Subperiodic Trigonometric Hyperinterpolation

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Dedicated to Ian H. Sloan on the occasion of his 80th birthday.

Abstract Using recent results on subperiodic trigonometric Gaussian quadrature and the construction of subperiodic trigonometric orthogonal bases, we extend Sloan's notion of hyperinterpolation to trigonometric spaces on subintervals of the period. The result is relevant, for example, to function approximation on spherical or toroidal rectangles.

1 Introduction

Trigonometric approximation in the absence of periodicity has been investigated along some apparently parallel paths: the theory of *Fourier Extensions* (also known as Fourier Continuations in certain applications), cf. [1, 2, 4, 6, 9, 27], the theory of *subperiodic* trigonometric interpolation and quadrature, e.g. [8, 12, 13, 14, 15, 40], and the recent developments of *nonperiodic* trigonometric approximation [39] and of *mapped* polynomial approximation [3]. It is also worth mentioning the study of monotone trigonometric approximations, which can arise only in the subperiodic setting, cf. [29].

Fourier Extensions emerged in the context of trigonometric approximation of nonperiodic functions, for example as a tool for circumventing the Gibbs phenomenon. Summarizing, a smooth nonperiodic real-valued function f defined in $[-1,1]$ is approximated by the restriction to $[-1,1]$ of a trigonometric polynomial

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periodic on $[-T,T]$, $T>1$,

$$\tau \in \mathscr{T}_n = \mathrm{span}\{1, \cos(k\pi u/T), \sin(k\pi u/T), 1 \le k \le n\ ,\ u \in [-T,T]\ ,\ T>1\}\ , \quad (1)$$

by solving the minimization problem

$$\min_{\tau \in \mathscr{T}_n} \|\tau - f\|_{L^2_\mu([-1,1])}\ , \quad (2)$$

where either μ is the Lebesgue measure on $[-1,1]$, or a discrete measure supported at a discrete set $X \subset [-1,1]$. In the first case one speaks of a "continuous Fourier extension", and in the second of a "discrete Fourier extension". When $\mathrm{card}(X) = 2n+1 = \dim(\mathscr{T}_n)$, the construction of the discrete Fourier extension becomes an interpolation problem.

A number of deep theoretical results have been obtained on Fourier extensions, concerning different aspects such as: choice of T, choice of interpolation nodes for discrete extensions, approximation power (with tight error estimates for analytic functions), resolution power on higly oscillatory functions, numerical stability; cf. e.g. [1, 2]. The main computational approach has been the solution of the least squares/interpolation linear systems working with the standard trigonometric basis (1), which leads to strongly ill-conditioned systems. Nevertheless, Fourier extensions are actually numerically stable when implemented in finite arithmetic, cf. [2] for an explanation of the phenomenon (a different approach has been recently proposed in [30]).

Let us now summarize the parallel path of *subperiodic trigonometric approximation*. In several recent papers, subperiodic trigonometric interpolation and quadrature have been studied, i.e., interpolation and quadrature formulas exact on

$$\mathbb{T}_n([-\omega,\omega]) = \mathrm{span}\{1, \cos(k\theta), \sin(k\theta), 1 \le k \le n\ ,\ \theta \in [-\omega,\omega]\}\ , \quad (3)$$

where $\mathbb{T}_n([-\omega,\omega])$ denote the $(2n+1)$-dimensional space of trigonometric polynomials restricted to the interval $[-\omega,\omega]$, $0<\omega\le\pi$; cf. e.g. [8, 14], and [12, 13] for the construction and application of subperiodic trigonometric Gaussian formulas. All these formulas are related by the simple nonlinear transformation

$$\theta(t) = 2\arcsin(\sin(\omega/2)t)\ ,\ t \in [-1,1]\ , \quad (4)$$

with inverse

$$t(\theta) = \frac{\sin(\theta/2)}{\sin(\omega/2)}\ ,\ \theta \in [-\omega,\omega]\ , \quad (5)$$

to polynomial interpolation and quadrature on $[-1,1]$, and have been called "subperiodic" since they concern subintervals of the period of trigonometric polynomials.

For example, trigonometric interpolation and quadrature by the transformed zeros of the $(2n+1)$-th Chebyshev polynomial $T_{2n+1}(t)$ has been studied in [8]. Moreover, in [14] stability of such Chebyshev-like subperiodic trigonometric interpolation has been studied, proving that its Lebesgue constant does not depend on ω and increases logarithmically in the degree.

In this article, we apply the following subperiodic quadrature [13].

Proposition 1. *Let* $\{(\xi_j,\lambda_j)\}_{1\leq j\leq n+1}$, *be the nodes and positive weights of the algebraic Gaussian quadrature formula on* $(-1,1)$ *induced by the weight function*

$$W_{-1/2}(t) = \frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)\,t^2}}\;,\;\; t\in(-1,1)\;. \tag{6}$$

Then

$$\int_{-\omega}^{\omega} f(\theta)\,d\theta = \sum_{j=1}^{n+1}\lambda_j f(\varphi_j)\;,\;\; \forall f\in\mathbb{T}_n([-\omega,\omega])\;,\;\; 0<\omega\leq\pi\;, \tag{7}$$

where

$$\varphi_j = 2\arcsin(\sin(\omega/2)\xi_j)\in(-\omega,\omega)\;,\;\; j=1,2,\ldots,n+1\;. \tag{8}$$

It is worth recalling that the key role played by the transformation (6) on subintervals of the period was also recognized in [7, E.3, p. 235], and more recently in [40], in the context of trigonometric polynomial inequalities. On the other hand, such a transformation (already introduced in [28]), is at the base of the recent studies on nonperiodic trigonometric approximations [39], and on “mapped” polynomial approximations [3].

The initial motivation of [8, 13] for the analysis of subperiodic interpolation and quadrature was different from that of Fourier extensions or other similar studies, since it arised from multivariate applications. The key observation is that a multivariate polynomial restricted to an arc of a circle becomes a univariate subperiodic trigonometric polynomial in the arclength. Then, multivariate polynomials on domains defined by circular arcs, such as sections of disk (circular sectors, segments, zones, lenses, lunes) and surface/solid sections of sphere, cylinder, torus (rectangles, collars, caps, slices) become in the appropriate coordinates elements of tensor-product spaces of univariate trigonometric and algebraic polynomials, where the angular variables are restricted to a subinterval of the period. This entails that approximation in polynomial spaces and in such product spaces are intimately related; cf. e.g. [12, 13, 36]. On the other hand, product approximations are simpler to construct.

Now, a closer look at Fourier extensions and subperiodic trigonometric approximation shows that we are speaking essentially of the same problem, and that the results obtained in one framework can be fruitfully adpated to the other one. Indeed, by the change of variables

$$\theta=\omega u\;,\;\; \omega=\frac{\pi}{T}\;,\;\; u\in[-1,1]\;, \tag{9}$$

we can immediately translate an extension problem into a subperiodic problem, and conversely.

In this paper we focus on a semi-discrete approximation in the subperiodic setting, namely *hyperinterpolation*, that is an orthogonal projection onto $\mathbb{T}_n([-\omega,\omega])$, discretized by means of the Gaussian quadrature formula of proposition 1 for exactness degree $2n$. In view of (9), this can be seen as a kind of discrete Fourier extension.

In order to generate such orthogonal projections, we need some theoretical tools that are recalled in Sections 2 and 3, that are: the extension of the notion of hyperinterpolation, originally introduced by Sloan in the seminal paper [35] for multivariate total-degree polynomial spaces, to a more general class of spaces, that we term *hyperinterpolation spaces*; the construction of a *subperiodic orthogonal* trigonometric basis. Moreover, in Section 3 we also discuss the main computational issues related to subperiodic orthogonality. Finally, in Section 4 we discuss the implementation of subperiodic trigonometric hyperinterpolation, together with examples and applications.

2 Hyperinterpolation Spaces

Hyperinterpolation is a powerful tool for total-degree polynomial approximation of multivariate continuous functions, introduced by Sloan in the seminal paper [35]. In brief, it corresponds to a truncated Fourier expansion in a series of orthogonal polynomials for some measure on a given multidimensional domain, where the Fourier coefficients are discretized by a positive algebraic cubature formula. Since then, theoretical as well as computational aspects of hyperinterpolation as an alternative to interpolation have attracted much interest, due to the intrinsic difficulties in finding good multivariate interpolation nodes, with special attention to the case of the sphere; cf., e.g., [17, 25, 26, 41, 42].

In order to generalize the notion of hyperinterpolation, we introduce the idea of *nested hyperinterpolation spaces*. We shall denote by $C(K)$ the space of real-valued continuous functions on a compact set $K \subset \mathbb{R}^d$.

Let $\{S_n\}$ be an increasing sequence of finite-dimensional subspaces $S_n \subset S_{n+1} \subset C(K)$, $n \geq 0$. Moreover, assume that

(i) if $u \in S_n$ and $v \in S_m$ then $uv \in S_{n+m}$;

(ii) the subalgebra $S = \bigcup_{n\geq 0} S_n$ is dense in $C(K)$ with respect to the uniform norm (by the Stone-Weierstrass theorem, if $\overline{S}$ contains the constants the latter is equivalent to the fact that $\overline{S}$ separates points in K, i.e., for every $x \in K$ there exist $u, v \in \overline{S}$ such that $u(x) \neq v(x)$; cf. e.g. [34]);

(iii) we know a sequence of positive quadrature rules $\mu_n = \{(X, \boldsymbol{w})\}$, $n \geq 0$, with nodes $X = \{x_j\}$ and weights $\boldsymbol{w} = \{w_j\}$, $1 \leq j \leq M$, that are exact in S_{2n} for a measure μ on K (for notational convenience we do not display the fact that the nodes X, the weights $\boldsymbol{w}$ and the cardinality M depend on n)

$$\int_K f(x)\, d\mu = \sum_{j=1}^{M} w_j f(x_j)\ ,\ \ \forall f \in S_{2n}\ . \tag{10}$$

Let $\{u_j\}$ be a μ-orthonormal basis of S_n, i.e. $S_n = \mathrm{span}\{u_1, ..., u_N\}$, $N = N_n = \dim(S_n)$ and

$$(u_i, u_j)_\mu = \int_K u_i(x) u_j(x)\, d\mu = \delta_{ij}. \tag{11}$$

Observe that such an orthonormal basis always exists by the Gram-Schmidt process applied to a given basis of S_n, that in view of (iii) can be performed using either the scalar product $(f,g)_\mu$ or equivalently its discrete counterpart $(f,g)_{\mu_n}$ defined below in (12). It is also worth observing that a quadrature formula like (10) always exists, by a generalized version of Tchakaloff theorem on positive algebraic formulas (whose proof however is not constructive, so that in pratice we have to get the formula in some other way); cf. [10] for the polynomial setting and [5, Thm. 5.1] for the extension to more general spaces.

Consider the “discrete inner product” in $C(K)$ (cf. (iii))

$$(f,g)_{\mu_n} = \sum_{j=1}^{M} w_j f(x_j) g(x_j) \tag{12}$$

together with the corresponding seminorm $\|f\|_{\ell^2_{\mathbf{w}}(X)} = \sqrt{(f,f)_{\mu_n}}$, and define the discrete orthogonal projection $\mathscr{L}_n : C(K) \to S_n$

$$\mathscr{L}_n f(x) = \sum_{i=1}^{N} (f, u_i)_{\mu_n} u_i(x) \ , \tag{13}$$

which solves the discrete weighted least squares problem

$$\|f - \mathscr{L}_n f\|_{\ell^2_{\mathbf{w}}(X)} = \min_{u \in S_n} \|f - u\|_{\ell^2_{\mathbf{w}}(X)} \ . \tag{14}$$

This construction was originally proposed in polynomial spaces by Sloan [35] with the name of “hyperinterpolation”, namely for $S_n = \mathbb{P}^d_n(K)$ (the space of total-degree polynomials in d real variables of degree not exceeding n, resticted to a compact set or manifold K).

All the relevant properties of the hyperinterpolation operator hold true also in our more general setting. We do not give the proofs, since “mutatis mutandis” they follow exactly the lines of those in [35]. We only observe that a key fact is the concidence of the original 2-norm and the discrete weighted 2-norm in S_n, in view of (i) and (iii), that is

$$\|u\|_{L^2_\mu(K)} = \|u\|_{\ell^2_{\mathbf{w}}(X)} \ , \ \ \forall u \in S_n \ . \tag{15}$$

It is worth collecting some of the most relevant features of hyperinterpolation in the following

Proposition 2. *(cf. [35] for the polynomial case), The hyperinterpolation operator $\mathscr{L}_n$ defined in (13) has the following properties:*

- *$M \geq \dim\left(S_n|_{supp(\mu)}\right)$ and if the equality holds, then $\mathscr{L}_n f$ interpolates f at the quadrature nodes;*
- *for every $f \in C(K)$*

$$\|\mathscr{L}_n f\|_{L^2_\mu(K)} \leq \sqrt{\mu(K)}\, \|f\|_{L^\infty(K)} \; ; \tag{16}$$

- *the $L^2_\mu(K)$ error can be estimated as*

$$\|f - \mathscr{L}_n f\|_{L^2_\mu(K)} \leq 2\sqrt{\mu(K)}\, E_{S_n}(f;K) \; , \tag{17}$$

  where $E_{S_n}(f;K) = \inf_{u \in S_n} \left\{ \|f-u\|_{L^\infty(K)} \right\}$.

Observe that by the density of the subalgebra $S = \bigcup_{n \geq 0} S_n$, (17) implies $L^2_\mu(K)$-convergence of the sequence of hyperinterpolants, since density is equivalent to $E_{S_n}(f;K) \to 0$, $n \to \infty$. To study $L^\infty(K)$-convergence, an estimate of uniform norm of the operator is needed, along with a Jackson-like theorem for approximation in S_n. The former can be obtained, in general, by estimating in the specific context the reciprocal Christoffel function

$$K_n(x,x) = \sum_{i=1}^{N} u_i^2(x) \; , \tag{18}$$

(i.e., the diagonal of the reproducing kernel $K_n(x,y) = \sum_{i=1}^{N} u_i(x) u_i(y)$ which does not depend on the orthonormal basis), as shown in [16] for total-degree polynomials. Indeed, one can easily prove that

$$\sup_{S_n \ni u \neq 0} \frac{\|u\|_{L^\infty(K)}}{\|u\|_{L^2_\mu(K)}} = \sqrt{\max_{x \in K} K_n(x,x)} \; , \tag{19}$$

and consequently by (16)

$$\begin{aligned} \|\mathscr{L}_n\| = \sup_{f \neq 0} \frac{\|\mathscr{L}_n f\|_{L^\infty(K)}}{\|f\|_{L^\infty(K)}} &\leq \sqrt{\max_{x \in K} K_n(x,x)}\, \sup_{f \neq 0} \frac{\|\mathscr{L}_n f\|_{L^2_\mu(K)}}{\|f\|_{L^\infty(K)}} \\ &\leq \sqrt{\mu(K) \max_{x \in K} K_n(x,x)} \; , \end{aligned} \tag{20}$$

cf. [16, Prop. 1.1 and Cor. 1.2].

On the other hand, by the representation

$$\mathscr{L}_n f(x) = \sum_{i=1}^{N} \left(\sum_{j=1}^{M} w_j u_i(x_j) f(x_j) \right) u_i(x) = \sum_{j=1}^{M} f(x_j) w_j K_n(x,x_j) \; , \tag{21}$$

we can obtain an explicit expression for the uniform norm of the hyperinterpolation operator

$$\|\mathscr{L}_n\| = \max_{x\in K} \sum_{j=1}^{M} w_j \left|K_n(x,x_j)\right| . \tag{22}$$

Observe that when $M = N = \dim\left(S_n|_{supp(\mu)}\right)$ and thus $\mathscr{L}_n$ is interpolant, the functions

$$\ell_j(x) = w_j K_n(x,x_j) \ , \ \ j = 1,\dots,N \ , \tag{23}$$

are the cardinal functions of interpolation at X in S_n, i.e. $\ell_j(x_k) = \delta_{jk}$, and $\|\mathscr{L}_n\|$ plays the role of the Lebesgue constant of polynomial interpolation.

We stress finally that (20) is usually an overestimate of the actual norm (22). Tighter estimates can be obtained on specific geometries and functional settings, see e.g. the case of the ball [41] and the cube [42] in the polynomial framework.

3 Subperiodic Orthogonality

In this paper we wish to apply the generalized notion of hyperinterpolation of Section 2 in the subperiodic trigonometric framework, namely $S_n = \mathbb{T}_n([-\omega,\omega])$, $0 < \omega \leq \pi$. To this end, since the trigonometric Gaussian quadrature formula (7) is at hand (cf. (iii)), we need to find a subperiodic orthonormal basis.

We can now state and prove the following Proposition (see [11] for a preliminary version)

Proposition 3. *An orthonormal basis in $L^2(-\omega,\omega)$ for the $(2n+1)$-dimensional space $\mathbb{T}_n([-\omega,\omega])$ is given by the trigonometric polynomials*

$$\tau_i(\theta) = \tau_i(\theta,\omega) \ , \ \ i = 0,1,...,2n \ , \tag{24}$$

$$\tau_{2k}(\theta) = p_{2k}\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right) \ , \ \ k = 0,\dots,n \tag{25}$$

where $\{p_j\}_{j\geq 0}$ are the algebraic orthonormal polynomials with respect to the weight function $W_{-1/2}(t)$ in (6) and

$$\tau_{2k-1}(\theta) = \cos(\theta/2)\, q_{2k-1}\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right) \ , \ \ k = 1,\dots,n \tag{26}$$

where $\{q_j\}_{j\geq 0}$ are the algebraic orthonormal polynomials with respect to the weight function

$$W_{1/2}(t) = 2\sin(\omega/2)\sqrt{1-\sin^2(\omega/2)t^2} \ , \ \ t \in (-1,1) \ . \tag{27}$$

Proof. First, observe that by basic trigonometric identities the functions τ_{2k} are a basis for the even trigonometric polynomials, whereas the functions τ_{2k-1} are a basis for the odd trigonometric polynomials. Indeed, an even power of a sine is a combination of cosines with frequencies up to the exponent, whereas an odd power of a sine is a combination of sines with frequencies up to the exponent, and $\cos(\theta/2)\sin(j\theta/2) = \frac{1}{2}\sin((j+1)\theta/2) + \sin((j-1)\theta/2)$ is a trigonometric polynomial of degree $(j+1)/2$ for odd j.

From the definition and the change of variable (4) it follows directly that τ_i has unit L^2-norm for even i, and that τ_i and τ_j with even i and j, $i \neq j$, are mutually orthogonal, whereas when i is even and j odd, or conversely, they are mutually orthogonal since their product is an odd function. To show that they are orthonormal also for odd i and j, write

$$\int_{-\omega}^{\omega} \tau_i(\theta)\tau_j(\theta)\,d\theta = \int_{-\omega}^{\omega} q_i\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right) q_j\left(\frac{\sin(\theta/2)}{\sin(\omega/2)}\right)\cos^2(\theta/2)\,d\theta\ .$$

By the change of variable $\theta = 2\arcsin(\sin(\omega/2)t)$ we get

$$\int_{-\omega}^{\omega} \tau_i(\theta)\tau_j(\theta)\,d\theta = \int_{-1}^{1} q_i(t)q_j(t)\,(1-\sin^2(\omega/2)t^2)\,\frac{2\sin(\omega/2)}{\sqrt{1-\sin^2(\omega/2)t^2}}\,dt$$

$$= \int_{-1}^{1} q_i(t)q_j(t)\,W_{1/2}(t)\,dt = \delta_{ij}\ . \quad \square$$

$\square$

We discuss now how to implement the computation of the subperiodic orthogonal basis of proposition 3, since this is the base for an algorithm that constructs subperiodic trigonometric hyperinterpolants.

Concerning univariate algebraic orthogonal polynomials, one of the most comprehensive and reliable tools is the Matlab OPQ suite by Gautschi [22]. For example, inside OPQ one finds the routine `chebyshev`, that computes in a stable way the recurrence coefficients for orthogonal polynomials with respect to a given measure by the modified Chebyshev algorithm, as soon as the modified Chebyshev moments (i.e., the moments of the Chebyshev basis with respect to the given measure) are known. As shown in [21, 22], the modified Chebyshev algorithm computes the recurrence coefficients for the orthogonal polynomials up to degree $k+1$ using the modified Chebyshev moments up to degree $2k+1$ (in our application $k = 2n$ is needed).

Our first step is then to compute the modified Chebyshev moments up to degree $2n$ for the weight functions $W_{-1/2}$ in (6) and $W_{1/2}$ in (27), namely

$$m_{2j} = 2\sin(\omega/2)\int_{-1}^{1} T_{2j}(t)\,(1-\sin^2(\omega/2)t^2)^{\beta}\,dt\ ,\quad j = 0,\ldots,2n\ , \tag{28}$$

where $\beta = -1/2$ or $\beta = 1/2$; observe that only the even moments have to be computed, since the weight functions are even and thus the odd moments vanish.

More generally, in [33] it is proved that the sequence of moments

$$I_{2j}(\alpha,\beta,s) = \int_{-1}^{1} T_{2j}(t)\,(\alpha+s^2+t^2)^{\beta}\,dt\ , \tag{29}$$

where $\alpha \in \mathbb{C}$, $s, \beta \in \mathbb{R}$, satisfies the recurrence relation

$$\left(\frac{1}{4} + \frac{\beta+1}{2(2j+1)}\right) I_{2j+2} + \left(\frac{1}{2} + \alpha^2 + s^2 - \frac{\beta+1}{4j^2-1}\right) I_{2j}$$

$$+ \left(\frac{1}{4} - \frac{\beta+1}{2(2j-1)}\right) I_{2j-2} = -\frac{2(1+\alpha^2+s^2)^{\beta+1}}{4j^2-1}\ ,\ \ j \geq 1\ . \qquad (30)$$

Now, setting $\alpha = i/\sin(\omega/2)$ where i is the imaginary unit ($i^2 = -1$), $s = 0$ and $\beta = \pm 1/2$, we have that $m_{2j} = 2\sin^{2\beta+1}(\omega/2)(-1)^{-\beta} I_{2j}(i/\sin(\omega/2), \beta, 0)$, from which follows that m_{2j} satisfies the recurrence relation

$$a_j m_{2j+2} + b_j m_{2j} + c_j m_{2j-2} = d_j\ ,\ \ j \geq 1\ , \qquad (31)$$

where

$$a_j = \left(\frac{1}{4} + \frac{\beta+1}{2(2j+1)}\right)\ ,\ \ b_j = \left(\frac{1}{2} - \frac{1}{\sin^2(\omega/2)} - \frac{\beta+1}{4j^2-1}\right)\ ,$$

$$c_j = \left(\frac{1}{4} - \frac{\beta+1}{2(2j-1)}\right)\ ,\ \ d_j = 4\,\frac{\cos^{2\beta+2}(\omega/2)}{\sin(\omega/2)}\,\frac{1}{4j^2-1}\ . \qquad (32)$$

Such a recurrence is however unstable, namely small errors on the starting values grow very rapidly increasing j. In order to stabilize it we have adopted the method that solves instead a linear system, with tridiagonal diagonally dominant matrix (of the recurrence coefficients) and the vector $(d_1 - c_1 m_0, d_2, \ldots, d_{2n-2}, d_{2n-1} - a_{2n-1} m_{4n})$ as right-hand side; cf. [11, 20]. We get immediately that $m_0 = 2\omega$ for $\beta = -1/2$ and $m_0 = \omega + \sin(\omega)$ for $\beta = 1/2$, whereas the last moment m_{4n} can be computed accurately in both cases by the `quadgk` Matlab function (adaptive Gauss-Kronrod quadrature).

Since the `chebyshev` routine, starting from the modified Chebyshev moments, returns the recurrence coefficients for the *monic* orthogonal polynomials, we have to modify the recurrence relation in the standard way (cf. [21, Thm. 1.29]) to get the orthonormal polynomials $\{p_{2k}\}$ for $W_{-1/2}$ and $\{q_{2k-1}\}$ for $W_{1/2}$, from which we compute the orthonormal subperiodic trigonometric basis $\{\tau_0, \ldots, \tau_{2n}\}$ as in proposition 3.

On the other hand, it turns out numerically (in double precision) that there is a moderate loss of orthogonality when n increases. Defining the Vandermonde-like matrix

$$V = V_n(\Theta, \omega) = (v_{ij}) = (\tau_{j-1}(\varphi_i))\ ,\ \ 1 \leq i, j \leq 2n+1\ , \qquad (33)$$

where $\Theta = \{\varphi_i\}$ are the $2n+1$ quadrature nodes for trigonometric exactness degree $2n$ (see proposition 1), we can then measure numerical orthogonality of the basis $\{\tau_{j-1}\}$ by computing $\varepsilon_n = \|(\sqrt{\Lambda}V)^t(\sqrt{\Lambda}V) - I\|_2$, where $\Lambda = \mathrm{diag}(\lambda_i)$ is the diagonal matrix of the quadrature weights. For example, we get $\varepsilon_{250} \approx 2 \cdot 10^{-13}$. In order to recover orthogonality at machine precision, it is sufficient to re-orthogonalize the basis by

$$\sqrt{\Lambda}\,V = QR\;,\;\;(\hat{\tau}_0(\theta),\dots,\hat{\tau}_{2n}(\theta)) = (\tau_0(\theta),\dots,\tau_{2n}(\theta))R^{-1}\;. \qquad (34)$$

In such a way, we can eventually compute in a stable way the orthonormal trigonometric basis of proposition 3. All the relevant codes are available in the Matlab package `HYPERTRIG` [38]. Notice that we could not have applied the QR based orthonormalization directly to the Vandermonde matrix in the canonical trigonometric basis (3), since such a matrix turns out to be extremely ill-conditioned already at moderate values of n for $\omega < \pi$. On the contrary, the Vandermonde-like matrix $\sqrt{\Lambda}V$ in (34) being quasi-orthogonal has a condition number very close to 1, and thus $Q = \sqrt{\Lambda}\,VR^{-1}$ is orthogonal at machine precision.

4 Subperiodic (Hyper)interpolation

By the tools developed in the previous sections, we can now construct a subperiodic trigonometric hyperinterpolation operator, whose properties are an immediate consequence of proposition 2 with $S_n = \mathbb{T}_n([-\omega,\omega])$. Notice that property (i) of hyperinterpolation spaces is immediate (n being the trigonometric degree) and concerning (ii) it is not difficult to show that subperiodic trigonometric polynomials separate points.

Corollary 1. *Consider the subperiodic trigonometric hyperinterpolation operator* $\mathscr{L}_n : C([-\omega,\omega]) \to \mathbb{T}_n([-\omega,\omega])$*,* $0 < \omega \le \pi$*, defined as*

$$\mathscr{L}_n f(\theta) = \sum_{i=0}^{2n} (f,\tau_i)_{\mu_n}\,\tau_i(\theta)\;,\;\;\theta \in [-\omega,\omega]\;,\;\;(f,\tau_i)_{\mu_n} = \sum_{j=1}^{2n+1} \lambda_j f(\varphi_j)\tau_i(\varphi_j)\;, \qquad (35)$$

where $\{\tau_i\}$ *is the orthonormal basis of proposition 3 and* $\{(\varphi_j,\lambda_j)\}$ *are the nodes and weights of the subperiodic Gaussian formula of proposition 1 for degree* $2n$*.*

Then, the following properties hold

- $\mathscr{L}_n f$ *interpolates* f *at the* $2n+1 = \dim(\mathbb{T}_n([-\omega,\omega]))$ *quadrature nodes*

$$\mathscr{L}_n f(\varphi_j) = f(\varphi_j)\;,\;\;1 \le j \le 2n+1\;; \qquad (36)$$

- *the* L^2*-error can be estimated as*

$$\|f - \mathscr{L}_n f\|_{L^2([-\omega,\omega])} \le 2\sqrt{2\omega}\,E_{\mathbb{T}_n([-\omega,\omega])}(f)\;. \qquad (37)$$

We can now give an estimate of the uniform norm of the hyperinterpolation operator, which we may call its Lebesgue constant since by Corollary 1 it is an interpolation operator.

Corollary 2. *The Lebesgue constant of the hyperinterpolation operator* $\mathscr{L}_n$ *of Corollary 1 can be estimated as*

$$\|\mathscr{L}_n\| \leq C_n \sim 2\sqrt{\pi}\, n + \frac{4}{\sqrt{3}}\, n^{3/2} \,,$$

$$C_n = \sqrt{\pi}(2n+1) + \sqrt{\frac{(2n+1)(2n+2)(4n+3)}{3}} \,. \tag{38}$$

Proof. In view of (20) with $K=[-\omega,\omega]$ and proposition 3, we are reduced to estimate the trigonometric reciprocal Christoffel function. Setting $t(\theta)=\sin(\theta/2)/\sin(\omega/2)\in[-1,1]$, we have

$$K_n(\theta,\theta) = \sum_{i=0}^{2n} \tau_i^2(\theta) = \sum_{even\ i} p_i^2(t(\theta)) + \cos^2(\theta/2) \sum_{odd\ i} q_i^2(t(\theta))$$

$$\leq \sum_{i=0}^{2n} p_i^2(t(\theta)) + \sum_{i=0}^{2n} q_i^2(t(\theta)) = \lambda_{2n}^{-1}(t(\theta);W_{-1/2}) + \lambda_{2n}^{-1}(t(\theta);W_{1/2}) \,, \tag{39}$$

where $\lambda_m^{-1}(t;W)$ denotes the reciprocal Christoffel function for algebraic degree n and weight function $W\in L_+^1(-1,1)$.

Now, in view of the basic property of monotonicity of Christoffel functions with respect to the underlying measure (cf., e.g., [31, Ch. 6]), we have that $\lambda_m^{-1}(t;W)$ is decreasing in W, in the sense that if $W_1\geq W_2$ a.e., then $\lambda_m^{-1}(t;W_1)\leq\lambda_m^{-1}(t;W_2)$. Observing that $W_{1/2}(t)\geq 2\sin(\omega/2)\sqrt{1-t^2}$ and $W_{-1/2}(t)\geq 2\sin(\omega/2)$, $t\in(-1,1)$, we get that the corresponding reciprocal Christoffel functions are bounded, up to a scaling by the factor $(2\sin(\omega/2))^{-1}$, by the reciprocal Christoffel functions of the Chebyshev measure of the second kind (sum of squares of the Chebyshev polynomials of the second kind) and of the Lebesgue measure (sum of squares of the Legendre polynomials), respectively. By well-known estimates for such reciprocal Christoffel functions (cf., e.g., [16]), we get

$$\lambda_{2n}^{-1}(t;W_{-1/2}) \leq \frac{(2n+1)^2}{4\sin(\omega/2)} \,,\ \ \lambda_{2n}^{-1}(t;W_{1/2}) \leq \frac{(2n+1)(2n+2)(4n+3)}{6\pi\sin(\omega/2)} \,, \tag{40}$$

and thus

$$\|\mathscr{L}_n\| \leq \sqrt{2\omega \max_{\theta\in[-\omega,\omega]} K_n(\theta,\theta)}$$

$$\leq 2\sqrt{\frac{\omega/2}{\sin(\omega/2)}} \left(\frac{2n+1}{\sqrt{2}} + \sqrt{\frac{(2n+1)(2n+2)(4n+3)}{6\pi}}\right) \,, \tag{41}$$

from which (38) follows since the function $y/\sin(y)$ is increasing and bounded by $\pi/2$ for $y=\omega/2\in[0,\pi/2]$. □ □

Even though (38) is clearly an overestimate of the actual growth, it provides a bound independent of ω and shows that the Lebesgue constant is slowly increasing with n. By (38) and the fact that $\mathscr{L}_n$ is a projection operator, we easily get the uniform error estimate

$$\|f-\mathscr{L}_n f\|_{L^\infty([-\omega,\omega])} \le (1+C_n)\, E_{\mathbb{T}_n([-\omega,\omega])}(f)\ ,\ \ \forall f\in C([-\omega,\omega])\ . \tag{42}$$

In Figure 1, we have plotted the Lebesgue constant computed numerically by (22) on a fine control grid in $[-\omega,\omega]$ for some values of ω, that is

$$\Lambda_n(\omega)=\|\mathscr{L}_n\|=\max_{\theta\in[-\omega,\omega]}\sum_{j=1}^{2n+1}|\ell_j(\theta)|\ , \tag{43}$$

where

$$\ell_j(\theta)=\lambda_j K_n(\theta,\varphi_j)\ ,\ \ 1\le j\le 2n+1\ ,\ \ K_n(\theta,\phi)=\sum_{i=0}^{2n}\tau_i(\theta)\tau_i(\phi)\ . \tag{44}$$

Observe that the Lebesgue constant appears to be decreasing in ω for fixed n and to converge to the Lebesgue constant of algebraic interpolation of degree $2n$ at the Gauss-Legendre nodes (for $\omega\to 0$), which as known is $\mathscr{O}(\sqrt{n})$ (upper solid line). This also shows that the bound (38) is a large overestimate of the actual values. It is also worth observing that for $\omega=\pi$ the Lebesgue constant is exactly that of algebraic interpolation of degree $2n$ at the Gauss-Chebyshev nodes, that is logarithmic in n, cf. [14].

Based on these and other numerical experiments that we do not report for brevity, we can then make the following

Conjecture 1. The Lebesgue constant $\Lambda_n(\omega)$ of trigonometric hyperinterpolation in $[-\omega,\omega]$, $0<\omega\le\pi$, cf. (35) and (43), is a decreasing function of ω for fixed degree. Moreover, its limit for $\omega\to 0$ (that is its supremum being bounded by (38)) is the Lebesgue constant of algebraic interpolation of degree $2n$ at the Gauss-Legendre nodes.

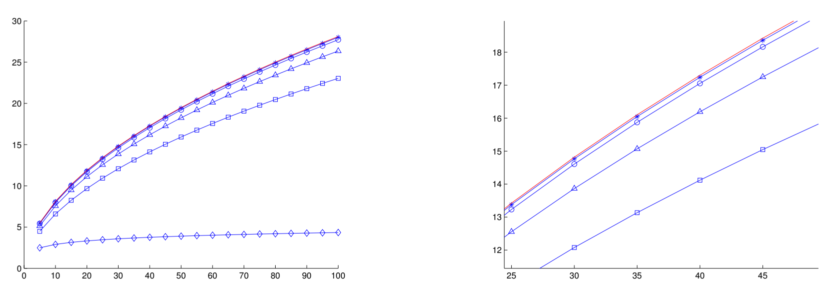

Fig. 1 the Lebesgue constant (43) for $n=5,10,\dots,95,100$ at some values of ω: from below, $\omega=\pi$ (diamonds), $\omega=3\pi/4$ (squares), $\omega=\pi/2$ (triangles), $\omega=\pi/4$ (circles), $\omega=\pi/8$ (asterisks); right: detail for $n=25,\dots,45$. The upper solid line is the Lebesgue constant of algebraic interpolation of degree $2n$ at the Gauss-Legendre nodes.

In order to show the performance of subperiodic trigonometric hyperinterpolation, in Figure 2 we have reported the errors in the uniform norm on six test functions with different regularity for some values of ω, namely

$$f_1(\theta) = (2+\cos(\theta)+\sin(\theta))^{30}\ ,\ \ f_2(\theta) = \exp(-\theta^2)\ ,\ \ f_3(\theta) = \exp(-5\theta^2)\ ,$$

$$f_4(\theta) = \frac{1}{1+25(\theta/\omega)^2}\ ,\ \ f_5(\theta) = |\theta|^{5/2}\ ,\ \ f_6(\theta) = (\omega-\theta)^{5/2}\ , \tag{45}$$

that are a positive trigonometric polynomial, two Gaussians centered at $\theta = 0$, a Runge-like function, a function with a singularity of the third derivative at $\theta = 0$ (where the nodes do not cluster), and one with with a singularity of the third derivative at $\theta = \omega$ (where the nodes cluster). The errors are measured in the relative ℓ_2-norm on a fine control grid in $[-\omega, \omega]$.

We see that convergence on the smooth functions f_1, f_2, f_3, f_4 (that are analytic) is faster by decreasing the interval length (in particular, the trigonometric polynomial f_1 is recovered at machine precision below the theoretical exactness degree $n = 30$), whereas the interval length has a substantial effect only at low degrees on the singular functions f_5, f_6 (observe that the clustering of sampling nodes at the singularity entails a faster convergence for f_6 compared to f_5).

The dependence of the convergence rate on ω for analytic functions is not surprising. Indeed, interpreting subperiodic trigonometric hyperinterpolation as a (discretized) Fourier extension, as discussed in the Introduction (cf. (2)-(9)), we may resort to deep convergence results in that theory. For example, in [1, Thm. 2.3] it is proved that the uniform convergence rate of Fourier extensions, and thus also that of the best trigonometric approximation in the relevant space if one uses (42), for "sufficiently" analytic functions (complex singularities not "too close" to the approximation interval) is (at least) exponential with order $\mathscr{O}(E(T)^{-n})$, where $E(T) = \cot^2(\pi/(4T)) = \cot^2(\omega/4)$ is a decreasing function of ω and $E(T) \to +\infty$ as $\omega \to 0^+$. This might give an explanation of the error behavior observed in Figure 2. We do not pursue further this aspect and refer the reader to [1] for a complete discussion on the convergence features of Fourier extensions.

To conclude, we observe that we can easily extend all the constructions above to any angular interval $[\alpha, \beta] \ni \theta$, $\beta - \alpha \le 2\pi$, by the change of variable

$$\theta' = \theta - \frac{\alpha+\beta}{2} \in [-\omega, \omega]\ ,\ \ \omega = \frac{\beta-\alpha}{2}\ , \tag{46}$$

namely using the orthonormal basis

$$t_i(\theta) = t_i(\theta, \alpha, \beta) = \tau_i(\theta', \omega)\ ,\ \ 0 \le i \le 2n\ , \tag{47}$$

and the subperiodic Gaussian quadrature formula with nodes and weights

$$\{(\theta_j, \lambda_j)\}\ ,\ \ \theta_j = \varphi_j + \frac{\alpha+\beta}{2}\ ,\ \ 1 \le j \le 2n+1\ , \tag{48}$$

cf. (35).

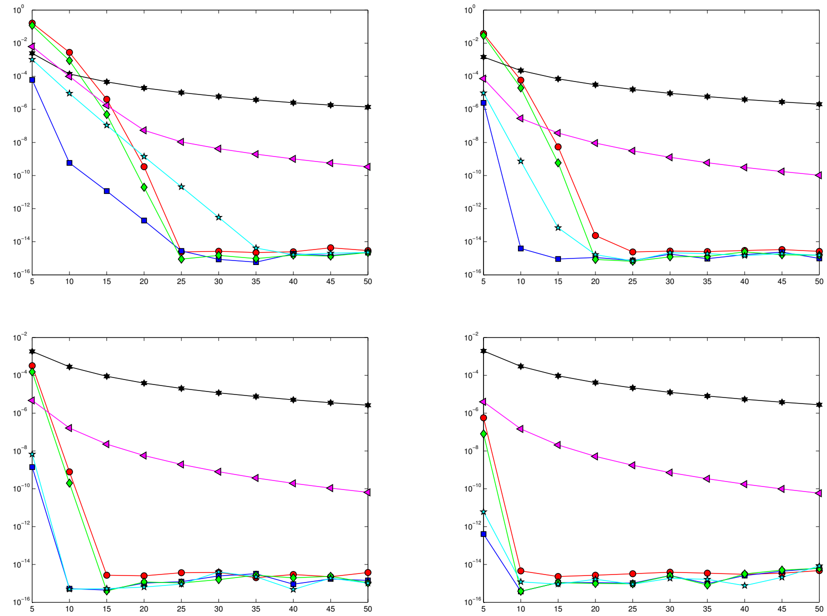

Fig. 2 Relative ℓ^2-errors of subperiodic trigonometric hyperinterpolation for degrees $n = 5, 10, \dots, 50$ on the test functions (45) on $[-\omega, \omega]$ with $\omega = 3\pi/4$ (top-left), $\omega = \pi/2$ (top-right), $\omega = \pi/4$ (bottom-left) and $\omega = \pi/8$ (bottom-right): f_1 (bullets), f_2 (squares), f_3 (diamonds), f_4 (stars), f_5 (asterisks), f_6 (triangles).

5 Product (Hyper)interpolation on Spherical Rectangles

The general setting of Section 2 allows to extend immediately subperiodic trigonometric hyperinterpolation to the tensor-product case. Indeed, consider the product basis

$$\{u_i(\theta)v_j(\phi)\}\ ,\ \ 0 \le i, j \le 2n\ ,\ \ (\theta, \phi) \in K = I_1 \times I_2 = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\ , \qquad (49)$$

where $u_i(\theta) = t_i(\theta, \alpha_1, \beta_1)$ and $v_j(\phi) = t_j(\phi, \alpha_2, \beta_2)$, cf. (47). Clearly (49) is a L^2-orthonormal basis of the tensor-product subperiodic trigonometric space

$$S_n = \mathbb{T}_n(I_1) \otimes \mathbb{T}_n(I_2)\ . \qquad (50)$$

Then, we can construct the product hyperinterpolant of $f \in C(I_1 \times I_2)$ as

$$\mathscr{L}_n f(\theta, \phi) = \sum_{i,j=0}^{2n} c_{ij}\, u_i(\theta) v_j(\phi)\ , \qquad (51)$$

with

$$c_{ij} = \sum_{h,k=1}^{2n+1} \lambda_{h1}\lambda_{k2} f(\theta_{h1},\theta_{k2})\, u_i(\theta_{h1}) v_j(\theta_{k2}) \;, \tag{52}$$

where $\{(\theta_{is},\lambda_{is})\}$ are the angular nodes and weights of the subperiodic trigonometric Gaussian formula on I_s, $s=1,2$, for exactness degree $2n$ (cf. proposition 1 and (48)).

All the relevant properties of hyperinterpolation apply, in particular the hyperinterpolant is a (product) interpolant (see also [32]). Moreover, the Lebesgue constant is the product of the one-dimensional constants, $\Lambda_n(\omega_1,\omega_2)=\Lambda_n(\omega_1)\Lambda_n(\omega_2)$, and the following error estimates hold

$$\|f-\mathscr{L}_n f\|_{L^2(I_1\times I_2)} \le 4\sqrt{\omega_1\omega_2}\, E_{S_n}(f;I_1\times I_2) \;,$$

$$\|f-\mathscr{L}_n f\|_{L^\infty(I_1\times I_2)} \le (1+\Lambda_n(\omega_1)\Lambda_n(\omega_2))\, E_{S_n}(f;I_1\times I_2) \;. \tag{53}$$

Concerning the implementation of subperiodic product hyperinterpolation, this can be constructed in a simple matrix form working on grids. Indeed, let

$$V_1=(u_{j-1}(\theta_{i1}))\;,\;\; V_2=(v_{j-1}(\theta_{i2}))\;,\;\; D_s=diag(\lambda_{is})\;,\;\; s=1,2\;,$$

$$F=(f(\theta_{i1},\theta_{j2}))\;,\;\; 1\le i,j\le 2n+1\;, \tag{54}$$

be the univariate Vandermonde-like matrices at the hyperinterpolation nodes in I_s, the diagonal matrices of the quadrature weights and the matrix of the function values at the bivariate hyperinterpolation grid $\{\theta_{i1}\}\times\{\theta_{j2}\}$, respectively. Moreover, let

$$U_1=(u_{j-1}(\hat{\theta}_\ell))\;,\;\; U_2=(v_{j-1}(\hat{\phi}_t))\;,\;\; 1\le \ell\le m_1\;,\;\; 1\le t\le m_2\;, \tag{55}$$

be univariate Vandermonde-like matrices at the points $\{\hat{\theta}_\ell\}\subset I_1$ and $\{\hat{\phi}_t\}\subset I_2$, respectively.

Then, it is easy to check that the hyperinterpolation coefficient matrix and the values of the hyperinterpolant at the target grid $\{\hat{\theta}_\ell\}\times\{\hat{\phi}_t\}$ can be computed by matrix products as

$$C=(c_{ij})=V_1^t D_1 F D_2 V_2\;,\;\; L=(\mathscr{L}_n f(\hat{\theta}_\ell,\hat{\phi}_t))=U_1 C U_2^t\;. \tag{56}$$

Subperiodic product hyperinterpolation via (56) has been implemented in the Matlab package [38].

Subperiodic product hyperinterpolation can be used, for example, to recover functions on spherical rectangles, in applications that require local approximation models. It is worth recalling that hyperinterpolation-like trigonometric approximation on the whole sphere, with applications in scattering theory, has been studied, e.g., in [18, 19].

Consider the spherical coordinates

$$(x,y,z)=\sigma(\theta,\phi)=(\cos(\theta)\sin(\phi),\sin(\theta)\sin(\phi),\cos(\phi))\;, \tag{57}$$

where θ is the azimuthal angle and ϕ the polar angle, $(\theta,\phi) \in [-\pi,\pi] \times [0,\pi]$, and a "geographic rectangle", that is

$$\Omega = \sigma(I_1 \times I_2)\ ,\ \ I_1 = [\alpha_1,\beta_1] \subseteq [-\pi,\pi]\ ,\ \ I_2 = [\alpha_2,\beta_2] \subseteq [0,\pi]\ . \tag{58}$$

Now, take a function $g \in C(\Omega)$, that we can identify with a continuous function in $I_1 \times I_2$ as $f(\theta,\phi) = g(\sigma(\theta,\phi))$. In order to estimate the hyperinterpolation errors in (53), we can observe that if $p \in \mathbb{P}_n^3(\Omega)$ then $p \circ \sigma \in S_n = \mathbb{T}_n(I_1) \otimes \mathbb{T}_n(I_2)$. Then, due to the surjectivity of the map σ, we have

$$\inf_{\psi \in S_n} \|f - \psi\|_{L^\infty(I_1 \times I_2)} \leq \inf_{p \in \mathbb{P}_n^3(\Omega)} \|f - p \circ \sigma\|_{L^\infty(I_1 \times I_2)}$$

$$= \inf_{p \in \mathbb{P}_n^3(\Omega)} \|g \circ \sigma - p \circ \sigma\|_{L^\infty(I_1 \times I_2)} = \inf_{p \in \mathbb{P}_n^3(\Omega)} \|g - p\|_{L^\infty(\Omega)}\ ,$$

that is

$$E_{S_n}(f; I_1 \times I_2) \leq E_{\mathbb{P}_n^3(\Omega)}(g;\Omega)\ . \tag{59}$$

Moreover, it is also clear that subperiodic product trigonometric hyperinterpolation reproduces total-degree polynomials, namely

$$\mathscr{L}_n(p \circ \sigma) = p \circ \sigma\ ,\ \forall p \in \mathbb{P}_n^3(\Omega)\ . \tag{60}$$

It is also worth observing that the orthonormal basis functions $\{u_i(\theta)v_j(\phi)\}$, and thus also the hyperinterpolant $\mathscr{L}_n f(\theta,\phi)$, correspond to continuous spherical functions on Ω, whenever Ω does not contain the north or south pole (i.e., $[\alpha_2,\beta_2] \subset (0,\pi)$ or in other words Ω is a nondegenerate rectangle).

To make an example, we have taken two geographic rectangles of the unit sphere. The first

$$\Omega_1 = \sigma\left(\left[-\frac{125}{180}\pi, -\frac{67}{180}\pi\right] \times \left[\frac{41}{180}\pi, \frac{65}{180}\pi\right]\right)\ ,\ \omega_{11} \approx 0.506\ ,\ \omega_{12} \approx 0.209\ , \tag{61}$$

corresponds in standard longitude-latitude to $67^o W$ - $125^o W$, $25^o N$ - $49^o N$, a vaste rectangle approximately corresponding to the contiguous continental USA, whereas the second

$$\Omega_2 = \sigma\left(\left[-\frac{109}{180}\pi, -\frac{102}{180}\pi\right] \times \left[\frac{49}{180}\pi, \frac{53}{180}\pi\right]\right)\ ,\ \omega_{21} \approx 0.061\ ,\ \omega_{22} \approx 0.035\ , \tag{62}$$

is the rectangle $102^o W$ - $109^o W$, $37^o N$ - $41^o N$, corresponding to Colorado.

In order to test the polynomial reproduction property, we have taken the positive test polynomials

$$p_n(x,y,z) = (ax + by + cz + 3)^n\ , \tag{63}$$

where a,b,c are random variables uniformly distributed in $[0,1]$. In Figure 3 we have reported the relative ℓ^2-errors (average of 100 samples) in the reconstruction of the polynomials by hyperinterpolation, computed on a 50×50 control grid. In particular, in Figure 3-right we see that the reconstruction of a fixed polynomial an overprecision phenomenon occurs, more pronounced with the smaller rectangle (where near exactness is obtained already at half the polynomial degree). This may be interpreted by the dependence on ω of the convergence rate of univariate subperiodic hyperinterpolation for analytic functions, as discussed above (cf. the convergence profile for f_1 in Figure 2).

It is worth stressing that the fact of being on a sphere is not essential, since similar results can be obtained for example also on rectangles of the torus, with angular intervals (in the usual poloidal-toroidal coordinates) of the same length of those in (61)-(62). On the other hand, subperiodic trigonometric hyperinterpolation could be useful also to construct mixed algebraic-trigonometric product formulas for solid sections of the sphere such as (truncated) spherical sectors with rectangular base, and also for planar circular sections, such as sectors, zones, lenses, lunes (via the transformations used in [12, 15]).

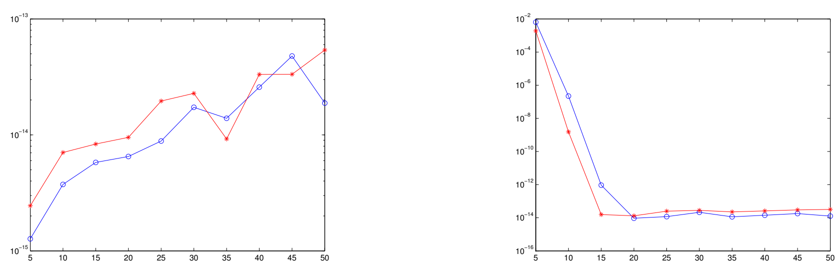

Fig. 3 Left: Average relative ℓ^2-errors of subperiodic trigonometric hyperinterpolation for degrees $n = 5, 10, \ldots, 50$ on the random test polynomials p_n in (63), with spherical rectangles corresponding to USA (circles) and Colorado (asterisks). Right: Average relative ℓ^2-errors in the reconstruction of the fixed polynomial p_{30}.

6 Comparison with Polynomial Hyperinterpolation

In a recent paper [24], hyperinterpolation on geographic rectangles has been studied and implemented in the usual hyperinterpolation setting of total-degree polynomials. As known, the dimension of the underlying polynomial space on the sphere is $(n+1)^2$ for degree n. Spherical harmonics, however, are no more an orthogonal basis on a portion of the sphere, so that a costly orthonormalization process has to be applied, based on the availability of algebraic quadrature formulas exact at degree $2n$ with $(2n+1)(2n+2)$ nodes and positive weights. Such an orthonormalization cost is much larger than the cost of the present approach, since here orthonormalization is univariate in the components. Moreover, a substantial drawback of polynomial hyperinterpolation on geographic rectangles is that the orthonormalization process suffers from severe ill-conditioning (of the relevant Vandermonde-like matrices) already at moderate degrees, cf. [24].

A good feature of polynomial hyperinterpolation is that we work by construction with continuous spherical functions, and this allows to have rectangles containing the north or south pole and even to work with spherical polar caps (with the appropriate transformation, cf. [24]). The number of sampling points on general rectangles is slightly bigger than that of the present approach, $(2n+1)(2n+2)$ versus $(2n+1)^2$, whereas the number of coefficients is smaller, namely $(n+1)^2$ versus $(2n+1)^2$. On the other hand, we expect smaller reconstruction errors from subperiodic product trigonometric hyperinterpolation, since we work here in a bigger space, indeed if $p \in \mathbb{P}_n^3(\Omega)$ then $p \circ \sigma \in S_n = \mathbb{T}_n(I_1) \otimes \mathbb{T}_n(I_2)$. Moreover, the hyperinterpolant is interpolant in the present context, whereas it is not in the polynomial case.

In order to make a numerical comparison of polynomial with superiodic product hyperinterpolation, we have considered the functions

$$g_1(P) = \exp\left(-5\|P-P_0\|_2^2\right)\,,\;\; g_2(P) = \|P-P_0\|_2^5\,,\;\; P=(x,y,z)\,, \tag{64}$$

on the two rectangles above corresponding to USA (61) and Colorado (62), $P_0 = (x_0, y_0, z_0)$ being the center of the rectangle (where the sampling points do not cluster). Notice that g_1 is smooth whereas g_2 has a singularity at P_0. In Figure 4 we have reported the relative ℓ^2-errors in the reconstruction of g_1 and g_2, computed on a 50×50 control grid. We see that subperiodic trigonometric hyperinterpolation is more accurate than polynomial hyperinterpolation (with essentially the same number of sampling points and a much lower computational cost).

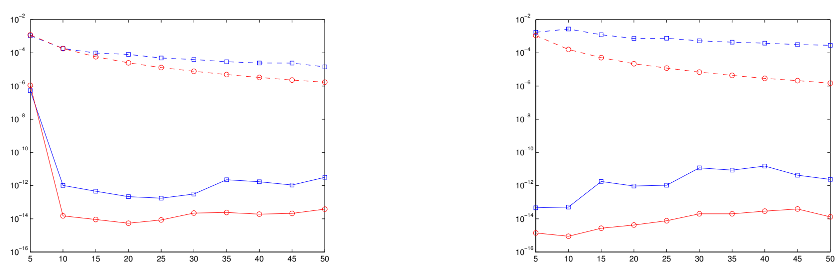

Fig. 4 Average relative ℓ^2-errors of polynomial (squares) and subperiodic trigonometric (circles) hyperinterpolation for degrees $n = 5, 10, \ldots, 50$ on the test functions g_1 (solid line) and g_2 (dashed line) in (64), with spherical rectangles corresponding to USA (left) and Colorado (right).

acknowledgements

Acknowledgements Supported by the Horizon 2020 ERA-PLANET European project "GEOEssential", by the EU project H2020-MSCA-RISE-2014-644175-MATRIXASSAY, by the DOR funds and by the biennial projects CPDA143275 and BIRD163015 of the University of Padova, and by the GNCS-INdAM.

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