A Generalization of the Cellina–Colombo Theorem for a Class of Non-convex Variational Problems

CARLO MARICONDA

Dipartimento di Matematica Pura e Applicata, via Belzoni 7,
35131 Padova, Italy

Submitted by Leonard D. Berkovitz

We state a condition under which the integral functional \( R(x) = \int_0^T L(t, x(t), x'(t)) \, dt \) attains a minimum under the assumption that \( x \mapsto L(t, x, x') \) is concave.

INTRODUCTION

This paper concerns the problem (P) of the existence of a minimum for the integral functional \( I \) defined by

\[
I(x) = \int_0^T L(t, x(t), x'(t)) \, dt
\]

on the set of functions \( x(\cdot) \) belonging to \( W^1, p([0, T], \mathbb{R}^n) \) \( (p \geq 1) \) such that \( x(0) = a, x(T) = b \), in the case where \( L \) does not necessarily satisfy Tonelli's classical assumption of convexity with respect to \( x' \).

In this situation, the most general result is the Cellina–Colombo theorem [2] stating that if \( L(t, x, x') = g(t, x) + h(t, x') \) and \( x \mapsto g(t, x) \) is concave for a.e. \( t \) then Problem (P) admits at least one solution. For the case where the integrand is not the sum of two functions whose arguments are \( t, x \) and \( t, x' \) separately, it is not known whether the concavity assumption on the map \( x \mapsto L(t, x, x') \) is sufficient for the existence of a solution to Problem (P). The purpose of this note is to consider this problem.

In Theorem 3 we prove that the functional \( I \) (under the concavity condition) attains a minimum if we assume further the existence of a solution

\[ (\tilde{x}, p_1, ..., p_{n+1}, v_1, ..., v_{n+1}) \]

to the associated relaxed problem (PR) satisfying

\[
\bigcap_{i=1}^{n+1} \partial_i (\tilde{x}; -L(t, \tilde{x}(t), v_i(t))) \neq \emptyset \quad \text{a.e.}
\]

(C)
Obviously, each solution to (P) is a solution to (PR') satisfying (C); the cases for which our theorem can be usefully applied are those where the converse does not hold. For instance, condition (C) is automatically satisfied (for each \( x, v_1, \ldots, v_{n+1} \)) when the integrand \( L \) is the sum of two functions whose arguments are \( t, x, t, x' \) separately. In this situation, Theorem 3 yields Cellina and Colombo's existence result; however, it is well known that a solution to the associated relaxed problem is not, in general, a solution to the original one.

As a further application of our condition we show that Problem (P) attains a minimum if \( L(t, x, x') = h(t, x) + f(t, x) g(t, x') \) and its bipolar \( L^{**}(t, x, \cdot) \) is locally constant on \( A(t, x) = \{ \xi : L(t, x, \xi) > L^{**}(t, x, \xi) \} \).

The main tools are basically the arguments of [2]: an extension of Liapunov's theorem on the range of a vector measure and a selection theorem.

**Assumptions and Preliminary Results**

The following hypothesis is considered:

**Hypothesis (H).** The set-valued map \( \Phi : [0, T] \to 2^{\mathbb{R}^n} \) is measurable [1, Def. III.1.1] with non-empty closed values. In addition we assume that there exists at least one \( v \in L^p([0, T], \mathbb{R}^n) \) such that \( v(t) \in \Phi(t) \) a.e. and \( \int_0^T v(t) \, dt = b - a \). The Carathéodory function \( L : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) satisfies the following growth assumption: if \( p = 1 \), there exist a convex l.s.c. monotonic function \( \psi : \mathbb{R}^n \to \mathbb{R}^n \), a constant \( \beta_1 \), and a function \( \zeta(\cdot) \) in \( L^1 \) such that

\[
L(t, x, \xi) \geq \alpha_1(t) - \beta_1 |x| + \psi(\xi)
\]

for each \( x, \xi \) and for a.e. \( t \). If \( \lim_{r \to +\infty} \frac{\psi(r)}{r} = +\infty \).

If \( p > 1 \), there exist a positive constant \( \gamma_p, \) a constant \( \beta_p (\beta_p/\gamma_p) \) is strictly smaller than the best Sobolev constant in \( W_p^{1,0}([0, T], \mathbb{R}^n) \), a function \( \alpha_p(\cdot) \) in \( L^1 \) such that

\[
L(t, x, \xi) \geq \alpha_p(t) - \beta_p |x|^p + \gamma_p |\xi|^p
\]

for each \( x, \xi \) and for a.e. \( t \).

For each \( t, x \) let us denote by \( L^{**}(t, x, \cdot) \) the bipolar of the map \( \zeta \mapsto L(t, x, \zeta) \) [4, Sect. 1.4.2]. For each function \( L \) satisfying Hypothesis (H), its bipolar fulfills Tonelli's classical assumptions for the existence of a solution to the relaxed problem (PR) associated to (P),

\[
\text{minimize } \int_0^T L^{**}(t, x(t), x'(t)) \, dt \quad \text{(PR)}
\]

on the subset of \( W^{1, p}([0, T], \mathbb{R}^n) \) of those functions \( x \) satisfying \( x(0) = a, x(T) = b \), \( x'(t) \in \Phi(t) \) a.e. Now, consider the problem

\[
\text{minimize } \int_0^T \sum_{i=1}^{n+1} p_i(t) L(t, x(t), v_i(t)) \, dt
\]

\[
p_i : [0, T] \to \mathbb{R}, \quad v_i : [0, T] \to \mathbb{R}^n \text{ measurable}
\]

\[
\sum_i p_i(t) = 1, \quad p_i \geq 0, \quad v_i(t) \in \Phi(t) \text{ a.e.} \quad \text{(PR')}
\]

Clearly,

\[
\min \text{PR} \leq \inf \text{PR'} \leq \inf \text{P}
\]

Moreover, we have the following:

**Theorem 1** [4, Th. IX.4.1, Sect. IX.4.5]. Let \( L \) satisfy Hypothesis (H). Then \( \min \text{PR'} = \min \text{PR} = \inf \text{P} \).

Let us denote by \( \chi_E(\cdot) \) the characteristic function of a set \( E \). Theorem 2 is an extension of Liapunov's theorem on the range of a vector measure [3, Chap. 16].

**Theorem 2** [2, 6, 9]. Let \( p_1, \ldots, p_n : [0, T] \to [0, 1], f_1, \ldots, f_n : [0, T] \to \mathbb{R}^l (l \geq 1) \) be measurable (\( \sum p_i = 1 \)) and bounded below by an integrable function. Let us further assume that \( \sum p_i f_i \in L^1 \). Then there exists a measurable partition \( E_i, \ldots, E_m \) of \([0, T]\) with the property that \( \sum_i f_i \chi_{E_i} \in L^1 \) and the following equality holds:

\[
\int_0^T \sum_i p_i f_i \, dt = \int_0^T \sum_i f_i \chi_{E_i} \, dt.
\]

Lemma 1 below concerns a property of the subdifferential of a convex function [4, Sect. I.5.1]; its proof follows directly from [2, Lemma 1].

Let us denote by \( \partial \phi(f(t, x, \xi)) \) the subdifferential of the function \( x \mapsto f(t, x, \xi) \). Also, for a subset \( Q \) of \( \mathbb{R}^n \), we write \( \|Q\| \) for the set \( \{ |q| : q \in Q \} \).

**Lemma 1.** Let \( f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be a Carathéodory function satisfying:
ON NON-CONVEX VARIATIONAL PROBLEMS

(i) \( f(t, x, \xi) \leq \alpha(t) + \beta |x|^p \) \( (\beta > 0, x \in L^1) \);
(ii) \( x \mapsto f(t, x, \xi) \) is convex for a.e. \( t \) and for each \( \xi \).

Let \( \hat{\xi} \) be continuous, \( v_1, \ldots, v_{n+1} \) be measurable and such that
\[
\mathcal{P}(t) = \int \partial_x f(t, \hat{\xi}(t), v_i(t)) \neq \emptyset \quad \text{a.e.}
\]

Then, the set-valued map \( \mathcal{P} \) admits an integrable selection.

**Remark.** The proof of [2, Lemma 1] points out the fact that an integrable selection of \( \mathcal{P} \) exists if, instead of (i), we assume that there exists a function \( \alpha(-) \) in \( L^1 \) and a function \( c : \mathbb{R}^+ \to \mathbb{R} \) such that
\[
|\partial_x f(t, x, \xi)| \leq \alpha(t) + c(A) \quad \text{for each } t, \xi, |x| \leq A.
\]

**Lemma 2.** Let \( f, g : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) and set \( h(t, x, \xi) = f(t, x) g(t, \xi) \).
Then for each \( t, x, \xi \),
\[
h^{**}(t, x, \xi) = f(t, x) G(t, \xi),
\]
where
\[
G(t, \xi) = \begin{cases} 
\varepsilon^{**}(t, \xi) & \text{if } f(t, x) \geq 0; \\
-(\varepsilon^{**}(t, \xi)) & \text{if } f(t, x) < 0.
\end{cases}
\]

**Proof.** Let us suppose \( f(t, x) < 0 \), the other case \( f(t, x) \geq 0 \) being similar. In this situation, the inequality
\[
-(\varepsilon^{**}(t, \xi)) \leq -g(t, \xi)
\]
implies
\[
-f(t, x)(-\varepsilon^{**}(t, \xi)) \leq f(t, x) g(t, \xi),
\]
whence
\[
f(t, x) G(t, \xi) \leq h^{**}(t, x, \xi).
\]

Conversely, let \( t, x \) be fixed and \( \hat{\xi} \) be any convex function satisfying
\[
\hat{\xi}(\xi) \leq f(t, x) g(t, \hat{\xi}) \quad \text{for each } \xi.
\]
Then
\[
-\frac{1}{f(t, x)} \hat{\xi}(\xi) \leq -g(t, \xi) \quad \text{for each } \xi,
\]
whence
\[
-\frac{1}{f(t, x)} \hat{\xi}(\xi) \leq (-g^{**}(t, \xi)).
\]

In particular, for \( \psi(\xi) = h^{**}(t, x, \xi) \),
\[
h^{**}(t, x, \xi) \leq f(t, x) G(t, \xi).
\]

The conclusion follows from (1) and (2).

**Main Result**

**Theorem 3.** Let \( L \) satisfy Hypothesis (H). Let us further suppose that the function \( x \mapsto L(t, x, \xi) \) is concave for each \( t, \xi \). Then, the problem
\[
\begin{align*}
\min & \int_0^T L(t, x(t), x'(t)) \, dt \\
\text{subject to} & x(0) = a, \ x(T) = b,
\end{align*}
\]
on the subset of \( W^{1,p} \) of those functions satisfying \( x(0) = a, \ x(T) = b, \ x'(t) \in \Phi(t) a.e. \) in \( [0, T] \) admits a solution if and only if there exists a solution \( (\hat{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1}) \) to the associated relaxed problem (PR') satisfying
\[
\bigcap_{i=1}^{n+1} \partial_x (L(t, \hat{x}(t), v_i(t))) \neq \emptyset \quad \text{a.e.}
\]
Note that, when \( L(t, x, \xi) \) is differentiable in \( x \), condition (C) reduces to
\[
\frac{\partial L}{\partial x} (t, \hat{x}(t), v_i(t)) = \frac{\partial L}{\partial x} (t, \hat{x}(t), v_i(t)) \quad \text{for each } i, j.
\]

**Proof of Theorem 3.** The necessity is due to the fact that each solution to (P) satisfies (C).

Conversely, let \( (\hat{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1}) \) be a solution to (PR') satisfying condition (C). By Lemma 1, let \( \delta(-) \in L^1 \) be a selection to
\[
t \mapsto \bigcap_{i=1}^{n+1} \partial_x (L(t, \hat{x}(t), v_i(t))).
\]

Then, for each \( y \in \mathbb{R}^n \) and \( i \in \{1, \ldots, n+1\} \), we have
\[
L(t, \hat{x}(t), v_i(t)) \geq L(t, v_i(t)) + \langle \delta(t), y - \hat{x}(t) \rangle,
\]
(\( \langle \cdot, \cdot \rangle \)) being the usual scalar product in \( \mathbb{R}^n \). Set
\[
B(t) = \int_0^t \delta(s) \, ds, \quad f_i(t) = (v_i(t), L(t, \hat{x}(t), v_i(t)), \langle v_i(t), B(t) \rangle).
\]
The growth assumptions on $L$ (Hypothesis (H)) imply that the condition concerning the functions $f_i$ stated in Theorem 2 are satisfied: let $E_1, \ldots, E_{n+1}$ be a measurable partition of $[0, T]$ such that

$$\sum_{i} v_{i, E} \in L^p, \quad \int_0^T \sum_i p_i(t) v_i(t) dt = \int_0^T \sum_i v_i(t) \chi_{E_i}(t) dt,$$

$$\int_0^T \sum_i p_i(t) \langle v_i(t), B(t) \rangle dt = \int_0^T \sum_i \langle v_i(t), B(t) \rangle \chi_{E_i}(t) dt,$$

$$\int_0^T \sum_i p_i(t) L(t, \bar{x}(t), v_i(t)) dt = \int_0^T \sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) dt,$$

and set $\bar{x}(t) = a + \int_0^t \sum_i v_i(s) \chi_{E_i}(s) ds$.

We show that $\bar{x}$ is a solution to (P). Clearly, by (4), $\bar{x}(T) = \bar{x}(T) = b$ and $\bar{x} \in W^{1, r}$. Furthermore, by (3),

$$\sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) \geq \sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) + \langle \delta(t), \bar{x}(t) - \bar{x}(t) \rangle.$$

The integration of the above inequality and (4) yield

$$\min(PR') = \int_0^T \sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) dt \geq \int_0^T \sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) dt + \int_0^T \langle \delta(t), \bar{x}(t) - \bar{x}(t) \rangle dt.$$

Let us remark that

$$\sum_i L(t, \bar{x}(t), v_i(t)) \chi_{E_i}(t) = L(t, \bar{x}(t), \bar{x}'(t))$$

and that the Tonelli–Fubini theorem and integration by parts give

$$\int_0^T \langle \delta(t), \bar{x}(t) \rangle dt = \int_0^T \sum_i (\chi_{E_i}(t) - p_i(t)) \langle v_i(t), B(t) \rangle dt.$$

Then, (4) and (5) together yield

$$\min(P) \geq \min(PR') \geq \int_0^T L(t, \bar{x}(t), \bar{x}'(t)) dt \geq \min(P);$$

the conclusion follows.

**THEOREM 4** (Cellina and Colomba [2]). Let $L(t, x, x') = g(t, x) + h(t, x)$ satisfy Hypothesis (H) and $x \rightarrow g(t, x)$ be concave for a.e. $t$. Then Problem (P) admits at least one solution.

**Proof.** Since we have $\tilde{\varphi}(a - L(t, x, \xi)) = \tilde{\varphi}(a - g(t, x))$ for each $t, x, \xi$, then condition (C) trivially holds. Theorem 3 yields the conclusion.

We shall assume the following hypotheses:

**HYPOTHESIS (A).** Set $A(t) = a + \int_0^t \sum_i \Phi(s) ds$ (see [7]). We assume that:

1. The functions $l, f, g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are such that $L(t, x, \xi) = l(t, x) + f(t, x) g(t, \xi)$ satisfies Hypothesis (H) for a.e. $t$, for each $\xi$, and for each $x \in A(t)$;
2. Either for a.e. $t$, $l(t, x) > 0$ for each $x \in A(t)$
3. or for a.e. $t$, $l(t, x) < 0$ for each $x \in A(t)$;
4. for a.e. $t$ and $x \in A(t)$, the set $A(t, x) = \{ \xi \in \Phi(t) : L^{**}(t, x, \xi) < L(t, x, \xi) \}$ is open and, on it, the function $\xi \mapsto L^{**}(t, x, \xi)$ is locally constant;
5. There exist a function $z(.)$ in $L^1$ and a function $c : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that for a.e. $t$:

$$|\tilde{\varphi}(f(t, x, \xi))| \leq z(t) + c(|\xi|)$$

for each $\xi \in \Phi(t), x \in A(t), |x| \leq A$.

Let us remark that the class of non-trivial functions satisfying the hypothesis quoted above is non-empty.

**EXAMPLE.** $\Phi(t) = \mathbb{R}^+, a = 0, L(t, x, \xi) = -y^2 + (1 + x)\xi - \phi(t)|\xi - \psi(t)|$ $(\phi, \psi \in \mathbb{R}^+, \phi, \psi \geq 0, \gamma$ being strictly smaller than the best Sobolev constant) satisfies Hypothesis (A).

As a further application of Theorem 3, we have the following

**THEOREM 5.** Let $f, g, \Phi$ satisfy Hypothesis (A). Then the problem

$$\minimize \quad I(x) = \int_0^T l(t, x(t)) dt + \int_0^T f(t, x(t)) g(t, x(t)) dt$$

(P)
on the subset of $W^{1,p}$ of those $x(\cdot)$ satisfying $x(0) = a$, $x(T) = b$, $x'(t) \in \Phi(t)$ a.e. in $[0, T]$ admits at least one solution.

Proof. Clearly, in view of Theorem 3, it is enough to prove the existence of a solution $(\tilde{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})$ to (PR') satisfying

$$L(t, \tilde{x}(t), v_j(t)) = L(t, \tilde{x}(t), v_j(t))$$

for each $t, x$ and $i, j \in \{1, \ldots, n+1\}$. For this purpose, let $(\tilde{x}, p_1, \ldots, p_{n+1}, w_1, \ldots, w_{n+1})$ be an arbitrary solution to (PR'). Then, by Theorem 1

$$L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) = \sum_i p_i(t) L(t, \tilde{x}(t), w_i(t)).$$

(6)

The map $\xi \mapsto L^{**}(t, \tilde{x}(t), \xi)$ being convex, we can assume

$$L^{**}(t, \tilde{x}(t), w_i(t)) = L(t, \tilde{x}(t), w_i(t))$$

a.e. (7)

Set $A = \{ t : L^{**}(t, \tilde{x}(t), \tilde{x}'(t)) < L(t, \tilde{x}(t), \tilde{x}'(t)) \} \cap \{ t : \tilde{x}'(t) \in A(t, \tilde{x}(t)) \}$. By $A(t)$, for a.e. $t \in A$, the convex function $L^{**}(t, \tilde{x}(t), \cdot)$ is constant in a neighbourhood of $\tilde{x}'(t)$. As a consequence

$$L^{**}(t, \tilde{x}(t), \xi) \geq L^{**}(t, \tilde{x}(t), \tilde{x}'(t))$$

for a.e. $t \in A$ and each $\xi \in \mathbb{R}^n$. In particular

$$L^{**}(t, \tilde{x}(t), w_i(t)) \geq L^{**}(t, \tilde{x}(t), \tilde{x}'(t))$$

for a.e. $t \in A$;

hence, by (6), we can assume

$$L^{**}(t, \tilde{x}(t), w_i(t)) = L^{**}(t, \tilde{x}(t), \tilde{x}'(t))$$

a.e. $t \in A$. (8)

Equalities (7) and (8) prove that, if we set

$$v_i = w_i \xi_A + \tilde{x}'(t) \xi_{[0, T] \setminus A}$$

then $(\tilde{x}, p_1, \ldots, p_{n+1}, v_1, \ldots, v_{n+1})$ is a solution to (PR') satisfying

$$L(t, \tilde{x}(t), v_i(t)) = L^{**}(t, \tilde{x}(t), \tilde{x}'(t))$$

a.e. (9)

By Lemma 2, there exists a function $G : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$L^{**}(t, x, \xi) = h(t, x) + f(t, x) G(t, \xi).$$

Thus (9) yields

$$g(t, v_i(t)) = G(t, \tilde{x}'(t))$$

a.e.

The claim is proved.