Improving the stability of a discrete least-squares method for membrane eigenvalue problems

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Abstract In this article it is shown how to improve the numerical stability of a discrete least-squares method for computing eigenvalue approximations of the Laplace operator defined on standard two-dimensional domains. Among many sets of matching points the smallest condition numbers of the corresponding matrices have been obtained by using the Morrow-Patterson and Padua points in the square case, the Fekete points in the triangular case, and the Chebyshev disc points for the unit disc. The approximations of the first few vibration frequencies of a nonhomogeneous square membrane, and homogeneous circular and triangular membranes are presented.

Keywords Discrete least-squares method · Padua and Fekete points · Matrix conditioning · Membrane eigenfrequencies · Algebraic polynomials

1. Introduction

Through the years of research in the area of numerical methods for computing the eigenvalues of differential operators one can perceive a great diversity of methods for computing the eigenfrequencies of membranes and plates. Among these methods the valuable results have been obtained by the finite difference methods, Rayleigh-Ritz method, and point matching methods as reported in Kuttler and Sigillito [1], discrete singular convolution method [2, 3], differential quadrature method [4, 5], boundary element method [6, 7], and by the methods of fundamental solution [8, 9, 10], to name only a few recently published papers.

To achieve better numerical stability of the corresponding computational processes and higher accuracy of the resulting eigenvalue approximations these methods use different trial functions ranging from the classical orthogonal polynomials, Bessel functions, and splines up to the recently discovered globally and compactly supported radial basis functions [11, 12, 13]. Consequently, the new methods are usually more reliable, more accurate, and more efficient than the methods published previously. On the other hand, sometimes it is useful to come back to older forgotten techniques in order to improve their computational properties to make the old techniques more important for practice and more attractive for further theoretical investigation. One such forgotten method, the discrete least-squares method for vibration analysis of membranes and plates proposed in 1985 by Dong and Lopez [14], has been rediscovered in 1989 by Zhang and He [15] and in 1996 by the present author [16]. The main drawback of this method consist in a difficult construction of suitable matching points ensuring well-conditioned matrices and high precision results.

In this article we will try to answer the question how to improve numerical stability of the above mentioned method for computing the vibration frequencies of an inhomogeneous membrane with square shape and homogeneous membranes with circular and triangular shapes using carefully selected matching points and algebraic polynomials orthogonal on the considered domains.

2. Discrete least-squares method

Let us consider the membrane eigenvalue problem

$$-\Delta u(\mathbf{x}) = \lambda \rho(\mathbf{x}) u(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1}$$

subject to the homogeneous Dirichlet boundary conditions

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{2}$$

where Δ denotes the two-dimensional Laplace operator ($\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$), Ω is an open two-dimensional bounded domain with the Lipschitz boundary $\partial\Omega$ and $\rho(\mathbf{x})$ is a sufficiently smooth positive function for all $\mathbf{x} \in \Omega \cup \partial\Omega$. As is well known, such an eigenvalue problem possesses infinite number of positive eigenvalues with one accumulation point at infinity.

The membrane eigenvalue problem describes many real-world problems from structural mechanics, electrical engineering, biomechanics, nuclear reactor design, acoustics and quantum mechanics [17-24]. The usual interpretation of this problem comes from the structural mechanics and represents the free transverse vibration of a fixed homogeneous ($\rho(\mathbf{x}) \equiv 1$) and inhomogeneous ($\rho(\mathbf{x}) \not\equiv 1$) membrane with the eigenvalue $\lambda = k^2$, where k is proportional to a principal frequency of vibration, and the eigenfunction u represents the shape of the corresponding mode of vibration [17]. In the electrical engineering Eqs. (1)-(2) represents the propagation of TM modes along a hollow electrical waveguide filled with a homogeneous ($\rho(\mathbf{x}) \equiv 1$) and inhomogeneous ($\rho(\mathbf{x}) \not\equiv 1$) dielectric medium. Here k ($\lambda = k^2$) is proportional to a cutoff frequency of a waveguide with the cross-section Ω and the eigenfunction u is the shape of the corresponding TM mode [17]. The theory of membrane vibrations has been applied in the field of biomechanics to study the human eardrum and mitral valve of the human heart [18-21] which behave like a thin membrane. For the reactor designers it is necessary to estimate the acoustically induced vibration levels and the corresponding stresses of reactor fuel rods of gas-cooled nuclear reactors [22]. In acoustics the vibration problems of membranes describes the propagation of sound along an acoustical waveguide and the considered problems occur also in the study of membraneous musical instruments like drums [20]. Finally, the eigenvalue problem (1)-(2) is investigated in quantum mechanics representing the chaotic behavior of quantum billiards [23, 24].

To approximate a few smallest eigenvalues of the problem under consideration we will use the discrete least-squares method considered in [14-16]. The main idea behind this method is to find approximate eigenvalues λ_i^n and the corresponding approximate eigenfunctions $u_i^n(\mathbf{x})$ minimizing a residual error of Eqs. (1)-(2), where the squared residuals are evaluated and summed at a finite set of m matching points

$\mathbf{x}_k \in \Omega$, where $m > n$. For this purpose we will consider the approximate eigenfunctions $u^n(\mathbf{x})$ in the form $u^n(\mathbf{x}) = \sum_{j=1}^{n} c_j\varphi_j(\mathbf{x})$, where the trial functions $\varphi_j(\mathbf{x})$ satisfy the prescribed boundary conditions (2), and try to minimize the following discrete weighted least-squares residual error function

$$E_\lambda^{m,n}(\mathbf{c}) = \sum_{k=1}^{m} w_k \left(\sum_{j=1}^{n} c_j(-\Delta\varphi_j(\mathbf{x}_k) - \lambda\rho(\mathbf{x}_k)\varphi_j(\mathbf{x}_k)) \right)^2 .$$

This residual error function may be written in the matrix form

$$E_\lambda^{m,n}(\mathbf{c}) = \mathbf{c}^T(K - \lambda L)^T W(K - \lambda L)\mathbf{c}, \tag{3}$$

where the elements of the $m \times n$ matrices K, and L are defined as follows

$$K_{i,j} = -\Delta\varphi_j(\mathbf{x}_i), \quad L_{i,j} = \rho(\mathbf{x}_i)\varphi_j(\mathbf{x}_i).$$

Here the notations $W = diag(w_i)$, $\mathbf{c} = (c_1, c_2, ..., c_n)$ are used. In this article we will consider only the case $w_i = 1, 1 \leq i \leq m$. By virtue of the necessary condition for $E_\lambda^{m,n}(\mathbf{c})$ to have a minimum

$$\frac{\partial E_\lambda^{m,n}(\mathbf{c})}{\partial c_i} = 0, \quad i = 1, 2, ..., n,$$

the used least-squares approach results in the quadratic matrix eigenvalue problem

$$R\mathbf{c} - \lambda S\mathbf{c} + \lambda^2 T\mathbf{c} = 0, \tag{4}$$

where

$$R = K^T W K, \quad S = L^T W K + K^T W L, \quad T = L^T W L.$$

The quadratic eigenvalue problem (4) possesses n pairs of complex conjugate eigenvalues $\lambda_k^+ = \lambda_k^n + i\omega_k^n$ and $\lambda_k^- = \lambda_k^n - i\omega_k^n$ with the corresponding eigenvectors $c_k^+ = c_k^n + iw_k^n$ and $c_k^- = c_k^n - iw_k^n$. The usual solution of the quadratic eigenvalue problem (4) consists in the Frobenius linearization to transform (4) to a linear generalized eigenvalue problem of the order $2n$ [25]. The results presented in [16] show that the imaginary parts ω_k^n exhibit important property

$$\lim_{n\to\infty} \omega_k^n = 0 \tag{5}$$

and, consequently, one can consider the values λ_k^n to be the desired approximations of the exact eigenvalues λ_k. On the other hand, owing to this property, if ω_k^n approaches computer accuracy, the eigenvalue λ_k^n becomes defective eigenvalue of the resulting generalized eigenvalue problem with the algebraic multiplicity 2 and the geometric multiplicity 1. As shown by Chaitin-Chatelin and Frayseé [26], an ϵ perturbation of a defective matrix with a double defective eigenvalue causes a perturbation of the corresponding eigenvalue approximation proportional to $\epsilon^{1/2}$. As a result, a double defective eigenvalue may be computed at best to 7 - 8 digits of accuracy instead of the full 15 - 16 digits accuracy in double precision arithmetic. Fortunately, the resulting approximations λ_k^n and c_k^n are sufficiently accurate to be considered as initial approximations for the Gauss-Newton method used in order to minimize the modified residual error function

$$F_\alpha^{m,n}(\mathbf{c}, \lambda) = E_\lambda^{m,n}(\mathbf{c}) + \alpha(1 - \sum_{i=1}^{n} c_i)^2, \tag{6}$$

where the condition $\sum_{i=1}^{n} c_i = 1$ is imposed on $E_{\lambda}^{m,n}(\mathbf{c})$ to avoid the zero solution.

3. Numerical results

The main intention of this article is to discuss the questions concerning the numerical stability of the presented discrete least-squares method and stress its crucial dependence on the choice of matching points. We will show and comment extensive numerical experiments with different sets of matching points for the membrane eigenvalue problems defined on rectangular, circular, and triangular domains. Although the optimal choice of matching points for the discrete least-squares method is not considered in the literature, this problem is similar to the optimal choice of interpolation nodes for the polynomial interpolation (if $m = n$ these two approximations coincide) and will be used as a starting point in our discussion.

Let us denote by P_n the n-dimensional space of algebraic polynomials and consider n points $\mathbf{x}_i \in \Omega$ such that for an arbitrary function f there is a unique polynomial $p \in P_n$ with the property $f(\mathbf{x}_i) = p(\mathbf{x}_i), i = 1, 2, ..., n$, i.e., p is the interpolating polynomial of f. Denoting $p \equiv I_n f$ and $\| f \| = max_{\mathbf{x}\in\Omega} \mid f(\mathbf{x}) \mid$ the Lebesque lemma shows how well f may be approximated by its interpolating polynomial $I_n f$ in the maximum norm

$$\| f - I_n f \| \leq (1 + \Lambda_n) \| f - g \|,$$

where $g \in P_n$ denotes the best polynomial approximation of f and Λ_n is known as the Lebesque constant. The optimal interpolating points that minimize the Lebesque constant are called the Lebesque points. Unfortunately, as written in [27]:"*Almost nothing seems to be known about Lebesque points in more than one dimension. Nor are we aware of a feasible method for computing them numerically* ." On the other hand, there are another points, known as Fekete points, that may be considered as an alternative to the Lebesque points.

To determine the Fekete points for a domain Ω we need to select a basis φ_j for P_n. Consequently, the Fekete points for a domain Ω are those points $\mathbf{x}_i \in \Omega$ which maximize the determinant of the Vandermonde matrix V with the elements $V_{ij} = \varphi_j(\mathbf{x}_i)$. Clearly, the Lebesque and Fekete points need not be the optimal choice with respect to our requirement for the matrices R, S, T to have the smallest possible condition numbers. However, as will be seen in our numerical experiments, the Fekete points for square and triangular domains produce well-conditioned matrices and the suitable matching points for the circular domain can easily be determined using the information on the distribution pattern of these point sets.

3.1 Rectangular domain

Whereas the homogeneous membrane eigenvalue problem defined on the rectangular domain exhibits very simple solution, we will solve a transformed membrane eigenvalue problem which in its original form (1)-(2) is defined on a domain with complicated boundary shape. One such domain may be created transforming the square $S = (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ by the conformal mapping $w = \tan(z/2)$. Then the original membrane eigenvalue problem (1)-(2) defined on a circular domain with two opposite circular ridges [1] is equivalent to an inhomogeneous fixed membrane eigenvalue problem of the form

$$-\Delta u(x,y) = \frac{\lambda}{(\cos x + \cosh y)^2} u(x,y) \quad in \quad S \tag{7}$$

$$u(x,y) = 0 \quad on \quad \partial S \tag{8}$$

defined on the square S. More details concerning this conformal mapping technique can be found in [1, 28].

In the numerical solution of the eigenvalue problem (7)-(8) the trial functions

$$\varphi_{k,l}^{\tau}(x,y) = (\pi^2/4 - x^2)(\pi^2/4 - y^2) J_k^{\tau}(2x/\pi) J_l^{\tau}(2y/\pi), \tag{9}$$

and four sets of matching points have been used. The functions $J_k^{\tau}(x)$ denote the Jacobi polynomials orthogonal on $[-1, 1]$ with the weight function $(1 - x^2)^{2\tau}$. In Table 1 the values of the spectral condition numbers of the matrices R, S, T (computed by the MATLAB function *cond*) are shown in the logarithmic form for the equidistant points P_{ij}^{eq}, Fekete points P_{ij}^{Fe} [29], and the Padua points P_{ij}^{Pa} [30]. These points are defined as follows

$$P_{ij}^{eq} = [-\pi/2 + ih, -\pi/2 + jh], \quad 1 \le i, j \le m_1, \quad h = \pi/(m_1 + 1) \tag{10}$$

and

$$P_{ij}^{Fe} = [\pi x_i/2, \pi x_j/2], \quad 1 \le i, j \le m_1. \tag{11}$$

Here x_i are the zeros of the m_1-*th* Lobatto polynomial $Lob_{m_1}(t) \equiv L'_{m_1+1}(t)$, where $L_{m_1+1}(t)$ is a Legendre polynomial and the prime denotes a derivative. The Padua points are given by the formula

$$P_{ij}^{Pa} = [\pi x_i/2, \pi y_j/2], \quad 1 \le i \le m_1 + 1, \quad 1 \le j \le m_1/2 + 1, \tag{12}$$

where

$$x_i = \cos\frac{(i-1)\pi}{m_1}, \quad y_j = \begin{cases} \cos\dfrac{(2j-2)\pi}{m_1+1} & i \ odd \\ \cos\dfrac{(2j-1)\pi}{m_1+1} & i \ even. \end{cases}$$

For $m_1 = 40$ this formula produces 780 points plotted in Figure 1.

The second column in Table 1 contains the smallest number of matching points for which $cond(R)$, $cond(S)$, and $cond(T)$ are less than 10^{10} using $n = 25, 49, 100, 196,$ and 400 trial functions (9). The next three columns report the approximate values $log_{10}(cond(R)), log_{10}(cond(S))$, and $log_{10}(cond(T))$ for the corresponding sets of matching points. The computations with the Morrow-Patterson points [31] are not reported here as they produce the matrices $R, S,$ and T with the same condition numbers as the Padua points. Finally, in Table 2 we report the approximations of the eigenvalues λ_1, λ_4, and λ_6 using n trial functions (9) and m Padua matching points (12). The values of the parameter *niter* denote the number of iterations of the Gauss-Newton method applied to the minimization problem (6). The initial approximations for the Gauss-Newton method have been computed by solving the quadratic matrix eigenvalue problem (4) for $n = 25$ using the MATLAB function *polyeig*. These initial approximations have been successively improved using $n = 49$, 100, 196, and 400 trial functions.

3.2. Circular domain

The fixed homogeneous circular membrane problem has been solved using the algebraic polynomials [32]

$$\varphi^{\tau}_{k,l}(x,y) = (1 - x^2 - y^2)J^{\tau}_l(xcos\frac{k\pi}{l+1} + ysin\frac{k\pi}{l+1}), \qquad 0 \le k \le l. \tag{13}$$

There are more possibilities how to construct matching points for the unit disc. One can transform suitable matching points for the square into the unit disc as done by Heinrichs [33]. This approach has been applied by using the Padua points $(x_i, y_i) \in (-1,1) \times (-1,1)$ transformed onto the unit disc D by the mapping [34]

$$(x^D_i, y^D_i) = (x_i, y_i\sqrt{1 - x^2_i}).$$

Table 3 reports the corresponding condition numbers of the matrices R, S, T in the logarithmic form.

The simpler and more effective solution may be obtained by creating the suitable matching points directly for the unit disc. According to the distribution pattern of the Fekete points for the square and triangular domains, where the points are concentrated near the boundary of the considered domains, the following simple construction of matching points for the unit disc has been proposed. The each set of the proposed matching points is determined by three parameters m_1, m_2, m_3. The corresponding matching points are situated on concentric circles with the radii $r_i = z^+_i$, where z^+_i are the positive roots of the Chebyshev polynomial of order $2m_3$. The smallest circle contains m_1 points and each following circle contains m_2 points more than the previous one. The resulting matching points for $m_1 = 3, m_2 = 3, m_3 = 19$ are plotted in Figure 2. The corresponding condition numbers of the matrices R, S, T are reported in Table 4 and the resulting approximations of $\lambda_1, \lambda_4 = \lambda_5$, and λ_6 are shown in Table 5. For the sake of completeness, the eigenvalues of a homogeneous circular membrane with the unit radius are of the form

$$\lambda_{k,l} = (r_{k,l})^2, \quad k = 0, 1, 2, ..., \quad l = 1, 2, 3, ...,$$

where $r_{k,l}$ is the *l-th* zero of the *k-th* Bessel function $J_k(r)$.

3.3 Triangular domain

Similarly as in the unit disc, the eigenvalues of a homogeneous fixed membrane defined on the equilateral triangle $T = \{(x,y)|0 \le x \le 1, 0 \le y \le \sqrt{3}(1-x), 0 \le y \le \sqrt{3}x\}$ are known in the explicit form [35]

$$\lambda_{i,j} = 16\pi^2(i^2 + ij + j^2)/9, \quad i, j = 1, 2, 3, ...$$

Surprisingly, the articles concerning the numerical solution of this nice problem are very rare in the literature [36]. In the presented numerical experiments the algebraic polynomials

$$\varphi_{l,k}^{\alpha,\beta,\gamma}(x,y) = J_{l-k}^{\alpha,\beta+\gamma+2k+1}(2x+\frac{2y}{\sqrt{3}}-1)J_k^{\beta,\gamma}(\frac{4y}{\sqrt{3}x+y}-1)\times$$
$$(\sqrt{3}x+y)^k(1-x-\frac{y}{\sqrt{3}})^\alpha(x-\frac{y}{\sqrt{3}})^\beta y^\gamma \tag{14}$$

orthogonal on T [37] and the Lobatto matching points proposed in [38] have been used. These Lobatto points are defined as follows

$$\xi_i = \frac{1}{3}(1+2v_i-v_j-v_p), \quad \eta_j = \frac{1}{3}(1+2v_j-v_i-v_p) \tag{15}$$

for $i = 1, 2, ..., q+1$, and $j = 1, 2, ..., q+2-i$, where $p = q+3-i-j$ and $v_i = (1+t_{i-1})/2$ for $i = 2, ..., q$, complemented by $v_1 = 0$ and $v_{q+1} = 1$. The points t_i are the zeros of the $(q-1)$-degree Lobatto polynomial.

The Lobatto matching points lying inside the equilateral triangle T for q=29 are plotted in Figure 3. Table 6 reports the condition numbers of the matrices R, S, T creating by n trial functions (14) for $\alpha = \beta = \gamma = 1$ and m matching points (15). The corresponding approximations of the eigenvalues $\lambda_1, \lambda_2 = \lambda_3$, and $\lambda_5 = \lambda_6$ are listed in Table 7.

4 Concluding remarks

There are two main obstacles which hamper to greater popularity of the least squares method in solving boundary value and eigenvalue problems. Whereas in the membrane case the Rayleigh-Ritz method requires the convergence of the eigenfunction approximations together with the convergence of their first partial derivatives, the least squares method requires in addition the convergence of the second partial derivatives of the corresponding eigenfunction approximations. This may be an essential drawback for membranes with a non-smooth boundary. As known, the eigenfunctions of the Laplace operator with the homogeneous Dirichlet boundary conditions defined on a region with corners formed by two straight lines meeting at an angle π/k for k integer are infinitely differentiable [39]. Consequently, one may expect a sufficient convergence of the least squares approximations at least for domains with a smooth boundary like the circle and ellipse, and domains with a piecewise-smooth boundary like the rectangle, and triangles with the angles $\pi/2, \pi/3, \pi/4$. The results presented in Table 2, Table 5, and Table 7 indicate that the method considered in this article is able to achieve sufficiently accurate approximations of the membrane eigenvalues for problems defined on the rectangular, circular and equilateral triangular domains. By virtue of these results one may expect at least 10 digits of accuracy for the first 10 eigenvalue approximations using less than 200 trial functions. Moreover, similarly as in the rectangular case, the conformal mapping technique may be used also for domains with exotic shapes generated by an arbitrary conformal mapping of the circular and triangular domains.

The second reason of small popularity of the least squares method is based on computational difficulties encountered in solving the corresponding matrix eigenvalue problems with ill-conditioned matrices. The condition number of the matrices resulting from the least squares method is essentially greater than the one resulting from the Rayleigh-Ritz method. As seen in the presented numerical experiments suitable matching points for the discrete least-squares method are easily available for the square, circular and triangular domains. The simple procedure for creating the Chebyshev disc points may certainly be generalized at least for simply connected and doubly connected domains with the boundaries parametrized in polar coordinates. In the case of general domains the more sophisticated methods computing the Fekete or Lebesgue points [27, 38, 40-45] may represent very promising alternative. On the other hand, the difficulties with selection of the optimal matching points for an arbitrarily shaped two-dimensional domain provide the source of our future research - to develop a simple and efficient method for finding the points minimizing the spectral condition numbers of the matrices R, S, and T.

Although the condition numbers in the triangular case are relatively large, there is only negligible influence of the roundoff error on the accuracy of the computed eigenvalue approximations shown in Table 7. This fact implies that the resulting accuracy need not be destroyed by the roundoff error in the full extent as may be expected according to the value of the condition numbers of the corresponding matrices.

Finally, to create the matrices R, S, T to be well-conditioned, one can try to use a local approximation scheme using the B-splines or compactly supported radial basis functions. Because of the property of the local approximation schemes to create better conditioned matrices than the global schemes, the results of such computations could be of interest.

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References

[1] Kuttler JR, Sigilitto VG (1984) Eigenvalues of the Laplacian in two dimensions. SIAM Rev 26(2):163–193

[2] Ng CHW, Zhao YB, Wei GW (2004) Comparison of discrete singular convolution and generalized differential quadrature for the vibration analysis of rectangular plates. Comput Methods Appl Mech Engrg 193(23-26):2483–2506

[3] Hou YS, Wei GW, Xiang Y (2005) DSC-Ritz method for the free vibration analysis of Mindlin plates. Int J Numer Methods Engng 62(2):262–288

[4] Karami G, Malekzadeh P (2002) Static and stability analysis of arbitrary straight-sided quadrilateral thin plates by DQM. Int J Solids Struct 39(19):4927–4947

[5] Wang XW, Gan LF, Wang YL (2006) A differential quadrature analysis of vibration and buckling of an SS-C-SS-C rectangular plate loaded by linearly varying in-plane stresses. J Sound Vib 298(1-2):420–431

[6] Chen JT, Lin JH, Kuo SR, Chyuan SW (2001) Boundary element analysis for the Helmholtz eigenvalue problems with a multiply connected domain. Proc R Soc Lond A 457(2014):2521–2546

[7] Chen JT, Liu LW, Chyuan SW (2004) Acoustic eigenanalysis for multiply-connected problems using dual BEM. Commun Numer Meth Engng 20(6):419–440

[8] Reutskiy SY (2005) The method of fundamental solutions for eigenproblems with Laplace and biharmonic operators. CMC - Comput Mater Continua 2(3):177–188.

[9] Alves CJS, Antunes PRS (2005) The method of fundamental solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes. CMC - Comput Mater Continua 2(4):251–265

[10] Tsai CC, Young DL, Chen CW, Fan CM (2006) The method of fundamental solutions for eigenproblems in domains with and without interior holes. Proc R Soc Lond A 462(2069):1443–1466

[11] Platte RB, Driscol TA (2004) Computing eigenmodes of elliptic operators using radial basis functions. Computers Math Applic 48(3-4):561–576

[12] Ferreira AJM, Fasshauer GE (2006) Computation of natural frequencies of shear deformable beams and plates by an RBF-pseudo spectral method. Comput Methods Appl Mech Engng 196(1-3):134–146

[13] Jiang PL, Li SQ, Chan CH (2002) Analysis of elliptical waveguides by a meshless collocation method with the Wendland radial basis functions. Microw Opt Techn Let 32(2):162–165.

[14] Dong SB, Lopez AE (1985) Natural vibrations of a clamped circular plate with rectilinear orthotropy by least-squares collocation. Int J Solids Structures 21(5):515–526

[15] Zhang YC, He XY (1989) Analysis of free vibration and buckling problems of beams and plates by discrete least-squares method using B_5-spline as trial function. Comput & Structures 31(2):115–119

[16] Žitňan P (1996) Vibration analysis of membranes and plates by a discrete least squares technique. J Sound Vib 195(4):595–605

[17] Mazumdar J (1975) A review of approximate methods for determining the vibrational modes of membranes. Shock Vib Dig 7(6):75–88

[18] Mazumdar J (1979) A review of approximate methods for determining the vibrational modes of membranes. Shock Vib Dig 11(2):25–29

[19] Mazumdar J (1982) A review of approximate methods for determining the vibrational modes of membranes. Shock Vib Dig 14(2):11–17

[20] Mazumdar J (1984) A review of approximate methods for determining the vibrational modes of membranes. Shock Vib Dig 16(10):9–16

[21] Williams KR, Lesser THJ (1993) Natural frequencies of vibration of a fibre supported human tympanic membrane analyzed by the finite element method. Clin Otolaryngol 18:75–386

[22] Hine MJ (1973) Acoustically induced vibrations of slender rods in a cylindrical duct. Trans ASME J Appl Mech 40(2):459–463

[23] de Menezes DD, Jar e Silva M, de Aguiar FM (2007) Numerical experiments on quantum chaotic billiards. Chaos 17:023116

[24] Barnett AH, Betcke T (2007) Quantum mushroom billiards. Chaos 17:043125

[25] Tisseur F, Meerbergen K (2001) The quadratic eigenvalue problem. SIAM Rev 43(2):235–286

[26] Chaitin-Chatelin F, Frayseé V (1996) Lectures on finite precision computations. Siam, Philadelphia

[27] Taylor MA, Wingate BA, Vincent RE (2000) An algorithm for computing Fekete points in the triangle. SIAM J Numer Anal 38(5):1707–1720

[28] Cureton LM, Kuttler JR (1999) Eigenvalues of the Laplacian on regular polygons and polygons resulting from their disection. J Sound Vib 220(1):83–98

[29] Bos L, Taylor MA, Wingate BA (2001) Tensor product Gauss-Lobatto points are Fekete points for the cube. Math Comput 70(236):1543–1547

[30] Caliari M, De Marchi S, Vianello M (2005) Bivariate polynomial interpolation on the square at new nodal sets. Appl Math Comput 165(2):261–274

[31] Morrow CR, Patterson TNL (1978) Construction of algebraic cubature rules using polynomial ideal theory. SIAM J Numer Anal 15(5):953–976

[32] Xu Y (2004) Lecture notes on orthogonal polynomials of several variables. In: zu Castell W, Filbir F, Forster B (eds) Advances in the Theory of Special Functions and Orthogonal Polynomials, Nova Science Publisher

[33] Heinrichs W (2004) Spectral collocation schemes on the unit disc. J Comput Phys 199(1):66-86

[34] Bos L, Caliari M, De Marchi S, Vianello M (2006) Bivariate interpolation at Xu points: results, extensions and applications. Elec Trans Numer Anal(ETNA) 25:1–16

[35] McCartin BJ (2003) Eigenstructure of the equilateral triangle, Part I: The Dirichlet problem. SIAM Rev 45(2):267–287

[36] Sun JC (2001) Orthogonal piece-wise polynomial basis on an arbitrary triangular domain and its applications. J Comput Math 19(1):55–66

[37] Koornwinder T (1975) Two-variable analogues of the classical orthogonal polynomials. In: Askey R (ed) Theory and Applications of Special Functions, Academic Press, New York

[38] Blyth MG, Pozrikidis C (2006) A Lobatto interpolation grid over the triangle. IMA J Appl Math 71(1):153–169

[39] Grisvard P (1985) Elliptic Problems in Nonsmooth Domains. Pitman Press, Boston

[40] Chen Q, Babuška I (1995) Approximate optimal points for polynomial interpolation of real functions in an interval and in a triangle. Comput Methods Appl Mech Engng 128(3-4):405-417

[41] Hesthaven JS (1998) From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex. SIAM J Numer Anal 35(2):655–676

[42] Heinrichs W (2005) Improved Lebesgue constants on the triangle. J Comput Phys 207(2):625–638

[43] Warburton T (2006) An explicit construction of interpolation nodes on the simplex. J Eng Math 53(3):247–262

[44] Blyth MG, Luo H, Pozrikidis C (2006) A comparison of interpolation grids over the triangle or the tetrahedron. J Eng Math 56(3):263–272

[45] Bendito E, Carmona A, Encinas AM, Gesto JM (2007) Estimation of Fekete points. J Comput Phys 225(2):2354–2376

Figure captions

Figure 1. 780 Padua matching points.

Figure 2. 570 Chebyshev disc matching points.

Figure 3. 378 Lobatto matching points.

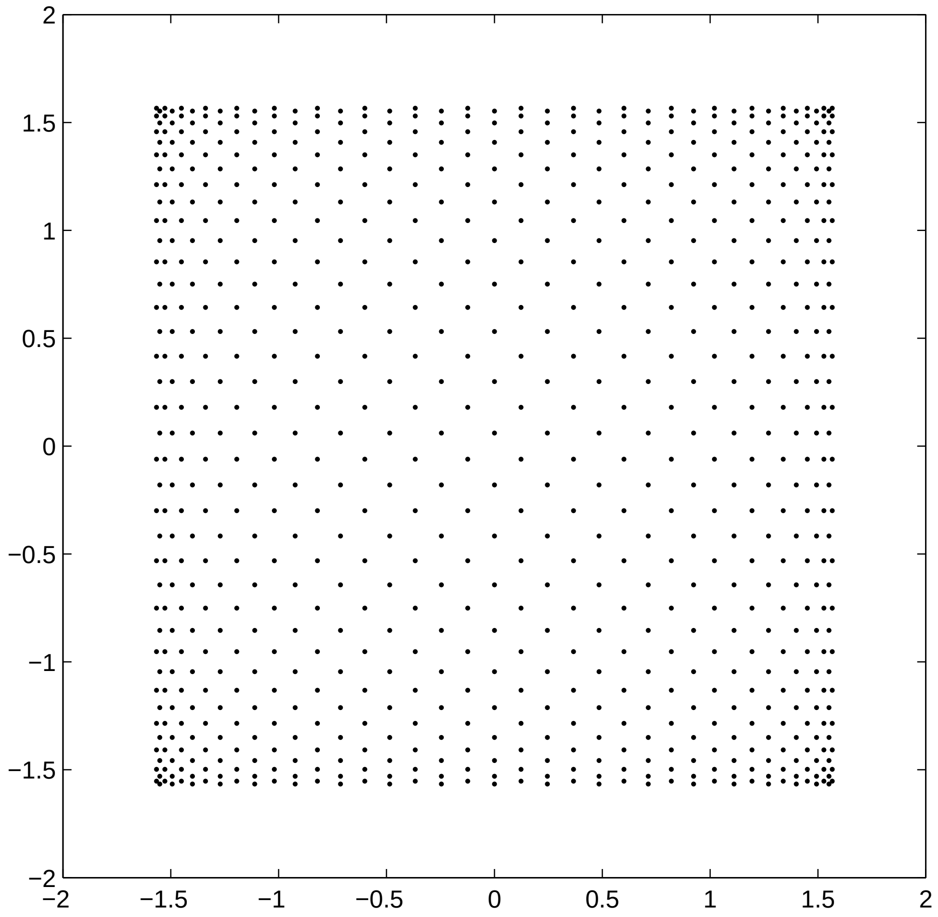

Figure 1.

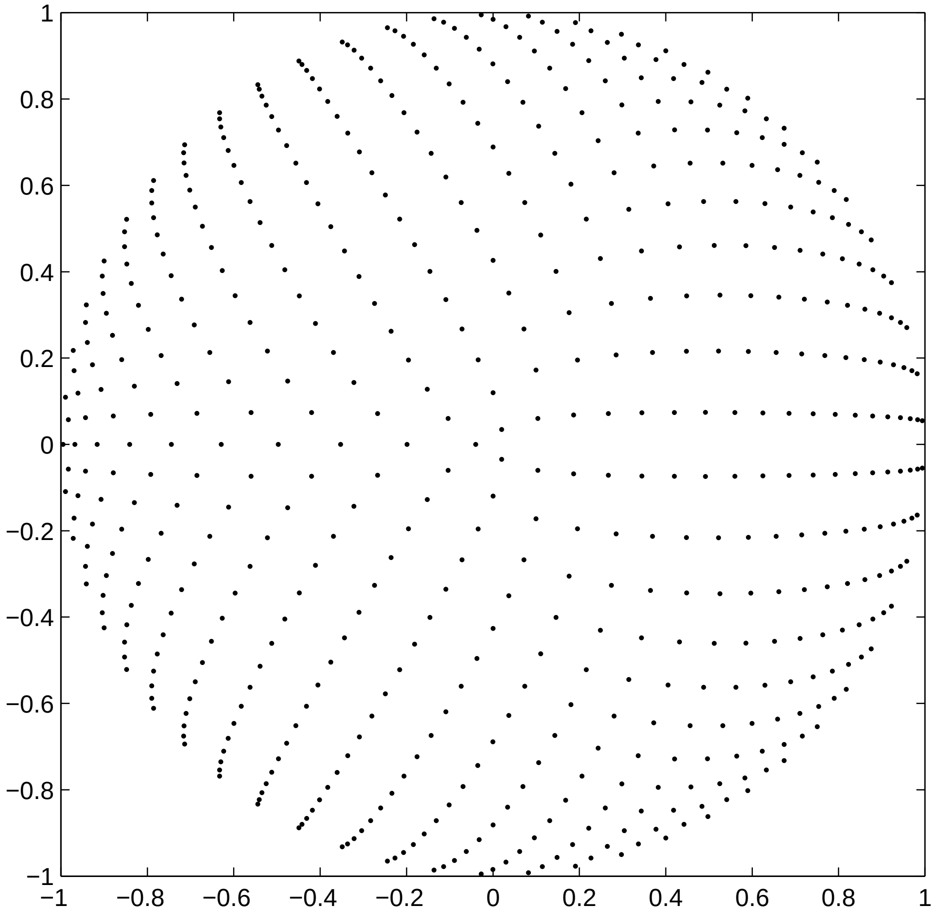

Figure 2.

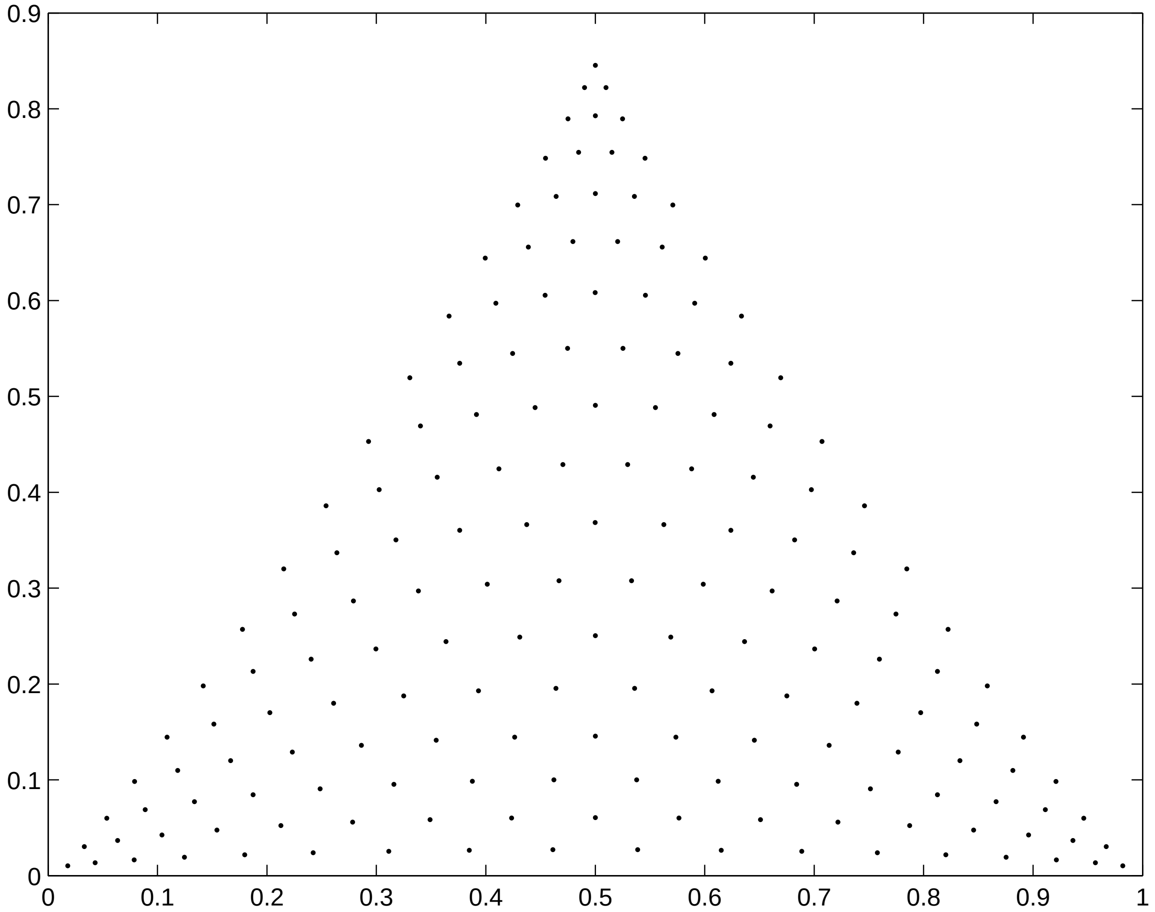

Figure 3.

Table 1: Spectral condition numbers of the matrices R, S, T using n trial functions (9) and m equidistant, Fekete, and Padua matching points.

n	m	$log_{10}(cond(R))$	$log_{10}(cond(S))$	$log_{10}(cond(T))$	τ
			equidistant points		
25	81	7	4	7	1
49	225	8	5	8	1
100	576	9	4	7	3
196	1600	11	5	9	3
			Fekete points		
25	81	3	1	2	1
49	169	3	2	3	1
100	361	4	2	3	1
196	729	5	2	4	1
400	1521	6	2	4	1
			Padua points		
25	45	3	1	2	1
49	91	4	2	3	1
100	190	5	2	3	1
196	378	5	2	4	1
400	780	6	3	4	1

Table 2: Approximations of the eigenvalues λ_1, λ_4 and λ_6 of the problem (7)-(8) using n trial functions (9), m Padua matching points (12), and *niter* number of the Gauss-Newton iterations.

n	m	$\lambda_1^{n,m}$	*niter*
25	45	7.569584 ...	–
49	91	7.56957684 ...	4
100	190	7.56957690163 ...	2
196	378	7.56957690182628	2
400	780	7.56957690182631	1
		$\lambda_4^{n,m}$	
25	45	29.107 ...	–
49	91	29.11536 ...	5
100	190	29.115433415 ...	4
196	378	29.1154334264 ...	2
400	780	29.11543342640719	2
		$\lambda_6^{n,m}$	
25	45	44.58 ...	–
49	91	44.834 ...	12
100	190	44.8401240 ...	7
196	378	44.84012692458 ...	3
400	780	44.84012692435061	2

Table 3: Spectral condition numbers of the matrices R, S, T using n trial functions (13) and m transformed Padua points

n	m	$log_{10}(cond(R))$	$log_{10}(cond(S))$	$log_{10}(cond(T))$	τ
25	45	3	1	1	1
49	91	4	1	2	1
100	190	5	2	2	1
196	378	6	2	3	1
400	780	7	2	3	1

Table 4: Spectral condition numbers of the matrices R, S, T using n trial functions (13) and m Chebyshev disc points.

n	m_1	m_2	m_3	m	$log_{10}(cond(R))$	$log_{10}(cond(S))$	$log_{10}(cond(T))$	τ
25	3	3	4	30	1	2	3	0
49	3	3	6	63	2	2	3	0
100	3	3	9	135	2	3	4	0
196	3	3	13	273	3	3	5	0
400	3	3	19	570	3	4	5	0

Table 5: Approximations of the eigenvalues λ_1, λ_4 and λ_6 of the fixed homogeneous circular membrane using n trial functions (13), m Chebyshev disc matching points, and *niter* number of the Gauss-Newton iterations.

n	m	$\lambda_1^{n,m}$	*niter*
25	30	5.78318585 ...	–
49	63	5.78318596294686	7
100	135	5.78318596294678	1
196	273	5.78318596294678	1
exact value		5.783185962946785	
		$\lambda_4^{n,m}$	
25	30	26.3752 ...	–
49	63	26.3746138 ...	5
100	135	26.37461642716344	2
196	273	26.37461642716339	1
exact value		26.37461642716339	
		$\lambda_6^{n,m}$	
25	30	30.4734 ...	–
49	63	30.47126228 ...	4
100	135	30.47126234366215	7
196	273	30.47126234366209	3
exact value		30.47126234366209	

Table 6: Spectral condition numbers of the matrices R, S, T using n trial functions (14) and m Lobatto matching points (15).

n	m	$log_{10}(cond(R))$	$log_{10}(cond(S))$	$log_{10}(cond(T))$	τ
25	45	4	3	3	
49	91	6	3	5	
100	190	8	3	7	
196	378	10	5	8	

Table 7: Approximations of the eigenvalues λ_1, λ_2 and λ_5 of the fixed triangular membrane using n trial functions (14), m Lobatto matching points (15), and *niter* number of the Gauss-Newton iterations.

n	m	$\lambda_1^{n,m}$	*niter*
25	45	52.637902 ...	–
49	91	52.637890171 ...	6
100	190	52.63789013914314	2
196	378	52.63789013914329	1
exact value		52.63789013914325	
		$\lambda_2^{n,m}$	
25	45	122.837 ...	–
49	91	122.821762 ...	4
100	190	122.82174365825 ...	3
196	378	122.8217436580014	2
exact value		122.8217436580009	
		$\lambda_5^{n,m}$	
25	45	228.86 ...	–
49	91	228.0948 ...	5
100	190	228.0975261 ...	3
196	378	228.0975239362881	2
exact value		228.09752393628745	