(\lhd^i_{A^{iN}}) \;\wedge\; \alpha(l) \;\to\; \alpha \between V \;) \to l \lhd^i_{A^{iN}} V \;) \end{array}$$

where $\alpha \between V$ *and* $\alpha \varepsilon Pt(\lhd^i_{A^{iN}})$ *are defined as in* emTT_0 *in definition 3.8.*
- The translation of $\mathrm{CT}_{\mathrm{tt}}$ *and* $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$*, called* $\mathrm{CT}^i_{\mathrm{tt}}$ *and* $\mathrm{AC!}^i_{\mathrm{N},\mathrm{N}}$ *are essentially the same as* $\mathrm{CT}_{\mathrm{tt}}$ *and* $\mathrm{AC!}_{\mathrm{N},\mathrm{N}}$ [5].

---

[5] Note that, according to the interpretation $(-)^i$ based on that in [Mai09], the emTT_0-set of natural numbers is interpreted as the mTT_0-set of natural numbers N equipped with the propositional equality of N. Hence, the support of the interpretation of the emTT_0-set N → N turns out to be the mTT_0-set N → N itself because all mTT_0-functions between natural numbers preserve the propositional equality on N.

Thanks to the interpretation of emTT_0 into mTT_0 and to prop. 3.21 we get:

Corollary 3.24
$\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$ *is inconsistent, and hence* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$, *where* $\mathrm{BI}^i_{\mathrm{fr}}$ *means* $\mathrm{BI}^i_{\mathrm{fr}}(A)$ *for all* mTT_0*-set* A *with an equivalence relation* $=_A$*, is inconsistent, too.*

Proof. Thanks to the interpretation $(-)^i$ a proof that falsum is true in $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N,N}}$ converts to the construction of a proof-term for falsum in $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}}(\mathrm{N}) + \mathrm{CT}^i_{\mathrm{tt}} + \mathrm{AC!}^i_{\mathrm{N,N}}$.

Corollary 3.25 *If* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *is consistent, then*

- mTT_0 *does not validate* $\mathrm{AC!}^i_{\mathrm{N,N}}$*;*
- $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}}$ *is consistent;*
- emTT_0 *does not validate unique choice on natural numbers* $\mathrm{AC!}_{\mathrm{N,N}}$.

4 The intermediate level mTT_0^{eq}

In building a realizability interpretation for mTT_0 validating $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ we encountered some technical obstacles in interpreting the indexed sum elimination constructor on proper collections. We can solve such difficulties if we adopt projections as indexed sum elimination constructors. But, in an intensional type theory as mTT_0, adopting projections as indexed sum elimination constructors does not seem to be equivalent to adopting the current elimination constructor $El_\Sigma(d,m)$. This is instead so if we replace the intensional propositional equality $\mathsf{Id}(A,a,b)$ with the extensional propositional equality $\mathsf{Eq}(A,a,b)$ as in [Mar84]. Therefore we give our realizability interpretation for an extension of mTT_0, called mTT_0^{eq}, where *the propositional equality* $\mathsf{Id}(A,a,b)$ *is replaced by the stronger extensional one* $\mathsf{Eq}(A,a,b)$ whose rules are the following

Extensional Propositional Equality

$$\text{F-Eq)}\ \frac{C\ \ col\quad c\in C\quad d\in C}{\mathsf{Eq}(C,c,d)\ prop}\qquad\qquad \text{I-Eq)}\ \frac{c\in C}{\mathsf{eq}\in\mathsf{Eq}(C,c,c)}$$

$$\text{E-Eq)}\ \frac{p\in\mathsf{Eq}(C,c,d)}{c=d\in C}\qquad\qquad \text{C-Eq)}\ \frac{p\in\mathsf{Eq}(C,c,d)}{p=\mathsf{eq}(c)\in\mathsf{Eq}(C,c,d)}$$

and we adopt projections as indexed sum elimination constructors both on collections and on sets together with β and η-conversion rules as follows:

Strong Indexed Sum elimination and conversion rules

$$\text{E}_1\text{-}\Sigma)\ \frac{d\in\Sigma_{x\in B}C(x)}{\pi_1(d)\in B}\qquad\qquad \text{E}_2\text{-}\Sigma)\ \frac{d\in\Sigma_{x\in B}C(x)}{\pi_2(d)\in C(\pi_1(d))}$$

$$\text{C}_1\text{-}\Sigma)\ \frac{b\in B\quad c\in C(b)}{\pi_1(\,\langle b,c\rangle\,)=b\in B}\qquad\qquad \text{C}_2\text{-}\Sigma)\ \frac{b\in B\quad c\in C(b)}{\pi_2(\,\langle b,c\rangle\,)=c\in C(b)}$$

$$\eta\text{-}\Sigma)\ \frac{d\in\Sigma_{x\in B}C(x)}{\langle\pi_1(\,d\,)\,,\,\pi_2(\,d\,)\,\rangle=d\in\Sigma_{x\in B}C(x)}$$

Luckily, the realizability interpretation we intend to build for mTT_0 validates the rules of mTT_0^{eq}. Indeed, this interpretation is based on Kleene's realizability interpretation for a version of Martin-Löf's type theory in [Tv88b], which was already known to validate the rules of $\mathsf{Eq}(A,a,b)$ in the absence of ξ-rule for λ-terms and in the presence of substitution rules in place of usual equality rules in [NPS90] (see [Mar75]).

Hence, we can interpret mTT_0 into mTT_0^{eq}, both extended with $\mathrm{BI}^i_{\mathrm{fr}}$ + $\mathrm{CT}^i_{\mathrm{tt}}$ as follows:

Proposition 4.1 *We can interpret* $\mathrm{mTT}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *into* $\mathrm{mTT}^{eq}_0 + \mathrm{BI}^i_{\mathrm{fr}} + \mathrm{CT}^i_{\mathrm{tt}}$ *as the identity on all constructors except for those of the propositional equality* Id *which are interpreted as those of the extensional one* Eq*, and for the indexed sum elimination constructor which is interpreted via projections.*

Proof. We interpret the indexed sum elimination constructor of mTT_0 in mTT^{eq}_0 as follows: given $d \in \Sigma_{x\in B} C(x)\ [\Gamma]$ and $m(x,y) \in M(\langle x,y\rangle)\ [\Gamma, x\in B, y \in C(x)]$ then

$$El_\Sigma(d,m) \equiv m(\pi_1(d), \pi_2(d))$$

that is of type $M(\langle \pi_1(d), \pi_2(d)\rangle)$ by definition. But by the substitution rules and the rule conv) [6] (see the rules of mTT in [Mai09]) and the above η-Σ) of mTT^{eq}_0 we conclude that it is of type $M(d)$ as well, as required.

Note that indexed sum projections can be defined as follows from the original indexed elimination constructor El_Σ:

$$\pi_1(z) \equiv El_\Sigma(z, (x,y).x) \qquad \pi_2(z) \equiv El_\Sigma(z, (x,y).y)$$

By the original conversion rule of Σ, they clearly satisfy C_1-Σ) and C_2-Σ) conversions. To validate η-Σ) we need to use the elimination on Eq (see [NPS90]) and the fact that inhabitation of $\mathsf{Eq}(C,c,d)$ yields that $c = d \in C$ holds also under a context.

Concerning the propositional equality: the constructor $\mathsf{id}_A(a)$ of mTT_0 is interpreted as eq of mTT^{eq}_0 and the elimination constructor $El_{\mathsf{Id}}(p, (x)c(x))$ as $c(a)$, given that its type $C(a,a,\mathsf{eq})$ happens to be equal to $C(a,b,p)$ by the rule conv) in [Mai09] since from $p \in \mathsf{Eq}(A,a,b)$ we get $a = b \in A$ and also $p = \mathsf{eq} \in \mathsf{Eq}(A,a,b)$ by the rules of Eq.

5 The model of mTT^{eq}_0 with $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$

Here we describe an interpretation of mTT^{eq}_0 into classical ZFC sets that validates Bar Induction $\mathrm{BI}^i_{\mathrm{fr}}$ and the formal Church thesis $\mathrm{CT}^i_{\mathrm{tt}}$. We call such an interpretation *bar recursive interpretation* of mTT^{eq}_0. Thanks to proposition 4.1 this gives an interpretation also for mTT_0.

The underlying idea of our recursive bar interpretation is *to interpret sets with their elements* in an *effective way*, namely as subsets of natural numbers considered as our data types and as computable functions respectively, *propositions* as *subsets of the zero singleton* $\{0\}$, and hence in ZFC as *their boolean value*, while *proper collections* (namely those collections that are not sets) *with their elements* are interpreted in a *non-effective* way, namely simply as suitable ZFC-sets and ZFC-functions. In particular, a dependent term is interpreted as a function from the interpretation of its context to the interpretation of its type.

Then, in order to validate the formal Church thesis we add a *computational requirement* saying that *a function interpreting an element of a set or of a proposition* has the property *to be computed by a family of programs* depending on the interpretation of the minimum context part containing all its proper collection assumptions. Indeed, the idea is to interpret *the dependency of a set element on a set assumption* as a *computable functional dependence*, computed by a program represented by a Gödel number. Instead we interpret *the dependency on a proper collection assumption* as a *functional dependence* with no *computational contents.*

Now, given that contexts are telescopic, a proper collection assumption could be in the middle of a context after some set assumptions. In this case the dependency on such set assumptions is treated simply as a functional dependence *by loosing its computational contents.*

For example, the term

$$x + y \in \mathrm{N}\ [x \in \mathrm{N}, y \in \mathrm{N}]$$

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[6] We just recall that this rule says that from $a \in A$ and $A = B$ *type* we get $a \in B$.

will be interpreted as the sum function computed by a program with the interpretation of both assumptions as inputs. But if we consider its weakening with a proper collection assumption, for example

$$x + y \in \mathrm{N}\ [x \in \mathrm{N}, V \in \mathsf{prop_s}, y \in \mathrm{N}]$$

the resulting term turns out to be interpreted as a *family of programs with one input depending on the interpretation of* V *and in turn also of* x *as* non computable *assumptions.* It is only after a substitution of V with some closed term that we get access to the sum function computed by a program with two inputs.

Given that we follow Kleene's interpretation of set constructors as in [Tv88b] for a version of Martin-Löf's type theory, this problem of restoring missing computable codes appears when interpreting the indexed sum elimination constructor of mTT in [Mai09] from a proper indexed sum collection toward a set. Luckily we can solve this problem for mTT_0^{eq} just because there we restricted its indexed sum collections to be only indexed sums of propositional functions. Indeed, in mTT_0^{eq} we can only eliminate from a proper collection toward a proper collection, which does not cause any problem of interpretation. Or we can eliminate from a proper collection toward a proposition, whose elements, even after substitutions, are always interpreted as the constant zero function. Hence we can assign to them the constant zero program in a *canonical* way: for example, after interpreting

$$t(z_1, z_2) \in \Sigma_{x \in \mathsf{prop_s}}\ \phi(x)\ [z_1 \in \Sigma_{x \in \mathsf{prop_s}}\ \phi(x), z_2 \in \mathrm{N}]$$

as a suitable ZFC-function, its second projection

$$\pi_2(t(z_1, z_2)) \in \phi(\pi_1(z_1))\ [z_1 \in \Sigma_{x \in \mathsf{prop_s}}\ \phi(x), z_2 \in \mathrm{N}]$$

turns out to be interpreted as its set-theoretic ZFC second projection. Moreover for each $w_1 \varepsilon (\Sigma_{x \in \mathsf{prop_s}}\ \phi(x))^{\mathcal{I}}$ the constant zero program code $\lfloor z_2 \mapsto 0 \rfloor$ computes the function

$$\pi_2(t(z_1, z_2))^{\mathcal{I}}(w_1, -) : w_2 \mapsto \pi_2(t(z_1, z_2))^{\mathcal{I}}(w_1, w_2)$$

on the computable input $w_2 \varepsilon \mathrm{N}^{\mathcal{I}}$.

In the more general case of indexed sums of a collection family indexed on a proper collection we are not able to assign canonical codes that can be restored by just looking at the type of the output. In order to interpret them we need to build a more complex realizability interpretation where the apparently forgotten codes of set inputs, on which proper collection assumptions depend, are *all stored* in order to use them after substitution. This more complex interpretation is left to future work.

The interpretation of propositions as their boolean values is also crucial to validate $\mathrm{BI}^i_{\mathrm{fr}}$, which is indeed a theorem classically! Recall also that we can not interpret propositions according to Kleene's realizability interpretation [Tv88a] because this validates the axiom of choice, and, hence, the axiom of unique choice $\mathrm{AC!}^i_{\mathrm{N,N}}$ that is inconsistent with $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$. Moreover, interpreting propositions as subsets of the zero singleton allows us to validate also the rules **prop-into-col** and **prop$_s$-into-set** in [Mai09].

Now we start to properly define the bar recursive interpretation of mTT_0^{eq} in ZFC-classical set theory. To this purpose we first fix some abbreviations regarding computable functions we are going to use. Given a natural number n in the ZFC-set of natural numbers $\mathcal{N}$, then $\{n\} : \mathcal{N} \to \mathcal{N}$ stands for the computable partial function with Gödel number n. Moreover, we will simply write

$$\{n\}(y) \varepsilon B \ \equiv\ \exists_{w \varepsilon \mathcal{N}}\, T(n, y, w)\ \wedge\ U(w) \varepsilon B$$

for a natural number n and for a subset B of $\mathcal{N}$. Conversely, given a partial computable function $z \mapsto f(z) : \mathcal{N} \to \mathcal{N}$, then $\lfloor z \mapsto f(z) \rfloor$ denotes a natural number such that

$$\{\ \lfloor z \mapsto f(z) \rfloor\ \}(x)\ =\ f(x)$$

Moreover the isomorphism of the set of natural numbers $\mathcal{N}$ with its binary product $\mathcal{N} \times \mathcal{N}$ is denoted by the following ZFC-functions:

$$\langle \mathsf{pr}_1, \mathsf{pr}_2 \rangle : \mathcal{N} \to \mathcal{N} \times \mathcal{N} \qquad \text{and} \qquad \mathsf{pair} : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$$

Finally, after recalling that the set of natural numbers $\mathcal{N}$ is isomorphic to the set of lists on itself, we denote its list structure as in type theory: the empty list is nil, which is 0 in $\mathcal{N}$, $\mathsf{cons}_{List(\mathcal{N})}(-,-)$ is the list constructor appending an element to a list and $Rec_{List(\mathcal{N})}(-,-,-)$ is the constructor defining a term by recursion on lists.

In order to validate $\mathrm{BI}^i_{\mathrm{fr}}$ we need to distinguish sets from proper collections. To this purpose we will use the following decidable function saying when a type is a *proper collection* (for short `is-pc`) that is neither a set nor a proposition for any mTT_0^{eq}-expression A for which we know that in mTT_0^{eq} A *type* $[\Gamma]$ is derivable [7]:

$$\texttt{is-pc}(A) \equiv \begin{cases} 0 & \text{if } A \textit{ set} \text{ or } A \textit{ prop} \\ 1 & \text{otherwise} \end{cases}$$

In mTT_0^{eq} we can decompose any context Γ in two parts: one, called Γ_p, is the minimal context part containing all proper collection assumptions, and hence the remaining part, called Γ_t, is made of set or proposition assumptions only:

Lemma 5.1 *Any context* Γ *can be decomposed into* Γ_p, Γ_t *where* $\Gamma_p \equiv \emptyset$ *or* $\Gamma_p \equiv x_1 \in A_1, \dots, x_n \in A_n$ *with* $\texttt{is-pc}\ (A_n) = 1$*, and* $\Gamma_t \equiv \emptyset$ *or* $\Gamma_t \equiv y_1 \in B_1, \dots \in B_m$ *made only of set or proposition assumptions, namely* $\texttt{is-pc}\ (B_i) = 0$ *for* $i = 1, \dots, m$.

Def. 5.2 (bar recursive interpretation of mTT_0^{eq}) Here we define the interpretation

$$(-)^{\mathcal{I}} : \mathrm{mTT}_0^{eq} \longrightarrow ZFC$$

A context Γ is interpreted by induction by means of disjoint unions as follows:

$$(\emptyset\ cont)^{\mathcal{I}} \equiv \{0\} \qquad (\Gamma, x \in A\ cont)^{\mathcal{I}} \equiv \biguplus_{z \varepsilon \Gamma^{\mathcal{I}}} A^{\mathcal{I}}(z)$$

A type judgement is interpreted as a set family

$$(\, B\ type\ [\Gamma]\,)^{\mathcal{I}} \equiv (\, B^{\mathcal{I}}(z)\,)_{z \varepsilon \Gamma^{\mathcal{I}}}$$

In particular, dependent sets turn out to be interpreted as families of subsets of natural numbers:

$$(\, B\ set\ [\Gamma]\,)^{\mathcal{I}} \equiv (\, B^{\mathcal{I}}(z)\,)_{z \varepsilon \Gamma^{\mathcal{I}}} \ \text{ such that } \ B^{\mathcal{I}}(z) \subseteq \mathcal{N}$$

and dependent propositions as subsets of the zero singleton $\{0\}$:

$$(\, B\ prop\ [\Gamma]\,)^{\mathcal{I}} \equiv (\, B^{\mathcal{I}}(z)\,)_{z \varepsilon \Gamma^{\mathcal{I}}} \ \text{ such that } \ B^{\mathcal{I}}(z) \subseteq \{\, n \varepsilon \mathcal{N} \mid n = 0\, \}$$

A type equality judgement is interpreted as the extensional equality between set families

$$(\, B = C\ type\ [\Gamma]\,)^{\mathcal{I}} \equiv \forall_{z \varepsilon \Gamma^{\mathcal{I}}}\ B^{\mathcal{I}}(z) = C^{\mathcal{I}}(z)$$

---

[7] After recalling that $type \in \{col, set, prop, prop_s\}$, more formally $\texttt{is-pc}(\, A\,)$ is defined by induction on type constructions as follows:
$\texttt{is-pc}(\, A\,) \equiv 0$ for $A \equiv N_0, N_1, List(C), B + C, \bot, B \wedge C, B \vee C, B \to C, \exists_{x \in B} C(x), \forall_{x \in B} C(x), \mathsf{Eq}(A, a, b)$
$\texttt{is-pc}(\, A\,) \equiv 0$ for $A \equiv \Sigma_{x \in B} C(x)$ iff $\texttt{is-pc}(\, B\,) = 0$ and $\texttt{is-pc}(\, C(x)\,) = 0$
$\texttt{is-pc}(\, p\,) \equiv 0$ for $p \in \mathsf{prop_s}$
$\texttt{is-pc}(\, A\,) \equiv 1$ for $A \equiv \mathsf{prop_s}, B \to \mathsf{prop_s}$.

A term judgement is interpreted as an element of a suitable dependent product

$$(\, b \in B \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ b^{\mathcal{I}} \in \Pi_{z \varepsilon \Gamma^{\mathcal{I}}}\, B^{\mathcal{I}}(z)$$

and, if $\texttt{is-pc}(B) = 0$, namely if B is a dependent set or a proposition, this must be equipped with a suitable family of program codes $b^{\sharp}(z_p)\varepsilon\mathcal{N}$ depending on $\Gamma_p^{\mathcal{I}}$, namely on the interpretation of the minimal context part including all proper collection assumptions, and computing the function $z_t \mapsto b^{\mathcal{I}}(z_p, z_t)$ [8]: i.e. we assume that there exists

$$z_p \varepsilon \Gamma_p^{\mathcal{I}} \mapsto b^{\sharp}(z_p)\varepsilon\mathcal{N} \qquad \text{s. t.} \qquad \forall_{z_t} \varepsilon \Gamma_t^{\mathcal{I}}(z_p) \ \ b^{\mathcal{I}}(z_p, z_t) = \{b^{\sharp}(z_p)\}(z_t)$$

where, in the case $\Gamma_t \equiv \emptyset$, i.e. there are no set inputs available, we simply consider the program code coinciding with the output.
In the next we will simply define $b^{\sharp}(z_p) \equiv \lfloor z_t \mapsto b^{\mathcal{I}}(z_p, z_t)\rfloor$ when applicable.
A term equality judgement is interpreted as the extensional equality of dependent functions:

$$(\, b = c \in B \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ \forall_{z \varepsilon \Gamma^{\mathcal{I}}}\ b^{\mathcal{I}}(z) = c^{\mathcal{I}}(z)$$

Now, we give the interpretation of mTT_0^{eq}-constructors. Actually this will be a partial interpretation of the so-called "raw syntax" in [Mai05a], namely of the syntax forming types and typed terms of mTT_0^{eq}, because term equalities are involved in the formation of types and typed terms.

Note that for simplicity, we interpret indexed sum of propositional functions indexed on a set in a computational way being mTT_0^{eq}-propositions interpreted as their boolean value.
Assumption of variables is interpreted as follows:

$$(\, x \in A \ [\Gamma, x \in A, \Delta]\,)^{\mathcal{I}} \ \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \pi_{n+1}(z_p, z_t)\rfloor & \text{if } \texttt{is-pc}(A) = 0 \\ \\ z \mapsto \pi_{n+1}(z) & \text{if } \texttt{is-pc}(A) = 1 \end{cases}$$

Collection and set constructors are interpreted as follows:

$$(\, \Sigma_{x \in B} D(x) \ col \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \begin{cases} (\, \{\ n\varepsilon\mathcal{N} \ \mid \ \mathsf{pr}_1(n)\varepsilon B^{\mathcal{I}}(z) \ \wedge \ \mathsf{pr}_2(n)\varepsilon C^{\mathcal{I}}(z, \mathsf{pr}_1(n)) \ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}} & \text{if } \texttt{is-pc}\ (B) = 0 \\ (\, \biguplus_{x \varepsilon B^{\mathcal{I}}(z)} C^{\mathcal{I}}(z, x)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} & \text{if } \texttt{is-pc}\ (B) = 1 \end{cases}$$

$$(\, \langle b, d\rangle \in \Sigma_{x \in B} D(x) \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pair}(\, b^{\mathcal{I}}(z_p, z_t)\,,\, c^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\ (B) = 0 \\ z \mapsto (b^{\mathcal{I}}(z), d^{\mathcal{I}}(z)) & \text{if } \texttt{is-pc}\ (B) = 1 \end{cases}$$

$$(\, \pi_1(d) \in B \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pr}_1(\, d^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\ (B) = 0 \\ z \mapsto \pi_1(d^{\mathcal{I}}(z)) & \text{if } \texttt{is-pc}\ (B) = 1 \end{cases}$$

$$(\, \pi_2(d) \in C(d) \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto \mathsf{pr}_2(\, d^{\mathcal{I}}(z_p, z_t)\,)\rfloor & \text{if } \texttt{is-pc}\ (\Sigma_{x \in B} C(x)) = 0 \\ z_p \mapsto \lfloor z_t \mapsto 0\rfloor & \text{if } \texttt{is-pc}\ (\Sigma_{x \in B} C(x)) = 1 \end{cases}$$

$(\, \mathsf{prop_s} \ col \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ (\, \{\, \emptyset, \{0\}\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, B \to \mathsf{prop_s} \ col \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ (\, B^{\mathcal{I}}(z) \to \{\, \emptyset, \{0\}\,\}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, \lambda x^B.C \in B \to \mathsf{prop_s} \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ z \mapsto (\, x \mapsto C^I(z, x)\,)$
$(\, \mathsf{Ap}(f, b) \in \mathsf{prop_s} \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ z \mapsto f^{\mathcal{I}}(z)\,(\, b^{\mathcal{I}}(z)\,)$
$(\, N_0 \ set \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ (\, \emptyset\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, \mathsf{emp_o}\,)^{\mathcal{I}} \ \equiv \ z_p \mapsto \lfloor z_t \mapsto 0\rfloor$
$(\, N_1 \ set \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ (\, \{\ 0\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$
$(\, * \in N_1 \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \ z_p \mapsto \lfloor z_t \mapsto 0\rfloor$

$$(\, El_{\mathsf{N_1}}(t, c)\,) \in M(t) \ [\Gamma]\,)^{\mathcal{I}} \ \equiv \begin{cases} z_p \mapsto \lfloor z_t \mapsto c^{\mathcal{I}}(z_p, z_t)\rfloor & \text{if } \texttt{is-pc}\ (M(t)) = 0 \\ z \mapsto c^{\mathcal{I}}(z) & \text{if } \texttt{is-pc}\ (M(t)) = 1 \end{cases}$$

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[8] From now on, when writing the dependency of a function on an indexed sum, we simply write $b^{\mathcal{I}}(z_p, z_t)$ instead of $b^{\mathcal{I}}(\,(z_p, z_t)\,)$, where (z_p, z_t) represents the pairing of z_p with components of z_t to become an element of $\Gamma^{\mathcal{I}}$ made of nested disjoint unions. The same we do for set families depending on a disjoint union, i.e. we write $C^{\mathcal{I}}(w, z)$ for $w\varepsilon A^{\mathcal{I}}, z\varepsilon B^{\mathcal{I}}(x)$ instead of $C^{\mathcal{I}}(\,(w, z)\,)$.

$(\, List(C)\ set\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n \varepsilon \mathcal{N}\ \mid\ \forall_{j\varepsilon\mathcal{N}}\ (\, 1 \leq j \leq \mathsf{lh}(n)\ \wedge\ \mathsf{p_j}(n)\varepsilon C^{\mathcal{I}}(z)\,)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

where $\mathsf{lh}(n)$ is the length of the list encoded by n and $\mathsf{p_j}(n)$ its ith-projection.

$(\, \epsilon \in List(C)\, [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \mathsf{cons}(s,c) \in List(C)\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto \mathsf{cons}_{List(\mathcal{N})}(\, s^{\mathcal{I}}(z_p,z_t)\, ,\ c^{\mathcal{I}}(z_p,z_t)\,)\rfloor$

$$(\, El_{List}(a,l,s) \in L(s)\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ \begin{cases} z_p \mapsto \lfloor z_t \mapsto \\ \qquad Rec_{List(\mathcal{N})}(\, a^{\mathcal{I}}(z_p,z_t)\, ,\ (x,y,w).\, l^{\mathcal{I}}(z_p,z_t,x,y,w)\, ,\ s^{\mathcal{I}}(z_p,z_t)\,)\rfloor \\ \qquad\quad \text{if } \mathtt{is\text{-}pc}\ \ (L(s)) = 0 \\ z \mapsto Rec_{List(\mathcal{N})}(\, a^{\mathcal{I}}(z),\ (x,y,w).\, l^{\mathcal{I}}(z,x,y,w)\, ,\ s^{\mathcal{I}}(z)\,) \\ \qquad\quad \text{if } \mathtt{is\text{-}pc}\ \ (L(s)) = 1 \end{cases}$$

$(\, B + C\ set\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\, n\varepsilon\mathcal{N}\ \mid\ (\, n = (0,y)\ \wedge\ y\varepsilon B^{\mathcal{I}}(z)\,)\ \vee\ (\, n = (1,y)\ \wedge\ y\varepsilon C^{\mathcal{I}}(z)\,)\, \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \mathsf{inl}(b) \in B + C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto \mathsf{pair}(0, b^{\mathcal{I}}(z_p,z_t)\,)\rfloor$

$(\, \mathsf{inl}(c) \in B + C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto \mathsf{pair}(1, c^{\mathcal{I}}(z_p,z_t)\,)\rfloor$

$$(\, El_{+}(d, a_B, a_{\{\,0,1\,\}^N}) \in A(w)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ \begin{cases} z_p \mapsto \lfloor z_t \mapsto \begin{cases} a^{\mathcal{I}}_B(z_p,z_t,y) & \text{if } d^{\mathcal{I}}(z_p,z_t) = \mathsf{pair}(0,y) \\ a^{\mathcal{I}}_C(z_p,z_t,y) & \text{if } d^{\mathcal{I}}(z_p,z_t) = \mathsf{pair}(1,y) \end{cases} \rfloor \\ \qquad\qquad \text{if } \mathtt{is\text{-}pc}\ \ (A(w)) = 0 \\ z \mapsto \begin{cases} a^{\mathcal{I}}_B(z,y) & \text{if } d^{\mathcal{I}}(z) = \mathsf{pair}(0,y) \\ a^{\mathcal{I}}_C(z,y) & \text{if } d^{\mathcal{I}}(z) = \mathsf{pair}(1,y) \end{cases} \\ \qquad\qquad \text{if } \mathtt{is\text{-}pc}\ \ (A(w)) = 1 \end{cases}$$

$(\, \Pi_{x\in B}C(x)\ set\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ \forall_{y\varepsilon\mathcal{N}}\ y\varepsilon B^{\mathcal{I}}(z) \rightarrow \{n\}(y)\varepsilon C^{\mathcal{I}}(z,y)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \lambda x^B.c \in \Pi_{x\in B}C(x)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto\ \lfloor z_t \mapsto S^1_m(c^{\sharp}(z_p)\, ,\ z_t\,)\rfloor$

with $\{\, S^1_m(\, c^{\sharp}(z_p)\, ,\ z_t\,)\}(x) = \{\, c^{\sharp}(z_p)\, \}(z_t, x)$ for all z_t, x by s-m-n theorem with m length of Γ_t

$(\, \mathsf{Ap}(f,b) \in C(b)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto \{\, f^{\mathcal{I}}(\, z_p, z_t\,)\, \}(\, b^{\mathcal{I}}(\, z_p, z_t\,)\,)\rfloor.$

Now, we give the interpretation of propositions:

$(\, \perp prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \emptyset\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \mathsf{r_o}(a)\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \exists_{x\in B}C(x)\ prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ n = 0\ \wedge\ \exists_{x\varepsilon\mathcal{N}}\, (\, x\varepsilon B^{\mathcal{I}}(z)\ \wedge\ 0\varepsilon C^{\mathcal{I}}(z,x)\,)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \langle b,_{\exists}\, c\rangle \in \exists_{x\in B}C(x)\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, El_{\Sigma}(d,m) \in M(d)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, B \vee C\ prop\ \Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ (\, n = 0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \vee\ 0\varepsilon C^{\mathcal{I}}(z)\,)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$\mathcal{I}(\, \mathsf{inl}_{\vee}(b) \in B \vee C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \mathsf{inr}_{\vee}(c) \in B \vee C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, El_{\vee}(d, a_B, a_C) \in A\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, B \wedge C\ prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ n = 0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \wedge\ 0\varepsilon C^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \langle b,_{\wedge}\, c\rangle \in B \wedge C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \pi^B_1(d) \in B\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \pi^C_2(d) \in C\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor^{\mathcal{I}}$

$(\, B \rightarrow C\ prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ n = 0\ \wedge\ 0\varepsilon B^{\mathcal{I}}(z)\ \rightarrow\ 0\varepsilon C^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\forall_{x\in B}C(x)\ prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\, n\varepsilon\mathcal{N}\ \mid\ n = 0\ \wedge\ \forall_{y\varepsilon\mathcal{N}}\ (\, y\varepsilon B^{\mathcal{I}}(z) \rightarrow 0\varepsilon C^{\mathcal{I}}(z,y)\,)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$(\, \lambda x^B.c \in \Pi_{x\in B}C(x)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \mathsf{Ap}(f,b) \in C(b)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, \mathsf{Eq}(A,a,b)\ prop\ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ (\, \{\ n\varepsilon\mathcal{N}\ \mid\ n = 0\ \wedge\ a^{\mathcal{I}}(z) = b^{\mathcal{I}}(z)\ \}\,)_{z\varepsilon\Gamma^{\mathcal{I}}}$

$\mathcal{I}(\, \mathsf{id_A}(a) \in \mathsf{Id}(A,a,a)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

$(\, El_{\mathsf{Id}}(p, (x)c(x)) \in C(a,b)\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ z_p \mapsto \lfloor z_t \mapsto 0 \rfloor$

In order to show the validity theorem, we need to show how weakening and substitutions are interpreted.

Lemma 5.3 *For any judgements* $B\ type\ [\Gamma]$ *and* $b \in B\ [\Gamma]$ *derived in* mTT^{eq} *and interpreted as*

$$(B\ type\ [\Gamma])^{\mathcal{I}} \ \equiv\ (\, B^{\mathcal{I}}(z_1, \ldots, z_n)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} \qquad \textit{and} \qquad (\, b \in B\ \ [\Gamma]\,)^{\mathcal{I}} \ \equiv\ b^{\mathcal{I}}\varepsilon\Pi_{z\varepsilon\Gamma^{\mathcal{I}}}\, B^{\mathcal{I}}(z_1, \ldots, z_n)$$

weakening *is interpreted as follows:*

$$\begin{aligned} (B\ \mathit{type}\ [\Gamma,\Delta]\,)^{\mathcal{I}} &= (\,B^{\mathcal{I}}(w)\,)_{w,z\varepsilon(\,\Gamma,\Delta\,)^{\mathcal{I}}} \\ \mathcal{I}(\,b \in B\ [\Gamma,\Delta]\,) &= w,z \mapsto b^{\mathcal{I}}(w) \\ &\qquad\qquad \mathit{in}\ \Pi_{w,z\varepsilon(\,\Gamma,\Delta\,)^{\mathcal{I}}}\, B^{\mathcal{I}}(w) \end{aligned}$$

Proof. By induction on the derivation of the judgements.

Now we show how substitution is interpreted via composition of functions by using the following abbreviations: given a context $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ with $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ then for every $a \in A_n\ [\Sigma]$ and for any type $B\ \mathit{type}\ [\Gamma]$ we simply write the type B after substitution of x_n with a in the form $B[x_n/a]\ \mathit{type}\ [\Sigma,\Delta_a]$ instead of the more correct form $B[x_n/a_n][x_i/x'_i]_{i=n+1,\dots,k}\ \mathit{type}\ [\Sigma,\Delta_a]$ where $\Delta_a \equiv x'_{n+1} \in A'_{n+1}, ..., x'_k \in A'_k$ and $A'_j \equiv A_j\ [x_n/a_n][x_i/x'_i]_{i=n+1,\dots,j-1}$ for $j = n+1, ..., k$. Similar abbreviations are used also for terms.

Lemma 5.4 *For any judgements* $B\ \mathit{type}\ [\Gamma]$ *and* $b \in B\ [\Gamma]$ *derived in* mTT_0^{eq} *and interpreted as*

$$(B\ \mathit{type}\ [\Gamma])^{\mathcal{I}} \equiv (\,B^{\mathcal{I}}(z_1,\dots,z_n)\,)_{z\varepsilon\Gamma^{\mathcal{I}}} \qquad \mathit{and} \qquad (\,b \in B\ [\Gamma]\,)^{\mathcal{I}} \equiv b^{\mathcal{I}} \in \Pi_{z\varepsilon\Gamma^{\mathcal{I}}}\, B^{\mathcal{I}}(z_1,\dots,z_n)$$

substitution *is interpreted as follows:*
supposed $\Gamma \equiv \Sigma, x_n \in A_n, \Delta$ *with* $\Delta \equiv x_{n+1} \in A_{n+1}, ..., x_k \in A_k$ *then for every* $a \in A_n\ [\Sigma]$ *interpreted as*

$$(\,a \in A_n\ [\Sigma]\,)^{\mathcal{I}} \equiv a^{\mathcal{I}}\varepsilon\Pi_{z\varepsilon\Sigma^{\mathcal{I}}}\, {A_n}^{\mathcal{I}}(z_1,\dots,z_{n-1})$$

and supposed the interpretations $(B[x_n/a]\ \mathit{type}\ [\Sigma,\Delta_a]\,)^{\mathcal{I}}$ *and* $(\,b[x_n/a] \in B[x_n/a]\ [\Sigma,\Delta_a]\,)^{\mathcal{I}}$ *well defined, then*

$$\begin{aligned} (B[x_n/a]\ \mathit{type}\ [\Sigma,\Delta_a]\,)^{\mathcal{I}} &= (\,B^{\mathcal{I}}(w,\ a^{\mathcal{I}}(w),\ w')\,)_{w,w'\varepsilon(\,\Sigma,\Delta_a\,)^{\mathcal{I}}} \\ (\,b[x_n/a] \in B[x_n/a]\ [\Sigma,\Delta_a]\,)^{\mathcal{I}} &= w,w' \mapsto b^{\mathcal{I}}(w,\ a^{\mathcal{I}}(w),\ w'\,) \\ &\qquad\qquad \mathit{in}\ \Pi_{w,w'\varepsilon(\,\Sigma,\Delta_a\,)^{\mathcal{I}}}\, B^{\mathcal{I}}(w,\ a^{\mathcal{I}}(w),\ w') \end{aligned}$$

Proof. By induction on the derivation of judgements.

Theorem 5.5 (bar recursive validity) *The calculus* mTT_0^{eq} *is validated by the bar recursive interpretation of definition 5.2, namely:*
If $A\ \mathit{type}\ [\Gamma]$ *is derivable in* mTT_0^{eq} *then* $(\,A\ \mathit{type}\ [\Gamma]\,)^{\mathcal{I}}$ *is well defined.*
If $a \in A\ [\Gamma]$ *is derivable in* mTT_0^{eq} *then* $(\,a \in A\ [\Gamma]\,)^{\mathcal{I}}$ *is well defined.*
Supposed $A\ \mathit{type}\ [\Gamma]$ *and* $B\ \mathit{type}\ [\Gamma]$ *derivable in* mTT_0^{eq}, *if* $A = B\ \mathit{type}\ [\Gamma]$ *is derivable in* mTT_0^{eq}, *then* $(A = B\ \mathit{type}\ [\Gamma]\,)^{\mathcal{I}}$ *is valid.*
Supposed $a \in A\ [\Gamma]$ *and* $b \in A\ [\Gamma]$ *derivable in* mTT_0^{eq}, *if* $a = b \in A\ [\Gamma]$ *is derivable in* mTT_0^{eq}, *then* $(\,a = b \in A\ [\Gamma]\,)^{\mathcal{I}}$ *is valid.*

Moreover, for any set A *in* mTT_0, *the bar recursive interpretation validates* CT^{i}_{tt} *and* BI^{i}_{fr} *when interpreting a generic* $\lhd^{i}_{A^N}$ *as the corresponding ZFC-tree formal topology over* $A^{\mathcal{I}}$ *quotiented under the translation of its equality* $=_{A^{\mathcal{I}}}$.
It also validates the principle of excluded middle EM, *written* $P \vee \neg P$ *for a proposition* P.

Proof. By induction on the derivation of judgements. Note that the interpretation of the second projection E_2-ip) is well defined given that a valid proposition is interpreted as the zero singleton. Indeed, if for each $z \in \Gamma^{\mathcal{I}}$ we have that $\pi_2(d^{\mathcal{I}}(z))\varepsilon C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z))$, this means that for $z \in \Gamma^{\mathcal{I}}$ then $C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z))$ is inhabited. Since it is a subset of $\{0\}$ we deduce $C^{\mathcal{I}}(z, \pi_1(d^{\mathcal{I}}(z)) = \{0\}$ and hence $\pi_2(d^{\mathcal{I}}(z)) = 0$. Therefore, for every $z_p\varepsilon\Gamma^{\mathcal{I}}_p$ and $z_t\varepsilon\Gamma^{\mathcal{I}}_t(z_p)$ we conclude $\pi_2(d^{\mathcal{I}}(z_p, z_t)) = \{\lfloor z_t \mapsto 0\rfloor\}(z_t)$ as wanted. Finally η-conversion, beside β-one is valid because $\langle \mathsf{pr}_1, \mathsf{pr}_2 \rangle$ is an isomorphism with pair.

The rules of Extensional Propositional Equality are valid: in particular the rule E-Eq) of mTT_0^{eq} is valid because from the assumption that $p^{\mathcal{I}}(z)\varepsilon \mathsf{Eq}(A,c,d)^{\mathcal{I}}$ for any $z\varepsilon\Gamma^{\mathcal{I}}$ we conclude $c^{\mathcal{I}}(z) = d^{\mathcal{I}}(z)$ for any $z\varepsilon\Gamma^{\mathcal{I}}$.
The interpretation of lambda abstraction in the rule I-Π) is well defined by s-m-n theorem.
EM is valid since propositions are interpreted as their boolean value.
$\mathrm{BI}^i_{\mathrm{fr}}$ is valid because it turns out to be interpreted as spatiality of a generic tree formal topology which is a ZFC theorem (actually ZF together with the axiom of dependent choices would be enough to validate $\mathrm{BI}^i_{\mathrm{fr}}$).
$\mathrm{CT}^i_{\mathrm{tt}}$ is valid because type-theoretic functions are interpreted as computable functions with a chosen program code.

Remark 5.6 Note from the above result we get also that emTT_0 is consistent with $\mathrm{BI}^i_{\mathrm{fr}}$ and $\mathrm{CT}^i_{\mathrm{tt}}$ and Markov principle

$$\forall x \in \mathrm{N}\ (\, P(x)\ \vee\ \neg P(x)\,)\quad \rightarrow\quad (\,\neg\neg\exists y \in \mathrm{N}\ P(x)\ \rightarrow\ \exists y \in \mathrm{N}\ P(x)\,)$$

given that in the realizability for mTT^{eq} the law of excluded middle is valid, and hence also Markov principle is valid, too.

This setting may provide a way to reconciliate Brouwer's intuitionism with Markov's mathematics, as soon as one drops the axiom of unique choice.

Remark 5.7 Note that, a direct proof that $\mathrm{AC!}_{\mathrm{N,N}}$ is not valid in mTT_0^{eq} and hence in mTT_0 and emTT_0, can be obtained from th. 5.5 and interpretation $(-)^i$ in section 2 with arguments like those used here where we replace $\mathrm{BI}_{\mathrm{fr}}$ (and $\mathrm{BI}^i_{\mathrm{fr}}$) with the principle of excluded middle EM. This is because $\mathrm{emTT}_0 + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N,N}}$ is inconsistent with EM, thanks to a proof similar to that in prop. 0.1 in [MS05] provided that one starts with the formula

$$\forall x \in \mathrm{N}\ \exists! y \in \mathrm{N}\ (\,(\,y = 1 \wedge P(x)\,)\ \vee\ (\,y = 0 \wedge \neg P(x)\,)\,)$$

Remark 5.8 Note that we can define our realizability interpretation for mTT_0^{eq} into an intuitionistic set theory equipped with Bar Induction, as Aczel's CZF or even emTT_0 itself. In this case the subsets of the zero singleton in which propositions are interpreted do not coincide with the boolean values any longer.

6 Conclusions: Choice sequences with and without unique choice

Thanks to the rereading of Kleene's result in our foundation emTT_0 as stated in cor. 3.21, $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N,N}}$ is inconsistent. However any combination of two such principles seems to be consistent with emTT_0, and it gives a specific behavior of choice sequences defined as functional relations between natural numbers.
Here, we list the various cases briefly:
- $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{CT}_{\mathrm{tt}}$: in this theory type-theoretic functions between natural numbers are recursive thanks to $\mathrm{CT}_{\mathrm{tt}}$, but they can not be identified with choice sequences given that $\mathrm{AC!}_{\mathrm{N,N}}$ does not hold (this is the extension proved consistent here);
- $\mathrm{emTT}_0 + \mathrm{BI}_{\mathrm{fr}} + \mathrm{AC!}_{\mathrm{N,N}}$: in this theory choice sequences are identified with type-theoretic functions between natural numbers, but then such type-theoretic functions can not be internally recursive given that $\mathrm{CT}_{\mathrm{tt}}$ does not hold (this happens if we interpret emTT_0 into classical set theory in the obvious way: sets and collections are interpreted into corresponding ZFC-sets and propositions as their boolean value);
- $\mathrm{emTT}_0 + \mathrm{CT}_{\mathrm{tt}} + \mathrm{AC!}_{\mathrm{N,N}}$: in this theory choice sequences are identified with type-theoretic functions between natural numbers that are also internally recursive but $\mathrm{BI}_{\mathrm{fr}}$ is not valid (a model of this is current work in progress).

In the future we intend to modify the realizability interpretation shown here to work for the whole original intensional level mTT in [Mai09].

It would be also interesting to investigate consistency with $\mathrm{BI}^i_{\mathrm{fr}}+\mathrm{CT}^i_{\mathrm{tt}}$ of impredicative theories as the Calculus of Constructions [Coq90].

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7 Appendix: The intensional level mTT_0

As mTT in [Mai09], the inference rules of mTT_0 involve judgements written in the style of Martin-Löf's type theory [Mar84, NPS90] that may be of the form:

$$A\ type\ [\Gamma] \qquad A = B\ type\ [\Gamma] \qquad a \in A\ [\Gamma] \qquad a = b \in A\ [\Gamma]$$

where types include collections, sets, propositions and small propositions, namely

$$type \in \{col, set, prop, prop_s\}$$

For easinesss, the piece of context common to all judgements involved in a rule is omitted and typed variables appearing in a context are meant to be added to the implicit context as the last one.
Note that to write the elimination constructors of our types we adopt the higher-order syntax in [NPS90]. According to this syntax the open term $a_B(x) \in A\ [x \in B]$ yields to $(x \in B)\, a_B(x)$ of higher type $(x \in B)\, A$. Then, by η-conversion among higher types, it follows that $(x \in B)\, a_B(x)$ is equal to a_B. Hence, we often simply write the short expression a_B to recall the open term where it comes from..
We also have a form of judgement to build contexts:

$$\Gamma\ cont$$

whose rules are the following

$$\emptyset\ cont \qquad\qquad \text{F-c}\frac{A\ type\ [\Gamma]}{\Gamma, x \in A\ cont}\ (x \in A \notin \Gamma)$$

Then, the first rule to build elements of type is the assumption of variables:

$$\text{var)}\ \frac{\Gamma, x \in A, \Delta \quad cont}{x \in A\ [\Gamma, x \in A, \Delta]}$$

Among types there are the following embeddings: sets are collections and propositions are collections

$$\textbf{set-into-col)}\ \frac{A\ set}{A\ col} \qquad\qquad \textbf{prop-into-col)}\ \frac{A\ prop}{A\ col}$$

Strong Indexed Sum of a propositional function

$$\text{F-ip)}\ \frac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x \in B} C(x)\ col} \qquad \text{I-ip)}\ \frac{b \in B \quad d \in C(b) \quad C(x)\ \ prop\ [x \in B]}{\langle b, d\rangle \in \Sigma_{x \in B} C(x)}$$

$$\text{E-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ d \in \Sigma_{x \in B} C(x) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) \in M(d)}$$

$$\text{C-ip)}\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(\ \langle b, c\rangle, m\ ) = m(b, c) \in M(\langle b, c\rangle)}$$

Sets are generated as follows:

Empty set

F-Em) $\mathsf{N}_0\ set$ E-Em) $\dfrac{a \in \mathsf{N}_0 \quad A(x)\ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$

Singleton

S) $\mathsf{N}_1\ set$ I-S) $\star \in \mathsf{N}_1$ E-S) $\dfrac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$ C-S) $\dfrac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$

Strong Indexed Sum set

F-Σ) $\dfrac{C(x)\ set\ \ [x \in B] \qquad B\ set}{\Sigma_{x \in B} C(x)\ set}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ set\ \ [x \in B] \qquad B\ set}{\langle b, c\rangle \in \Sigma_{x \in B} C(x)}$

E-Σ) $$\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ d \in \Sigma_{x \in B} C(x) \quad m(x, y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) \in M(d)}$$

C-Σ) $$\dfrac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x, y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(\langle b, c\rangle, m) = m(b, c) \in M(\langle b, c\rangle)}$$

List set

F-list) $\dfrac{C\ set}{List(C)\ set}$ I$_1$-list) $\dfrac{List(C)\ set}{\epsilon \in List(C)}$ I$_2$-list) $\dfrac{s \in List(C) \quad c \in C}{\mathsf{cons}(s, c) \in List(C)}$

E-list) $$\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) \in L(s)}$$

C$_1$-list) $$\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \qquad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\epsilon, a, l) = a \in L(\epsilon)}$$

C$_2$-list) $$\dfrac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s \in List(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(\mathsf{cons}(s, c), a, l) = l(s, c, El_{List}(s, a, l)) \in L(\mathsf{cons}(s, c))}$$

Disjoint Sum set

F-+) $\dfrac{B\ set \quad C\ set}{B + C\ set}$ I$_1$-+) $\dfrac{b \in B \qquad B\ set \qquad C\ set}{\mathsf{inl}(b) \in B + C}$ I$_2$-+) $\dfrac{c \in C \qquad B\ set \qquad C\ set}{\mathsf{inr}(c) \in B + C}$

E-+) $$\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ w \in B + C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_{+}(w, a_B, a_C) \in A(w)}$$

C$_1$-+) $$\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_{+}(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$$

C$_2$-+) $$\dfrac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_{+}(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$$

Dependent Product set

$$\text{F-}\Pi)\quad \frac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x\in B}C(x)\ set} \qquad\qquad \text{I-}\Pi)\quad \frac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^{B}.c(x) \in \Pi_{x\in B}C(x)}$$

$$\text{E-}\Pi)\quad \frac{b \in B \quad f \in \Pi_{x\in B}C(x)}{\mathsf{Ap}(f,b) \in C(b)}$$

$$\beta\text{C-}\Pi)\quad \frac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^{B}.c(x), b) = c(b) \in C(b)}$$

Propositions are generated as follows:

Falsum

$$\text{F-Fs})\quad \bot\ prop \qquad\qquad \text{E-Fs})\quad \frac{a \in \bot \quad A\ prop}{\mathsf{r_o}(a) \in A}$$

Disjunction

$$\text{F-}\vee)\quad \frac{B\ prop \quad C\ prop}{B \vee C\ prop} \qquad \text{I}_1\text{-}\vee)\quad \frac{b \in B \qquad B\ prop \qquad C\ prop}{\mathsf{inl}_{\vee}(b) \in B \vee C} \qquad \text{I}_2\text{-}\vee)\quad \frac{c \in C \qquad B\ prop \qquad C\ prop}{\mathsf{inr}_{\vee}(c) \in B \vee C}$$

$$\text{E-}\vee)\quad \frac{\begin{array}{l} A\ prop \\ w \in B \vee C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(w, a_B, a_C) \in A}$$

$$\text{C}_1\text{-}\vee)\quad \frac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ b \in B \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inl}_{\vee}(b), a_B, a_C) = a_B(b) \in A}$$

$$\text{C}_2\text{-}\vee)\quad \frac{\begin{array}{l} A\ prop \qquad B\ prop \qquad C\ prop \\ c \in C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inr}_{\vee}(c), a_B, a_C) = a_C(c) \in A}$$

Conjunction

$$\text{F-}\wedge)\quad \frac{B\ prop \qquad C\ prop}{B \wedge C\ prop} \qquad\qquad \text{I-}\wedge)\quad \frac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\langle b,_{\wedge} c\rangle \in B \wedge C}$$

$$\text{E}_{\mathbf{1}}\text{-}\wedge)\quad \frac{d \in B \wedge C}{\pi_1^{B}(d) \in B} \qquad\qquad \text{E}_{\mathbf{2}}\text{-}\wedge)\quad \frac{d \in B \wedge C}{\pi_2^{C}(d) \in C}$$

$$\beta_1\ \text{C-}\wedge)\quad \frac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_1^{B}(\langle b,_{\wedge} c\rangle) = b \in B} \qquad\qquad \beta_2\ \text{C-}\wedge)\quad \frac{b \in B \quad c \in C \qquad B\ prop \qquad C\ prop}{\pi_2^{C}(\langle b,_{\wedge} c\rangle) = c \in C}$$

Implication

$$\text{F-}\rightarrow)\quad \frac{B\ prop \qquad C\ prop}{B \rightarrow C\ prop}$$

$$\text{I-}\rightarrow)\quad \frac{c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\lambda_{\rightarrow} x^{B}.c(x) \in B \rightarrow C} \qquad\qquad \text{E-}\rightarrow)\quad \frac{b \in B \quad f \in B \rightarrow C}{\mathsf{Ap}_{\rightarrow}(f,b) \in C}$$

$$\beta\text{C-}\rightarrow)\quad \frac{b \in B \quad c(x) \in C\ [x \in B] \qquad B\ prop \qquad C\ prop}{\mathsf{Ap}_{\rightarrow}(\lambda_{\rightarrow} x^{B}.c(x), b) = c(b) \in C}$$

Existential quantification

$$\text{F-}\exists)\ \frac{C(x)\ prop\ \ [x \in B]}{\exists_{x\in B}C(x)\ prop} \qquad \text{I-}\exists)\ \frac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ [x \in B]}{\langle b,_{\exists} c\rangle \in \exists_{x\in B}C(x)}$$

$$\text{E-}\exists)\ \frac{\begin{array}{l} M\ prop \\ d \in \exists_{x\in B}C(x) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(d,m) \in M}$$

$$\text{C-}\exists)\ \frac{\begin{array}{l} M\ prop \qquad C(x)\ prop\ [x \in B] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(\langle b,_{\exists} c\rangle, m) = m(b,c) \in M}$$

Universal quantification

$$\text{F-}\forall)\ \frac{C(x)\ prop\ [x \in B]}{\forall_{x\in B}C(x)\ prop} \qquad \text{I-}\forall)\ \frac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\lambda_{\forall}x^{B}.c(x) \in \forall_{x\in B}C(x)}$$

$$\text{E-}\forall)\ \frac{b \in B \quad f \in \forall_{x\in B}C(x)}{\mathsf{Ap}_{\forall}(f,b) \in C(b)} \qquad \beta\text{C-}\forall)\ \frac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ prop\ [x \in B]}{\mathsf{Ap}_{\forall}(\lambda_{\forall}x^{B}.c(x), b) = c(b) \in C(b)}$$

Propositional Equality

$$\text{F-Id})\ \frac{A\ col \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop} \qquad \text{I-Id})\ \frac{a \in A}{\mathsf{id}_{\mathsf{A}}(a) \in \mathsf{Id}(A,a,a)}$$

$$\text{E-Id})\ \frac{\begin{array}{l} C(x,y)\ prop\ [x : A, y \in A] \\ a \in A \quad b \in A \quad p \in \mathsf{Id}(A,a,b) \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(p,(x)c(x)) \in C(a,b)}$$

$$\text{C-Id})\ \frac{\begin{array}{l} C(x,y)\ prop\ [x : A, y \in A] \\ a \in A \quad c(x) \in C(x,x)\ [x \in A] \end{array}}{El_{\mathsf{Id}}(\mathsf{id}_{A}(a),(x)c(x)) = c(a) \in C(a,a)}$$

Then, small propositions are generated as follows:

$$\bot\ prop_s \qquad \frac{B\ prop_s \quad C\ prop_s}{B \vee C\ prop_s} \qquad \frac{B\ prop_s \qquad C\ prop_s}{B \to C\ prop_s} \qquad \frac{B\ prop_s \qquad C\ prop_s}{B \wedge C\ prop_s}$$

$$\frac{C(x)\ prop_s\ \ [x \in B] \qquad B\ set}{\exists_{x\in B}C(x)\ prop_s} \qquad \frac{C(x)\ prop_s\ [x \in B] \qquad B\ set}{\forall_{x\in B}C(x)\ prop_s} \qquad \frac{A\ set \quad a \in A \quad b \in A}{\mathsf{Id}(A,a,b)\ prop_s}$$

And we add rules saying that a small proposition is a proposition and that a small proposition is a set:

$$\textbf{prop}_s\textbf{-into-prop)}\ \frac{A\ prop_s}{A\ prop} \qquad \textbf{prop}_s\textbf{-into-set)}\ \frac{A\ prop_s}{A\ set}$$

Then, we also have the collection of small propositions and function collections from a set toward it:

Collection of small propositions

$$\text{F-Pr)}\ \mathsf{props_s}\ col \qquad \text{I-Pr)}\ \frac{B\ prop_s}{B \in \mathsf{props_s}} \qquad \text{E-Pr)}\ \frac{B \in \mathsf{props_s}}{B\ prop_s}$$

Function collection to $\mathsf{props_s}$

$$\text{F-Fun)}\ \frac{B\ set}{B \rightarrow \mathsf{props_s}\ col} \qquad \text{I-Fun)}\ \frac{c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\lambda x^{B}.c(x) \in B \rightarrow \mathsf{props_s}}$$

$$\text{E-Fun)}\ \frac{b \in B \quad f \in B \rightarrow \mathsf{props_s}}{\mathsf{Ap}(f,b) \in \mathsf{props_s}} \qquad \beta\text{C-Fun)}\ \frac{b \in B \quad c(x) \in \mathsf{props_s}\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^{B}.c(x), b) = c(b) \in \mathsf{props_s}}$$

Equality rules include those saying that type equality is an equivalence relation and substitution of equal terms in a type:

$$\text{ref)}\ \frac{A\ type}{A = A\ type} \qquad \text{sym)}\ \frac{A = B\ type}{B = A\ type} \qquad \text{tra)}\ \frac{A = B\ type \quad B = C\ type}{A = C\ type}$$

$$\text{subT)}\ \frac{\begin{array}{c} C(x_1,\ldots,x_n)\ type\ [\,x_1 \in A_1,\ \ldots,\ x_n \in A_n(x_1,\ldots,x_{n-1})\,] \\ a_1 = b_1 \in A_1\ \ldots\ a_n = b_n \in A_n(a_1,\ldots,a_{n-1}) \end{array}}{C(a_1,\ldots,a_n) = C(b_1,\ldots,b_n)\ type}$$

where $type \in \{col, set, prop, prop_s\}$ with the same choice both in the premise and in the conclusion.

8 Appendix: The intermediate typed calculus mTT_0^{eq}

The typed calculus mTT^{eq} is an extension of mTT where the propositional equality Id is replaced by the extensional equality Eq defined as follows:

Extensional Propositional Equality

$$\text{I-Eq)}\ \frac{C\ col \quad c \in C \quad d \in C}{\mathsf{Eq}(C,c,d)\ prop} \qquad \text{I-Eq)}\ \frac{c \in C}{\mathsf{eq}(c) \in \mathsf{Eq}(C,c,c)}$$

$$\text{E-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{c = d \in C} \qquad \text{C-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{p = \mathsf{eq_C}(c) \in \mathsf{Eq}(C,c,d)}$$

Then the rules for indexed sums on collections and sets are the following:

Strong Indexed Sum of a propositional function

$$\text{F-ip)}\ \frac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x\in B}C(x)\ col} \qquad \text{I-ip)}\ \frac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ [x \in B]}{\langle b, c\rangle \in \Sigma_{x\in B}C(x)}$$

$$\text{E}_1\text{-ip)}\ \frac{d \in \Sigma_{x\in B}C(x)}{\pi_1(d) \in B} \qquad \text{E}_2\text{-ip)}\ \frac{d \in \Sigma_{x\in B}C(x)}{\pi_2(d) \in C(\pi_1(d))}$$

$$\text{C}_1\text{-ip)}\ \frac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\pi_1(\,\langle b, c\rangle\,) = b \in B} \qquad \text{C}_2\text{-ip)}\ \frac{b \in B \quad c \in C(b) \qquad C(x)\ prop\ \ [x \in B]}{\pi_2(\,\langle b, c\rangle\,) = c \in C(b)}$$

$$\eta\text{-ip)}\ \frac{d \in \Sigma_{x\in B}C(x)}{\langle \pi_1(\,d\,)\,,\ \pi_2(\,d\,)\,\rangle = d \in \Sigma_{x\in B}C(x)}$$

Strong Indexed Sum set

F-Σ) $$\frac{C(x)\ \ set\ [x \in B]}{\Sigma_{x \in B} C(x)\ set}$$ I-Σ) $$\frac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$$

E$_1$-Σ) $$\frac{d \in \Sigma_{x \in B} C(x)}{\pi_1(d) \in B}$$ E$_2$-Σ) $$\frac{d \in \Sigma_{x \in B} C(x)}{\pi_2(d) \in C(\pi_1(d))}$$

C$_1$-Σ) $$\frac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\pi_1(\ \langle b, c \rangle\ ) = b \in B}$$

C$_2$-Σ) $$\frac{b \in B \quad c \in C(b) \qquad B\ set \qquad C(x)\ set\ [x \in B]}{\pi_2(\ \langle b, c \rangle\ ) = c \in C(b)}$$

η-Σ) $$\frac{d \in \Sigma_{x \in B} C(x)}{\langle \pi_1(\,d\,)\,,\ \pi_2(\,d\,)\,\rangle = d \in \Sigma_{x \in B} C(x)}$$

9 Appendix: The extensional level emTT$_0$

As emTT to build types and terms of emTT$_0$ we use the same kinds of judgements used in mTT$_0$.
Contexts are generated by the same context rules of mTT$_0$.
Also here, the only change we do on emTT$_0$ with respect to emTT is to allow only strong indexed sums of propositional functions as generic collection constructors:

Strong Indexed Sum of a propositional function

F-ip) $$\frac{C(x)\ \ prop\ [x \in B]}{\Sigma_{x \in B} C(x)\ col}$$ I-ip) $$\frac{b \in B \quad c \in C(b) \quad C(x)\ \ prop\ [x \in B]}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$$

E-*ip*) $$\frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \\ d \in \Sigma_{x \in B} C(x) \quad m(x, y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) \in M(d)}$$

C-*ip*) $$\frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x \in B} C(x)] \qquad C(x)\ prop\ [x \in B] \\ b \in B \quad c \in C(b) \quad m(x, y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(\ \langle b, c \rangle, m\ ) = m(b, c) \in M(\langle b, c \rangle)}$$

Sets are generated as follows:

Empty set

F-Em) N_0 *set* E-Em) $$\frac{a \in \mathsf{N}_0 \quad A(x)\ \ col\ [x \in \mathsf{N}_0]}{\mathsf{emp_o}(a) \in A(a)}$$

Singleton set

S) N_1 *set* I-S) $\star \in \mathsf{N}_1$ E-S) $$\frac{t \in \mathsf{N}_1 \quad M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(t, c) \in M(t)}$$ C-S) $$\frac{M(z)\ col\ [z \in \mathsf{N}_1] \quad c \in M(\star)}{El_{\mathsf{N}_1}(\star, c) = c \in M(\star)}$$

Strong Indexed Sum set

F-Σ) $\dfrac{C(x)\ \mathit{set}\ [x \in B] \qquad B\ \mathit{set}}{\Sigma_{x \in B} C(x)\ \mathit{set}}$ I-Σ) $\dfrac{b \in B \quad c \in C(b) \qquad C(x)\ \mathit{set}\ [x \in B] \qquad B\ \mathit{set}}{\langle b, c \rangle \in \Sigma_{x \in B} C(x)}$

E-Σ) $\dfrac{\begin{array}{l} M(z)\ \mathit{type}\ [z \in \Sigma_{x \in B} C(x)] \\ d \in \Sigma_{x \in B} C(x) \quad m(x,y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)] \end{array}}{\mathit{El}_{\Sigma}(d, m) \in M(d)}$

C-Σ) $\dfrac{\begin{array}{l} M(z)\ \mathit{type}\ [z \in \Sigma_{x \in B} C(x)] \\ b \in B \quad c \in C(b) \quad m(x,y) \in M(\langle x, y \rangle)\ [x \in B, y \in C(x)] \end{array}}{\mathit{El}_{\Sigma}(\langle b, c \rangle, m) = m(b, c) \in M(\langle b, c \rangle)}$

List set

F-list) $\dfrac{C\ \mathit{set}}{\mathit{List}(C)\ \mathit{set}}$ I$_1$-list) $\dfrac{\mathit{List}(C)\ \mathit{set}}{\epsilon \in \mathit{List}(C)}$ I$_2$-list) $\dfrac{s \in \mathit{List}(C) \quad c \in C}{\mathsf{cons}(s, c) \in \mathit{List}(C)}$

E-list) $\dfrac{\begin{array}{c} L(z)\ \mathit{col}\ [z \in \mathit{List}(C)] \quad s \in \mathit{List}(C) \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in \mathit{List}(C), y \in C, z \in L(x)] \end{array}}{\mathit{El}_{\mathit{List}}(s, a, l) \in L(s)}$

C$_1$-list) $\dfrac{\begin{array}{c} L(z)\ \mathit{col}\ [z \in \mathit{List}(C)] \qquad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in \mathit{List}(C), y \in C, z \in L(x)] \end{array}}{\mathit{El}_{\mathit{List}}(\epsilon, a, l) = a \in L(\epsilon)}$

C$_2$-list) $\dfrac{\begin{array}{c} L(z)\ \mathit{col}\ [z \in \mathit{List}(C)] \quad s \in \mathit{List}(C) \quad c \in C \quad a \in L(\epsilon) \\ l(x,y,z) \in L(\mathsf{cons}(x,y))\ [x \in \mathit{List}(C), y \in C, z \in L(x)] \end{array}}{\mathit{El}_{\mathit{List}}(\mathsf{cons}(s,c), a, l) = l(s, c, \mathit{El}_{\mathit{List}}(s, a, l)) \in L(\mathsf{cons}(s, c))}$

Disjoint Sum set

F-+) $\dfrac{B\ \mathit{set} \quad C\ \mathit{set}}{B + C\ \mathit{set}}$ I$_1$-+) $\dfrac{b \in B \qquad B\ \mathit{set} \qquad C\ \mathit{set}}{\mathsf{inl}(b) \in B + C}$ I$_2$-+) $\dfrac{c \in C \qquad B\ \mathit{set} \qquad C\ \mathit{set}}{\mathsf{inr}(c) \in B + C}$

E-+) $\dfrac{\begin{array}{l} A(z)\ \mathit{col}\ [z \in B + C] \\ w \in B + C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{\mathit{El}_{+}(w, a_B, a_C) \in A(w)}$

C$_1$-+) $\dfrac{\begin{array}{l} A(z)\ \mathit{col}\ [z \in B + C] \\ b \in B \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{\mathit{El}_{+}(\mathsf{inl}(b), a_B, a_C) = a_B(b) \in A(\mathsf{inl}(c))}$

C$_2$-+) $\dfrac{\begin{array}{l} A(z)\ \mathit{col}\ [z \in B + C] \\ c \in C \quad a_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{\mathit{El}_{+}(\mathsf{inr}(c), a_B, a_C) = a_C(c) \in A(\mathsf{inr}(c))}$

Dependent Product set

$$\text{F-}\Pi)\ \frac{C(x)\ set\ [x \in B] \qquad B\ set}{\Pi_{x\in B}C(x)\ set} \qquad\qquad \text{I-}\Pi)\ \frac{c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\lambda x^{B}.c(x) \in \Pi_{x\in B}C(x)}$$

$$\text{E-}\Pi)\ \frac{b \in B \quad f \in \Pi_{x\in B}C(x)}{\mathsf{Ap}(f,b) \in C(b)}$$

$$\beta\text{C-}\Pi)\ \frac{b \in B \quad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\mathsf{Ap}(\lambda x^{B}.c(x), b) = c(b) \in C(b)}$$

$$\eta\text{C-}\Pi\ \ \frac{f \in \Pi_{x\in B}C(x)}{\lambda x^{B}.\mathsf{Ap}(f,x) = f \in \Pi_{x\in B}C(x)}(x\ not\ free\ in\ f)$$

Quotient set

$$\text{Q)}\ \frac{\begin{array}{ll} A\ set & R(x,y)\ \in \mathsf{props_s}\ [x \in A, y \in A] \\ & \mathsf{true} \in R(x,x)\ [x \in A] \\ \mathsf{Equiv}(R) & \mathsf{true} \in R(y,x)\ [x \in A,\ y \in A,\ u \in R(x,y)] \\ & \mathsf{true} \in R(x,z)\ [x \in A,\ y \in A,\ z \in A, \\ & \qquad\qquad\qquad u \in R(x,y),\ v \in R(y,z)] \end{array}}{A/R\ set}$$

$$\text{I-Q)}\ \frac{a \in A \qquad A/R\ set}{[a] \in A/R} \qquad\qquad \text{eq-}Q)\ \frac{a \in A \quad b \in A \quad \mathsf{true} \in R(a,b) \qquad A/R\ set}{[a] = [b] \in A/R}$$

$$\text{E-Q)}\ \frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p \in A/R \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(p,l) \in L(p)}$$

$$\text{C-Q)}\ \frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ a \in A \quad l(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x,y)] \end{array}}{El_Q(l,[a]) = l(a) \in L([a])}$$

Effectiveness

$$\text{eff)}\ \frac{a \in A \quad b \in A \quad [a] = [b] \in A/R \qquad A/R\ set}{\mathsf{true} \in R(a,b)}$$

emTT$_0$ propositions are mono, namely they are inhabited by at most a canonical proof-term:

$$\textbf{prop-mono})\ \frac{A\ prop \qquad p \in A \quad q \in A}{p = q \in A} \qquad\qquad \textbf{prop-true})\ \frac{A\ prop \qquad p \in A}{\mathsf{true} \in A}$$

Propositions are generated as follows:

Falsum

$$\text{F-Fs)}\ \ \bot\ prop \qquad\qquad \text{E-Fs)}\ \frac{\mathsf{true} \in \bot \quad A\ prop}{\mathsf{true} \in A}$$

Extensional Propositional Equality

$$\text{F-Eq)}\ \frac{C\ \ col \quad c \in C \quad d \in C}{\mathsf{Eq}(C,c,d)\ prop} \qquad \text{I-Eq)}\ \frac{c \in C}{\mathsf{true} \in \mathsf{Eq}(C,c,c)}$$

$$\text{E-Eq)}\ \frac{\mathsf{true} \in \mathsf{Eq}(C,c,d)}{c = d \in C} \qquad \text{C-Eq)}\ \frac{p \in \mathsf{Eq}(C,c,d)}{p = \mathsf{eq_C}(c) \in \mathsf{Eq}(C,c,d)}$$

Implication

$$\text{F-Im)}\ \frac{B\ \mathit{prop} \quad C\ \mathit{prop}}{B \to C\ \mathit{prop}} \qquad \text{I-Im)}\ \frac{\mathsf{true} \in C\ [x \in B] \qquad B\ \mathit{prop} \qquad C\ \mathit{prop}}{\mathsf{true} \in B \to C}$$

$$\text{E-Im)}\ \frac{\mathsf{true} \in B \quad \mathsf{true} \in B \to C}{\mathsf{true} \in C}$$

$$\beta\text{C-}\to\ \ \frac{B\ \mathit{prop} \qquad b \in B \quad c \in C\ [x \in B]}{\mathsf{Ap}_{\to}(\lambda_{\to} x^{B}.c, b) = c(b) \in C}$$

$$\eta\text{C-}\to\ \ \frac{f \in B \to C}{\lambda_{\to} x^{B}.\mathsf{Ap}_{\to}(f, x) = f}(x\ not\ free\ in\ f)$$

Conjunction

$$\text{F-}\wedge)\ \frac{B\ \mathit{prop} \quad C\ \mathit{prop}}{B \wedge C\ \mathit{prop}} \qquad \text{I-}\wedge)\ \frac{\mathsf{true} \in B \quad \mathsf{true} \in C \qquad B\ \mathit{prop} \qquad C\ \mathit{prop}}{\mathsf{true} \in B \wedge C}$$

$$\text{E}_{\mathbf{1}}\text{-}\wedge)\ \frac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in B} \qquad \text{E}_{\mathbf{2}}\text{-}\wedge)\ \frac{\mathsf{true} \in B \wedge C}{\mathsf{true} \in C}$$

Disjunction

$$\text{F-}\vee)\ \frac{B\ \mathit{prop} \quad C\ \mathit{prop}}{B \vee C\ \mathit{prop}} \qquad \text{I}_1\text{-}\vee)\ \frac{\mathsf{true} \in B \qquad B\ \mathit{prop} \qquad C\ \mathit{prop}}{\mathsf{true} \in B \vee C} \qquad \text{I}_2\text{-}\vee)\ \frac{\mathsf{true} \in C \qquad B\ \mathit{prop} \qquad C\ \mathit{prop}}{\mathsf{true} \in B \vee C}$$

$$\text{E-}\vee)\ \frac{A\ \mathit{prop} \qquad \mathsf{true} \in B \vee C \quad \mathsf{true} \in A\ [x \in B] \quad \mathsf{true} \in A\ [y \in C]}{\mathsf{true} \in A}$$

$$\text{C}_1\text{-}\vee)\ \ \frac{\begin{array}{l} A\ \mathit{prop} \\ b \in B \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inl}_{\vee}(b), a_B, a_C) = a_B(b) \in A}$$

$$\text{C}_2\text{-}\vee)\ \ \frac{\begin{array}{l} A\ \mathit{prop} \\ c \in C \quad a_B(x) \in A\ [x \in B] \quad a_C(y) \in A\ [y \in C] \end{array}}{El_{\vee}(\mathsf{inr}_{\vee}(c), a_B, a_C) = a_C(c) \in A}$$

$$\eta\text{-}\vee\ \ \frac{t \in A\ [z \in C + D]}{El_{\vee}(z, (x)t(\mathsf{inl}_{\vee}(x)), (y)t(\mathsf{inr}_{\vee}(x))) = t(z) \in A}$$

Existential quantification

$$\text{F-}\exists)\ \frac{C(x)\ \mathit{prop}\ \ [x \in B]}{\exists_{x \in B} C(x)\ \mathit{prop}} \qquad \text{I-}\exists)\ \frac{b \in B \quad \mathsf{true} \in C(b) \qquad C(x)\ \mathit{prop}\ \ [x \in B]}{\mathsf{true} \in \exists_{x \in B} C(x)}$$

$$\text{E-}\exists)\ \frac{M\ \mathit{prop} \qquad \mathsf{true} \in \exists_{x \in B} C(x) \quad \mathsf{true} \in M\ [x \in B, y \in C(x)]}{\mathsf{true} \in M}$$

$$\text{C-}\exists)\ \frac{\begin{array}{l} M\ \mathit{prop} \\ b \in B \quad c \in C(b) \quad \mathsf{true} \in M\ [x \in B, y \in C(x)] \end{array}}{El_{\exists}(\langle b,_{\exists} c\rangle, m) = m(b, c) \in M}$$

Universal quantification

$$\text{F-}\forall)\ \frac{C(x)\ \mathit{prop}\ [x\in B]}{\forall_{x\in B}C(x)\ \mathit{prop}} \qquad \text{I-}\forall)\ \frac{\mathsf{true}\in C(x)\ [x\in B] \qquad C(x)\ \mathit{prop}\ \ [x\in B]}{\mathsf{true}\in \forall_{x\in B}C(x)}$$

$$\text{E-}\forall)\ \frac{b\in B\quad \mathsf{true}\in \forall_{x\in B}C(x)}{\mathsf{true}\in C(b)} \qquad \beta\text{C-}\forall)\ \frac{b\in B\quad c(x)\in C(x)\ [x\in B]}{\mathsf{Ap}_{\forall}(\lambda_{\forall}x^{B}.c(x),b)=c(b)\in C(b)}$$

$$\eta\text{C-}\forall)\ \frac{f\in \forall_{x\in B}C(x)}{\lambda_{\forall}x^{B}.\mathsf{Ap}_{\forall}(f,x)=f\in \forall_{x\in B}C(x)}$$

As in mTT_0, small propositions are generated as follows:

$$\bot\ \mathit{prop}_s \qquad \frac{B\ \mathit{prop}_s\quad C\ \mathit{prop}_s}{B\vee C\ \mathit{prop}_s} \qquad \frac{B\ \mathit{prop}_s \qquad C\ \mathit{prop}_s}{B\rightarrow C\ \mathit{prop}_s} \qquad \frac{B\ \mathit{prop}_s \qquad C\ \mathit{prop}_s}{B\wedge C\ \mathit{prop}_s}$$

$$\frac{C(x)\ \mathit{prop}_s\ [x\in B] \qquad B\ \mathit{set}}{\exists_{x\in B}C(x)\in \mathit{prop}_s} \qquad \frac{C(x)\ \mathit{prop}_s\ [x\in B] \qquad B\ \mathit{set}}{\forall_{x\in B}C(x)\ \mathit{prop}_s} \qquad \frac{A\ \mathit{set}\quad a\in A\quad b\in A}{\mathsf{Eq}(A,a,b)\ \mathit{prop}_s}$$

Contrary to mTT_0, in emTT_0 we do not have the intensional collection of small propositions but the quotient of the collection of small propositions under equiprovability representing the power collection of the singleton:

Power collection of the singleton

$$\text{F-P)}\ \ \mathcal{P}(1)\ \mathit{col} \qquad \text{I-P)}\ \frac{B\ \mathit{prop}_s}{[B]\in\mathcal{P}(1)} \qquad \text{eq-P)}\ \frac{\mathsf{true}\in B\leftrightarrow C}{[B]=[C]\in\mathcal{P}(1)} \qquad \text{eff-P)}\ \frac{[B]=[C]\in\mathcal{P}(1)}{\mathsf{true}\in B\leftrightarrow C}$$

$$\frac{U\in\mathcal{P}(1) \qquad V\in\mathcal{P}(1)}{\mathsf{Eq}(\mathcal{P}(1),U,V)\ \mathit{prop}_s} \qquad \eta\text{-P)}\ \frac{U\in\mathcal{P}(1)}{U=[\,\mathsf{Eq}(\mathcal{P}(1),U,[\mathsf{tt}])\,]}$$

where $\mathsf{tt}\equiv\bot\rightarrow\bot$ represents the truth constant.

Then, we have also function collections from a set toward $\mathcal{P}(1)$:

Function collection to $\mathcal{P}(1)$

$$\text{F-Fc)}\ \frac{B\ \mathit{set}}{B\rightarrow\mathcal{P}(1)\ \mathit{col}} \qquad \text{I-Fc)}\ \frac{c(x)\in\mathcal{P}(1)\ [x\in B] \qquad B\ \mathit{set}}{\lambda x^{B}.c(x)\in B\rightarrow\mathcal{P}(1)}$$

$$\text{E-Fc)}\ \frac{b\in B\quad f\in B\rightarrow\mathcal{P}(1)}{\mathsf{Ap}(f,b)\in\mathcal{P}(1)} \qquad \beta\text{C-Fc)}\ \frac{b\in B\quad c(x)\in\mathcal{P}(1)\ [x\in B] \qquad B\ \mathit{set}}{\mathsf{Ap}(\lambda x^{B}.c(x),b)=c(b)\in\mathcal{P}(1)}$$

$$\eta\text{C-Fc)}\ \frac{f\in B\rightarrow\mathcal{P}(1)}{\lambda x^{B}.\mathsf{Ap}(f,x)=f\in B\rightarrow\mathcal{P}(1)}(x\ \mathit{not\ free\ in}\ f)$$

Then, as in mTT_0 we add the embedding rules of sets into collections **set-into-col**, of propositions into collections **prop-into-col**, of small propositions into sets **prop$_s$-into-set** and of small propositions into propositions **prop$_s$-into-prop**.
Moreover, we also add the equality rules ref), sym), tra) both for types and for terms saying that type and term equalities are equivalence relations, and the rules conv), conv-eq).
Contrary to mTT_0, we add all the equality rules about collections and sets saying that their constructors preserve type equality as follows:

Strong Indexed Sum Collection-eq

$$\text{eq-}ip)\ \frac{C(x) = D(x)\ prop\ \ [x \in B] \qquad B = E\ col}{\Sigma_{x\in B}C(x) = \Sigma_{x\in E}D(x)\ col}$$

Function collection-eq

$$\text{eq-Fc)}\ \frac{B = E\ set}{B \to \mathcal{P}(1) = E \to \mathcal{P}(1)\ col}$$

Lists-eq

$$\text{eq-list)}\ \frac{C = D\ set}{List(C) = List(D)\ set}$$

Strong Indexed Sum set-eq

$$\text{eq-}\Sigma)\ \frac{C(x) = D(x)\ set\ \ [x \in B] \qquad B = E\ set}{\Sigma_{x\in B}C(x) = \Sigma_{x\in E}D(x)\ set}$$

Disjoint Sum-eq

$$\text{eq-+)}\ \frac{B = D\ set \quad C = E\ set}{B + C = D + E\ set}$$

Dependent Product-eq

$$\text{eq-}\Pi)\ \frac{C(x) = D(x)\ set\ [x \in B] \qquad B = E\ set}{\Pi_{x\in B}C(x) = \Pi_{x\in E}D(x)\ set}$$

Quotient set-eq

$$\text{eq-Q)}\ \frac{A = B\ set \qquad R(x,y) = S(x,y)\ \ prop_s\ [x \in A, y \in A] \qquad \mathsf{Equiv}(R) \qquad \mathsf{Equiv}(S)}{A/R = B/S\ \ set}$$

Then, emTT$_0$ includes the following equality rules about propositions:

Disjunction-eq

$$\text{eq-}\vee)\ \frac{B = D\ prop \quad C = E\ prop}{B \vee C = D \vee E\ prop}$$

Implication-eq

$$\text{eq-}\to)\ \frac{B = D\ prop \qquad C = E\ prop}{B \to C = D \to E\ prop}$$

Conjunction-eq

$$\text{eq-}\wedge)\ \frac{B = D\ prop \qquad C = E\ prop}{B \wedge C = D \wedge E\ prop}$$

Propositional equality-eq

$$\text{eq-Eq)}\ \frac{A = E\ col \quad a = e \in A \quad b = c \in A}{\mathsf{Eq}(A, a, b) = \mathsf{Eq}(E, e, c)\ prop}$$

Existential quantification-eq

$$\text{eq-}\exists)\ \frac{C(x) = D(x)\ prop\ \ [x \in B] \qquad B = E\ col}{\exists_{x\in B}C(x) = \exists_{x\in E}D(x)\ prop}$$

Universal quantification-eq

$$\text{eq-}\forall)\ \frac{C(x) = D(x)\ prop\ [x \in B] \qquad B = E\ col}{\forall_{x\in B}C(x) = \forall_{x\in E}D(x)\ prop}$$

Analogously, we add eq-$\vee$), eq-$\to$), eq-$\wedge$), eq-Eq), eq-$\exists$), eq-$\forall$) restricted to small propositions. Moreover, equality of propositions is that of collections, that of small propositions coincides with that of $\mathsf{prop_s}$ and is that of propositions and that of sets:

$$\textbf{prop-into-col eq)}\ \frac{A = B\ prop}{A = B\ col} \qquad \textbf{prop}_s\textbf{-eq1)}\ \frac{A = B\ prop_s}{A = B \in \mathsf{prop_s}} \qquad \textbf{prop}_s\textbf{-eq2)}\ \frac{A = B \in \mathsf{prop_s}}{A = B\ prop_s}$$

$$\textbf{prop}_s\textbf{-into-prop eq)}\ \frac{A = B\ prop_s}{A = B\ prop} \qquad \textbf{prop}_s\textbf{-into-set eq)}\ \frac{A = B\ prop_s}{A = B\ set}$$

Equality of sets is that of collections:

$$\textbf{set-into-col eq)}\ \frac{A = B\ set}{A = B\ col}$$

Contrary to mTT$_0$, also for terms we add equality rules saying that all the constructors preserve equality as in [NPS90]:

$$\text{I-eq }\Sigma)\ \frac{b = b' \in B \quad c = c' \in C(b) \qquad C(x)\ set\ [x \in B] \qquad B\ set}{\langle b, c\rangle = \langle b', c'\rangle \in \Sigma_{x\in B}C(x)}$$

$$\text{E-eq }\Sigma)\ \frac{\begin{array}{l} M(z)\ col\ [z \in \Sigma_{x\in B}C(x)] \\ d = d' \in \Sigma_{x\in B}C(x) \quad m(x,y) = m'(x,y) \in M(\langle x, y\rangle)\ [x \in B, y \in C(x)] \end{array}}{El_{\Sigma}(d, m) = El_{\Sigma}(d', m') \in M(d)}$$

$$\text{E-eq Em})\ \frac{a = a' \in \mathsf{N_0} \quad A(x)\ col\ [x \in \mathsf{N_0}]}{\mathsf{emp_o}(a) = \mathsf{emp_o}(a') \in A(a)} \qquad \text{E-eq S})\ \frac{t = t' \in \mathsf{N_1} \quad M(z)\ col\ [z \in \mathsf{N_1}] \quad c = c' \in M(\star)}{El_{\mathsf{N_1}}(t, c) = El_{\mathsf{N_1}}(t', c') \in M(t)}$$

$$\text{I}_2\text{-eq list})\ \frac{s = s' \in List(C) \quad c = c' \in C}{\mathsf{cons}(s, c) = \mathsf{cons}(s', c') \in List(C)}$$

$$\text{E-eq list})\ \frac{\begin{array}{l} L(z)\ col\ [z \in List(C)] \quad s = s' \in List(C) \quad a = a' \in L(\epsilon) \\ l(x, y, z) = l'(x, y, z) \in L(\mathsf{cons}(x, y))\ [x \in List(C), y \in C, z \in L(x)] \end{array}}{El_{List}(s, a, l) = El_{List}(s', a', l') \in L(s)}$$

$$\text{I-eq Q})\ \frac{a = a' \in A \qquad A/R\ set}{[a] = [a'] \in A/R}$$

$$\text{E-eq Q})\ \frac{\begin{array}{l} L(z)\ col\ [z \in A/R] \\ p = p' \in A/R \quad l(x) = l'(x) \in L([x])\ [x \in A] \quad l(x) = l(y) \in L([x])\ [x \in A, y \in A, d \in R(x, y)] \end{array}}{El_Q(p, l) = El_Q(p', l') \in L(p)}$$

$$\text{I}_1\text{-eq +})\ \frac{b = b' \in B \qquad B\ set \qquad C\ set}{\mathsf{inr}(b) = \mathsf{inr}(b') \in B + C} \qquad \text{I}_2\text{-eq +})\ \frac{c = c' \in C \qquad B\ set \qquad C\ set}{\mathsf{inl}(c) = \mathsf{inl}(c') \in B + C}$$

$$\text{E-eq +})\ \frac{\begin{array}{l} A(z)\ col\ [z \in B + C] \\ d = d' \in B + C \quad a_B(x) = a'_B(x) \in A(\mathsf{inl}(x))\ [x \in B] \quad a_C(y) = a'_C(y) \in A(\mathsf{inr}(y))\ [y \in C] \end{array}}{El_{+}(d, a_B, a_C) = El_{+}(d', a'_B, a'_C) \in A(w)}$$

$$\text{I-eq }\Pi)\ \frac{c(x) = c'(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in \Pi_{x\in B}C(x)} \qquad \text{E-eq }\Pi)\ \frac{b = b' \in B \quad f = f' \in \Pi_{x\in B}C(x)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in C(b)}$$

$$\text{I-eq Fc})\ \frac{c(x) = c'(x) \in \mathcal{P}(1)\ [x \in B] \quad B\ set}{\lambda x^B.c(x) = \lambda x^B.c'(x) \in B \to \mathcal{P}(1)} \qquad \text{E-eq Fc})\ \frac{b = b' \in B \quad f = f' \in B \to \mathcal{P}(1)}{\mathsf{Ap}(f, b) = \mathsf{Ap}(f', b') \in \mathcal{P}(1)}$$

Analogously, we define I-eq ip), E-eq ip) for indexed sum collections of propositional functions as I-eq Σ) and E-eq Σ).
Note that I-eq Π) is the so-called ξ-rule in [Mar75].