Near optimal interpolation and quadrature in two variables: the Padua points

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Abstract

The Padua points, recently studied during an international collaboration at the University of Padua, are the first known example of near-optimal point set for bivariate polynomial interpolation of total degree. The associate algebraic cubature formulas are both, in some sense, near-optimal.

Which are the Padua points?

Chebyshev-Lobatto points in $[-1, 1]$

$$C_{n+1} = \{- \cos((j+1/2)\pi/n), j = 0, \ldots, n\}$$

$$\text{card}(C_{n+1}) = n + 1 = \dim(P_n)$$

Padua points in $[-1, 1] \times [-1, 1]$

$$\text{Pada}_n = (C_{n+1} \times C_{n+1}) \cup (C_{n+2} \times C_{n+1}) \subset C_{n+1} \times C_{n+1}$$

$$\text{card}(\text{Pada}_n) = (n + 1)(n + 2) = \dim(P_{2n})$$

Alternative representation as self-intersections and boundary contacts of the generating curve

$$g(t) := (-\cos((n+1)t), -\cos(nt)), \quad t \in [0, \pi]$$

Figure 1. The Padua points with their generating curve for $n = 12$ (left, 91 points) and $n = 13$ (right, 105 points), also as union of two Chebyshev-Lobatto subgrids (open bullets and filled bullets).

There are 4 families of such points, corresponding to successive rotations of 90 degrees.

Interpolation at the Padua points

Trigonometric quadrature on the generating curve

$$w_\xi = \frac{1}{n(n+1)} \begin{cases} 1/2 & \text{if } \xi \text{ is a vertex point} \\ 1 & \text{if } \xi \text{ is an edge point} \\ 2 & \text{if } \xi \text{ is an interior point} \end{cases}$$

near exactness in $P_{2n}$

Lagrange interpolation formula

$$L_{\text{Pada}_n}(f)(x) = \sum_{\xi \in \text{Pada}_n} f(\xi) L_n(\xi) - \delta_0$$

$$L_n(x) = w_\xi (K_n(\xi, x) - T_n(\xi)/T_n(x_1))$$

$T_n(\cdot) = \cos(n \arccos(\cdot))$ and $K_n(x, y)$ reproducing kernel of the product Chebyshev orthonormal basis

The Lebesgue constant

Unisolveness gives the projection operator

$$L_{\text{Pada}_n} : C([-1, 1]^2, \|\cdot\|_\infty) \to P_{2n}([-1, 1]^2, \|\cdot\|_\infty)$$

Theorem

$$\|L_{\text{Pada}_n}\| = \max_{f \in [-1, 1]^2} \sum_{\xi \in \text{Pada}_n} |L_n(f)| = \mathcal{O}(\log^2(n))$$

i.e., the Lebesgue constant has optimal order of growth (cf. [7]).