ON BANG–BANG CONSTRAINED SOLUTIONS OF A CONTROL SYSTEM∗

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Abstract. Given ϕ1, ϕ2 ∈ L1([0, T]) and a function x ∈ W^{2,1}([0, T]) solving the control problem (P) x'' + a_{1}(t)x' + a_{0}(t)x ∈ Φ(t) = [ϕ_{1}(t), ϕ_{2}(t)], (x(0), x'(0), x(T), x'(T)) = (x_{0}, v_{0}, x_{1}, v_{1}), there exists a bang–bang solution y to (P) satisfying y ≤ x; moreover there exists a finite union of intervals E such that y'' + a_{1}(t)y' + a_{0}(t)y = ϕ_{1}χ_{E} + ϕ_{2}χ_{[0,T]\setminus E}. The reachable set of bang–bang constrained solutions is convex: an application to the calculus of variations.

Key words. bang–bang, linear control system, range of a vector measure, reachable set, calculus of variations

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1. Introduction. We consider the family of bidimensional linear control systems (P) described by a generic second-order equation subject to a scalar control:

x'' + a_{1}(t)x' + a_{0}(t)x ∈ Φ(t) = [ϕ_{1}(t), ϕ_{2}(t)], (x(0), x'(0), x(T), x'(T)) = (x_{0}, v_{0}, x_{1}, v_{1}),

where ϕ_{1} ≤ ϕ_{2} ∈ L^{1}([0, T]) and a_{1}, a_{0} ∈ C([0, T]), x_{0}, v_{0}, x_{1}, v_{1} ∈ ℝ, x ∈ W^{2,1}([0, T]).

The function y is said to be a bang–bang solution to (P) if it solves (P) and, moreover,

(1.1) y'' + a_{1}(t)y' + a_{0}(t)y ∈ extr Φ(t) = {ϕ_{1}(t), ϕ_{2}(t)} a.e.

Existence of bang–bang solutions has been proved, for instance, by Cesari [4, Thm. 16.3]. The purpose of this paper is to prove that, given an arbitrary solution x to (P), there exists a bang–bang solution y such that

(1.2) ∀t ∈ [0, T] y(t) ≤ x(t)

and, in addition, y'' + a_{1}(t)y' + a_{0}(t)y steers from ϕ_{1} to ϕ_{2} only a finite number of times.

Motivation of such a problem was to study the reachable set

X_{T} = \{(y(T), y'(T)) : y ≤ c, y'' + a_{1}(t)y' + a_{0}(t)y ∈ extr Φ(t), (y(0), y'(0)) = (x_{0}, v_{0})\},

where c is an arbitrary function. A consequence of Theorem 3.1 is that X_{T} coincides with

\bar{X}_{T} = \{(y(T), y'(T)) : y ≤ c, y'' + a_{1}(t)y' + a_{0}(t)y ∈ Φ(t), (y(0), y'(0)) = (x_{0}, v_{0})\}.

Notice that X_{T} is convex, so the above assumption implies that \bar{X}_{T} is convex too. Another motivation arises from nonconvex problems of the calculus of variations (see [1]).

A possible approach in finding bang–bang solutions is to use the Lyapunov Theorem on the range of a vector measure [4, §16.1].

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Here, the solution of \( x'' + a_1(t)x' + a_0(t)x = \rho(t), \quad x(0) = x'(0) = 0 \) is given by

\[
x(t) = \int_0^t h(t, s)\rho(s) \, ds,
\]

where \( h \in C^1([0, T] \times [0, T]) \), and for each \( s \in [0, T] \) the function \( h_s(.) = h(., s) \in C^2([0, T]) \) is the solution to the associated homogeneous differential equation satisfying the initial conditions \( h_s(s) = 0, \quad h_s'(s) = 1 \). The Lyapunov Theorem yields the existence of a measurable subset \( E \) of \( [0, T] \) such that

\[
(1.3) \quad \int_0^T h(T, s)\rho(s) \, ds = \int_0^T h(T, s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0, T] \setminus E}(s)) \, ds,
\]

\[
(1.4) \quad \int_0^T \frac{\partial h}{\partial t}(T, s)\rho(s) \, ds = \int_0^T \frac{\partial h}{\partial t}(T, s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0, T] \setminus E}(s)) \, ds.
\]

Clearly, by differentiating under the integral sign, the function \( y \) defined by

\[
y(t) = \int_0^t h(t, s)(\phi_1(s)\chi_E(s) + \phi_2(s)\chi_{[0, T] \setminus E}(s)) \, ds
\]

is a bang–bang solution. However, this approach does not give any information on the behaviour of \( y \) with respect to \( x \) on \([0, T]\).

Here we prove a new Lyapunov-type theorem concerning the range of a two-dimensional vector measure whose densities are such that their quotient is monotone; in this case, the set \( E \) can be chosen in the form \([\alpha, \beta]\). Note that this is not true in general; for instance, there are no \( \alpha, \beta \in [0, 3\pi] \) satisfying

\[
\int_\alpha^\beta \sin t \, dt = \int_0^{3\pi} \sin t\chi_{[0, \pi] \cup [2\pi, 3\pi]}(t) \, dt, \quad \int_\alpha^\beta 1 \, dt = \int_0^{3\pi} 1\chi_{[0, \pi] \cup [2\pi, 3\pi]}(t) \, dt.
\]

In our application, the equalities \( h(s, s) = 0 \) and \( \frac{\partial h}{\partial t}(s, s) = 1 \) imply that the monotonicity condition is locally fulfilled; this allows us to build a set \( E \) satisfying (1.3)–(1.4) as a finite union of intervals and, in the case where \( \phi_1 < \rho < \phi_2 \) are continuous, to choose \( E \) in such a way that neither 0 nor \( T \) belong to its closure.

These facts, together with a decomposition of the kernel \( h(t, s) \) into a linear combination of linearly independent functions, are the main tools that we use to show that the bang–bang solution \( y \) defined by (1.5) satisfies the inequality \( y \leq x \).

As an application, we consider the problem of minimizing the integral functionals

\[
I(x, u) = \int_0^T f(t, x(t), u(t)) \, dt,
\]

where \( x : [0, T] \to \mathbb{R}^2 \) is such that \( x(0), \ x'(0), \ x(T), \ x'(T) \) are fixed and \( u \) is a control belonging to \( U(t, x) \subset \mathbb{R}^2 \). The classical approach to obtain existence of a minimum is to impose conditions in order to have the lower semicontinuity of \( I \) with respect to \( u \) (for instance convexity of \( u \mapsto f(t, x, u) \)).

Recently, in an effort to provide existence criteria other than convexity in \( u \), some sufficient conditions have been given: for problems of the calculus of variations \( (x' = u \) in the above setting) and for maps of the form \( f(t, x, x') = g(t, x) + h(t, x') \), existence of solutions has been obtained by requiring that the real map \( x \mapsto g(t, x) \) be monotone \([5]\) or, for \( x \) in \( \mathbb{R}^n \), that the same function be concave \([2]\). Optimal control problems escaping to convexity conditions have been handled in \([6]\).
It has been proved further in [3] that there exists a dense subset $D$ of $C(\mathbb{R})$ such that, for $g$ in it, the problem

$$\minimize \int_0^T g(x(t)) \, dt + \int_0^T h(x'(t)) \, dt : \quad x(0) = x_0, \ x(T) = x_1$$

admits a solution for every lower semicontinuous $h$ satisfying growth conditions.

Our theorem gives a straightforward generalization of the above result.

2. Assumptions and preliminary results. Let $\phi_1, \phi_2 \in L^1[0, T], \phi_1 \leq \phi_2$, and put $\Phi(t) = [\phi_1(t), \phi_2(t)] \subset \mathbb{R}$. We are interested in the solutions of the following control problem.

**Problem $P$.**

$$a_1, a_0 \in C([0, T]), \quad x_0, x_1, v_0, v_1 \in \mathbb{R}, \quad x \in W^{2,1}([0, T]),$$

$$(P) \quad x'' + a_1(t)x' + a_0(t)x \in \Phi(t) \ \text{a.e.,}$$

$$x(0) = x_0, \quad x'(0) = v_0, \quad x(T) = x_1, \quad x'(T) = v_1.$$  

By extr $\Phi$ we mean the extreme points of $\Phi$, i.e., extr $\Phi(t) = \{\phi_1(t), \phi_2(t)\}$.

**Definition 2.1.** A function $y \in W^{2,1}([0, T])$ is said to be a bang–bang solution to $(P)$ if $y$ solves $(P)$ and, moreover,

$$y'' + a_1(t)y' + a_0(t)y \in \text{extr } \Phi(t) \ \text{a.e.}$$

The following representation formula of the solutions to $(P)$ will be used later.

**Proposition 2.1.** There exists a function $h \in C^1([0, T] \times [0, T])$ satisfying Property $S$ below such that, for each function $\rho \in L^1([0, T])$, the solution of

$$(P_{\rho}) \quad x'' + a_1(t)x' + a_0(t)x = \rho(t), \quad x(0) = x'(0) = 0$$

is given by the formula

$$(2.1) \quad x(t) = \int_0^t h(t, s)\rho(s) \, ds.$$  

Moreover, for each $s \in [0, T]$, the function $h(., s)$ is of class $C^2([0, T])$.

**Property $S$.**

(1) There exist $w_1, w_2 \in C^2([0, T])$, $z_1, z_2 \in C^1([0, T])$ such that

$$\forall s, t \in [0, T] \quad h(t, s) = w_1(t)z_1(s) + w_2(t)z_2(s)$$

and $W(w_1, w_2, t) = \det \begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix} \neq 0$.

For each $t_0$ in $[0, T]$ there exists $\delta > 0$ such that if we set $I_\delta = [t_0 - \delta, t_0 + \delta] \cap [0, T]$ then:

(2) $\forall s, t \in I_\delta \quad h(t, s) > 0$ if $s < t, \quad h(t, s) < 0$ if $t < s$ (whence $h(s, s) = 0$);

(3) $\forall s, t \in I_\delta \quad \frac{\partial h}{\partial t}(t, s) > 0$;

(4) $\forall t \in I_\delta \quad s \mapsto h(t, s)/\frac{\partial h}{\partial t}(t, s)$ is decreasing on $I_\delta$.

**Proof of Proposition 2.1.** For each $s \in [0, T]$, let $h_s(., s) = h(., s) \in C^2([0, T])$ be the solution to

$$h_s''(t) + a_1(t)h_s'(t) + a_0(t)h_s(t) = 0, \quad h_s(s) = 0, \ h_s'(s) = 1.$$
Set \( z(t) = \int_0^t h(t, s)\rho(s)\,ds \). Differentiation under the integral sign shows that \( z \) is a solution to \((P_\rho)\) whence, by uniqueness, \( z = x \).

To prove the second part of the claim, let \( w_1, w_2 \in C^2([0, T]) \) be two solutions of the differential equation

\[
(2.3) \quad x'' + a_1(t)x' + a_0(t)x = 0
\]

such that their Wronskian

\[
W(w_1, w_2, t) = \det \begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(t) & w_2'(t) \end{vmatrix}
\]

is nonzero for every \( t \). Such functions exist because the set of the solutions of a second-order linear differential equation is a two-dimensional vector space. Since for each \( s \in [0, T] \) the function \( h_s \) is a solution to (2.3), there exist \( z_1, z_2 \) defined on \([0, T]\) such that

\[
(2.4) \quad \forall s, t \in [0, T] \quad h_s(t) = w_1(t)z_1(s) + w_2(t)z_2(s).
\]

Conditions on \( h_s \) at \( s \) and equation (2.4) yield

\[
\begin{cases}
  h_s(s) = 0 = w_1(s)z_1(s) + w_2(s)z_2(s), \\
  h_s'(s) = 1 = w_1'(s)z_1(s) + w_2'(s)z_2(s).
\end{cases}
\]

Since \( W(w_1, w_2, s) \neq 0 \) for each \( s \), we find

\[
z_1(s) = -\frac{w_2(s)}{W(w_1, w_2, s)}, \quad z_2(s) = \frac{w_1(s)}{W(w_1, w_2, s)},
\]

so that \( z_1, z_2 \in C^1([0, T]) \); hence \( h(t, s) = h_s(t) \) belongs to \( C^1([0, T] \times [0, T]) \).

By construction

\[
\forall s \in [0, T] \quad h(s, s) = 0 \quad \text{and} \quad \frac{\partial h}{\partial t}(s, s) = 1
\]

implying

\[
\forall s \in [0, T] \quad \frac{d}{ds} h(s, s) = 0 \iff \forall s \in [0, T] \quad \frac{\partial h}{\partial t}(s, s) + \frac{\partial h}{\partial s}(s, s) = 0
\]

\[
\iff \forall s \in [0, T] \quad \frac{\partial h}{\partial s}(s, s) = -1.
\]

As a consequence,

\[
\forall s \in [0, T] \quad \frac{\partial}{\partial s} \left( \frac{h}{\partial t} \right)(s, s) = -1.
\]

By continuity for a fixed \( t_0 \) in \([0, T]\), there exists \( \delta > 0 \) such that

\[
\forall s, t \in [t_0 - \delta, t_0 + \delta] \cap [0, T] \quad \frac{\partial h}{\partial t}(t, s) > 0 \quad \text{and} \quad \frac{\partial}{\partial s} \left( \frac{h}{\partial t} \right)(t, s) < 0;
\]

for this \( \delta \) (2), (3), and (4) in Property S are satisfied.
Assume, for instance, \( \Phi(t) = [0, \phi(t)] \) and let \( \rho \in L^1([0, T]) \) be such that \( 0 \leq \rho \leq \phi \). For a solution \( x \) to \((P_\rho)\) formula (2.1) yields, in particular,

\[
(2.5) \quad x(T) = \int_0^T h(T, s) \rho(s) \, ds,
\]

\[
(2.6) \quad x'(T) = \int_0^T \partial_t h(T, s) \rho(s) \, ds.
\]

Let us point out that the classical Lyapunov Theorem on the range of a vector measure [4, §16.1] allows us to find a bang–bang solution. In fact, its application yields the existence of a measurable subset \( E \) of \([0, T]\) such that

\[
(2.7) \quad \int_0^T h(T, s) \rho(s) \, ds = \int_0^T h(T, s) \phi(s) \chi_E(s) \, ds,
\]

\[
(2.8) \quad \int_0^T \partial_t h(T, s) \rho(s) \, ds = \int_0^T \partial_t h(T, s) \phi(s) \chi_E(s) \, ds,
\]

so that the function \( \bar{x} \) defined by

\[
\bar{x}(t) = \int_0^t h(t, s) \phi(s) \chi_E(s) \, ds
\]

is, by Proposition 2.1, a bang–bang solution to \((P)\) (with \( \phi_1 = 0, \phi_2 = \phi, x_0 = v_0 = 0 \)). However, for \( 0 < t < T \), the Lyapunov Theorem does not give any information on the relative positions of \( \bar{x} \) and the original solution \( x \).

The purpose of Proposition 2.2 below is to show that if \( s \mapsto (h/\partial h/\partial t)(t, s) \) is monotone on \([0, T]\) then the measurable subset \( E \) can be chosen to be an interval \([\alpha, \beta]\) with \( 0 \leq \alpha \leq \beta \leq T \). Taking into account Property S (4), this will allow us to define in §3 a bang–bang solution \( y \) satisfying \( y(t) \leq x(t) \) for each \( t \).

In what follows \([a, b]\) is an interval of \( \mathbb{R}\), \( \rho \) and \( \phi \) are two functions belonging to \( L^1([a, b]) \) satisfying \( 0 \leq \rho \leq \phi \). We say that \( r \in \mathbb{R}\) is positive (resp. negative) if \( r \geq 0 \) (resp. \( r \leq 0 \)).

We consider the following hypothesis.

**Hypothesis H.** The functions \( f, g \) belong to \( L^\infty([a, b]) \) and are positive almost everywhere. Moreover there exists a strictly monotone positive function \( k \) such that

\[
g(t) = k(t)f(t) \text{ a.e.}
\]

We have the following Lyapunov-type result.

**Proposition 2.2.** Let \( f, g \) satisfy Hypothesis H. Then there exist \( \alpha, \beta \in \mathbb{R} \) such that, if we put \( E = [\alpha, \beta] \), we have

\[
(2.9) \quad \int_a^b \rho(s)f(s) \, ds = \int_\alpha^\beta \phi(s)f(s) \, ds = \int_a^b \phi(s)f(s)\chi_E(s) \, ds,
\]

\[
(2.10) \quad \int_a^b \rho(s)g(s) \, ds = \int_\alpha^\beta \phi(s)g(s) \, ds = \int_a^b \phi(s)g(s)\chi_E(s) \, ds.
\]

Moreover, \( \alpha \) and \( \beta \) are unique if \( \rho, \phi, f, g \) are continuous, and \( 0 < \rho < \phi, f > 0, g > 0 \).

To prove Proposition 2.2, we need the following fundamental lemma.
Lemma 2.1. Assume that $f$, $g$ satisfy Hypothesis H and let $\alpha, \beta \in [a, b]$ be such that

\begin{align}
(2.11) \quad \int_{\alpha}^{\beta} \phi(s)f(s)\,ds &= \int_{\alpha}^{\beta} \rho(s)f(s)\,ds, \\
(2.12) \quad \int_{\alpha}^{\beta} \phi(s)f(s)\,ds &= \int_{\alpha}^{\beta} \rho(s)f(s)\,ds.
\end{align}

Then, if $k$ is increasing, we have

\begin{align}
(2.13) \quad \int_{\alpha}^{\beta} \phi(s)g(s)\,ds &\geq \int_{\alpha}^{\beta} \rho(s)g(s)\,ds, \\
(2.14) \quad \int_{\alpha}^{\beta} \phi(s)g(s)\,ds &\leq \int_{\alpha}^{\beta} \rho(s)g(s)\,ds.
\end{align}

If $k$ is decreasing on $[a, b]$, inequalities (2.13) and (2.14) are reversed. Moreover, inequalities (2.13)–(2.14) are strict if $0 < \rho < \phi$ and $f > 0$, $g > 0$ a.e.

Proof of Lemma 2.1. Assume for instance that $k$ is increasing. To prove (2.14) let $f_\phi, f_\rho$ be the monotone functions defined by

\[ f_\phi(t) = \int_{\alpha}^{t} \phi(s)f(s)\,ds, \quad f_\rho(t) = \int_{\alpha}^{t} \rho(s)f(s)\,ds. \]

The Lebesgue–Stieltjes formula for integration by parts yields

\[
\int_{\alpha}^{\beta} \rho(s)g(s)\,ds = \int_{\alpha}^{\beta} \rho(s)k(s)f(s)\,ds \\
= \int_{\alpha}^{\beta} k(s)\,d f_\rho(s) \\
= k(b)f_\rho(b) - k(a)f_\rho(a) - \int_{\alpha}^{\beta} f_\rho(s)\,dk(s);
\]

analogously we have

\[
\int_{\alpha}^{\beta} \phi(s)g(s)\,ds = k(\beta)f_\phi(\beta) - k(a)f_\phi(a) - \int_{\alpha}^{\beta} f_\phi(s)\,dk(s).
\]

Taking into account that $f_\phi(a) = f_\rho(a) = 0$ and that by (2.12) $f_\rho(b) = f_\phi(\beta)$, we are thus led to show that

\begin{align}
(2.15) \quad \int_{\alpha}^{\beta} f_\rho(s)\,dk(s) - \int_{\alpha}^{\beta} f_\phi(s)\,dk(s) &\leq (k(b) - k(\beta))f_\rho(b).
\end{align}

By our assumptions we have

\begin{align}
(2.16) \quad \forall t \in [a, b] \quad f_\phi(t) &\geq f_\rho(t);
\end{align}

therefore,

\begin{align}
(2.17) \quad \int_{\alpha}^{\beta} f_\rho(s)\,dk(s) - \int_{\alpha}^{\beta} f_\phi(s)\,dk(s) &\leq \int_{\alpha}^{\beta} f_\rho(s)\,dk(s).
\end{align}

Furthermore, since functions $f_\rho$ and $k$ are increasing we have

\[ \int_{\beta}^{b} f_\rho(s)\,dk(s) \leq (k(b) - k(\beta))f_\rho(b). \]
which, together with (2.17), gives (2.15).

To prove the final part of the lemma, it is enough to remark that if \( f > 0 \) and \( \rho > 0 \) then, by (2.12), \( \beta \neq a \); if, moreover, \( 0 < \rho < \phi \) a.e., then inequality (2.16) is strict for every \( t > a \) so that (2.17) is strict too (\( k \) being increasing). Similar arguments prove (2.13).

**Proof of Proposition 2.2.**

(i) **Existence.** (a) Assume first \( 0 < \rho < \phi \) and \( f > 0, g > 0 \) a.e. Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in [a, b] \) be such that

\[
\int_{\alpha_1}^{b} \phi(s)f(s) \, ds = \int_{a}^{b} \rho(s)f(s) \, ds,
\]

\[
\int_{\alpha_2}^{b} \phi(s)g(s) \, ds = \int_{a}^{b} \rho(s)g(s) \, ds,
\]

\[
\int_{\alpha}^{\beta_1} \phi(s)f(s) \, ds = \int_{a}^{b} \rho(s)f(s) \, ds,
\]

\[
\int_{\alpha}^{\beta_2} \phi(s)g(s) \, ds = \int_{a}^{b} \rho(s)g(s) \, ds.
\]

Assume for instance that \( k \) is decreasing on \([a, b]\). In this situation Lemma 2.1 yields

\[
\beta_2 \leq \beta_1, \quad \alpha_2 \leq \alpha_1.
\]

The function \( v \) defined by

\[
v(x) = \int_{a}^{x} \phi(s)f(s) \, ds
\]

is continuous and increasing with values in \([0, v(b)]\): let \( v^{-1} \) denote its inverse function. Set

\[
m = \int_{a}^{b} \rho(s)f(s) \, ds.
\]

Since, by (2.18), \( v(b) = v(\alpha_1) + m \), then \( v(\alpha) + m \in [0, v(b)] \) if and only if \( a \leq \alpha \leq \alpha_1 \); this allows us to introduce the continuous function \( \xi_1 \) defined by the formula

\[
\forall \alpha \in [a, \alpha_1] \quad \xi_1(\alpha) = v^{-1}(v(\alpha) + m).
\]

By definition, we have

\[
\forall \alpha \in [a, \alpha_1] \quad \int_{a}^{\xi_1(\alpha)} \phi(s)f(s) \, ds = v(\xi_1(\alpha)) - v(\alpha) = m = \int_{a}^{b} \rho(s)f(s) \, ds
\]

so that, by (2.20) and (2.22), we deduce

\[
\forall \alpha \in [a, \alpha_1] \quad \xi_1(\alpha) \geq \beta_1 \geq \beta_2.
\]

Similarly, (2.21) allows us to define a continuous function \( \xi_2 : [\beta_2, b] \to \mathbb{R} \) such that we have

\[
\forall \beta \geq \beta_2 \quad \int_{\xi_2(\beta)}^{\beta} \phi(s)g(s) \, ds = \int_{a}^{b} \rho(s)g(s) \, ds,
\]

from which, together with (2.19) and (2.22), we deduce

\[
\forall \beta \geq \beta_2 \quad \xi_2(\beta) \leq \alpha_2 \leq \alpha_1.
\]
We deduce from (2.24) and (2.26) that the composed application
\[ \xi_2 \circ \xi_1 : [a, \alpha_1] \xrightarrow{\xi_1} [\beta_2, b] \xrightarrow{\xi_2} [a, \alpha_1] \]
is defined and continuous from $[a, \alpha_1]$ into itself and, therefore, admits a fixed point $\bar{\alpha}$. Thus, if we set $\bar{\beta} = \xi_1(\bar{\alpha})$ we have $\bar{\alpha} = \xi_2(\bar{\beta})$. Equalities (2.23) and (2.25) with $\alpha, \beta$ replaced by $\bar{\alpha}, \bar{\beta}$ yield the conclusion.

(b) Let $\rho_n = \rho + \frac{1}{n}, \phi_n = \phi + \frac{2}{n}, f_n = f + \frac{1}{n}$ so that $0 < \rho_n < \phi_n$ and $f_n > 0$ a.e., and set $g_n = kf_n$ so that the monotonicity of $k$ implies that $g_n > 0$ a.e. and $f_n, g_n$ satisfy H. By (a) there exist $\alpha_n, \beta_n$ such that
\begin{align*}
(2.27) \quad \int_a^b \rho_n(s)f_n(s)\,ds &= \int_{\alpha_n}^{\beta_n} \phi_n(s)f_n(s)\,ds, \\
(2.28) \quad \int_a^b \rho_n(s)g_n(s)\,ds &= \int_{\alpha_n}^{\beta_n} \phi_n(s)g_n(s)\,ds.
\end{align*}
By compactness we may assume $\alpha_n \to \alpha, \beta_n \to \beta$. The conclusion follows by passing through the limit in (2.27) and (2.28).

(ii) Uniqueness. Assume that $0 < \rho < \phi, f > 0, g > 0$ are continuous and that, for instance, $k$ is decreasing. By (i(a)) the points $\alpha_i$ such that there exists $\beta_i$ satisfying (2.11) and (2.12), are the fixed points of the composed map $\xi_2 \circ \xi_1$. By definition the functions $\xi_1, \xi_2$ are differentiable and we have
\begin{align*}
\forall \alpha \in [a, \alpha_1] \quad \xi_1'(\alpha) &= \frac{v'(\alpha)}{v'(\xi_1(\alpha))} = \frac{\phi(\alpha)f(\alpha)}{\phi(\xi_1(\alpha))f(\xi_1(\alpha))}, \\
\forall \beta \in [\beta_2, b] \quad \xi_2'(\beta) &= \frac{\phi(\beta)g(\beta)}{\phi(\xi_2(\beta))g(\xi_2(\beta))}.
\end{align*}
To prove the claim we notice that if $\alpha$ satisfies $\xi_2 \circ \xi_1(\alpha) = \alpha$ then
\[ (\xi_2 \circ \xi_1)'(\alpha) = \xi_2'(\xi_1(\alpha))\xi_1'(\alpha) = \frac{k(\xi_1(\alpha))}{k(\alpha)}. \]
By (2.23) we have $\xi_1(\alpha) > \alpha$ so that the strict monotonicity of $k$ implies $k(\xi_1(\alpha)) < k(\alpha)$ and thus $(\xi_2 \circ \xi_1)'(\alpha) < 1$ whenever $\xi_2 \circ \xi_1(\alpha) = \alpha$. Let $S = \{ \alpha \in [a, b] : \xi_2 \circ \xi_1(\alpha) = \alpha \}$. Clearly, $S$ is compact and nonempty by (i); moreover, taking (2.29) into account, for each $\alpha \in S$ there exists $\eta$ such that
\[ \forall t \in [\alpha - \eta, \alpha] \quad \xi_2 \circ \xi_1(t) > t, \]
\[ \forall t \in [\alpha, \alpha + \eta] \quad \xi_2 \circ \xi_1(t) < t. \]
As a consequence, the set $S$ has no accumulation points and is therefore finite.

Let $\alpha_1 = \min S$ and assume $S \neq \{ \alpha_1 \}$; let $\alpha_2 = \min S \setminus \{ \alpha_1 \}$. Then by (2.30) there exist $t_1 < t_2 \in [\alpha_1, \alpha_2]$ such that $\xi_2 \circ \xi_1(t_1) < t_1$ and $\xi_2 \circ \xi_1(t_2) > t_2$. Therefore there exists $\bar{t} \in [t_1, t_2]$ such that $\xi_2 \circ \xi_1(\bar{t}) = \bar{t}$, a contradiction. \[ \square \]

3. Main result.

**Theorem 3.1.** Let $x \in W^{2,1}([0, T])$ be a solution to (P). Then there exists a bang–bang solution $y$ to (P) satisfying
\[ \forall t \in [0, T] \quad y(t) \leq x(t). \]
Moreover, there exists a set $E$ which is a finite union of intervals such that
\[ y'' + a_1(t)y' + a_0(t)y = \phi_1(t)x_E(t) + \phi_2(t)x_{[0,T] \setminus E}(t) \text{ a.e.} \]
COROLLARY 1. Under the above assumption, there exists a bang-bang solution $y$ satisfying
\[ \forall t \in [0, T] \quad y(t) \geq x(t). \]

Proof of Corollary 1. Let $-\Phi$ be defined by the equality $(-\Phi)(t) = -\Phi(t)$. Clearly, $\bar{x} = -x$ solves
\[ \bar{x}'' + a_1(t)\bar{x}' + a_0(t)\bar{x} \in -\Phi(t) \text{ a.e.} \]
By Theorem 3.1 there exists a bang-bang solution $\tilde{y}$ satisfying the same boundary conditions as $\bar{x}$ and satisfying
\[ \forall t \in [0, T] \quad \tilde{y}(t) \leq \bar{x}(t). \]
Then the function $y$ defined by
\[ \forall t \in [0, T] \quad y(t) = -\tilde{y}(t) \]
is a solution to our problem. \[ \square \]

Proof of Theorem 3.1. Let $h$ be the function defined in Proposition 2.1. (i) We show that it is not restrictive to assume
\[ \Phi(t) = [0, \phi(t)] \quad (\phi \in L^1([0, T]), \phi > 0 \text{ a.e.}) \quad \text{and} \quad x_0 = v_0 = 0. \]
In fact, let $\Phi(t) = [\phi_1(t), \phi_2(t)]$ and $x$ satisfy
\[ x'' + a_1(t)x' + a_0(t)x \in \Phi(t) \text{ a.e.} \]
Then the function $\bar{x}$ defined by
\[ \bar{x}(t) = x(t) - x'(0)t - x(0) \]
satisfies $\bar{x}(0) = \bar{x}'(0) = 0$ and
\[ \bar{x}'' + a_1(t)\bar{x}' + a_0(t)\bar{x} \in [\psi_1(t), \psi_2(t)] \text{ a.e.,} \]
where
\[ \psi_1(t) = \phi_1(t) - a_0(t)x'(0)t - a_1(t)x'(0) - a_0(t)x(0), \]
\[ \psi_2(t) = \phi_2(t) - a_0(t)x'(0)t - a_1(t)x'(0) - a_0(t)x(0). \]
Moreover, by Proposition 2.1, the function $\bar{x}$ defined by
\[ \bar{x}(t) = \bar{x}(t) - \int_0^t h(t, s)\psi_1(s) ds \]
satisfies $\bar{x}(0) = 0$, $\bar{x}'(0) = 0$ and
\[ \bar{x}'' + a_1(t)\bar{x}' + a_0(t)\bar{x} \in [0, \psi_2(t) - \psi_1(t)] \text{ a.e.} \]
If we assume that Theorem 3.1 holds for such an interval and initial boundary conditions, there exists a function $\tilde{y}$ satisfying
\[ \tilde{y}(0) = \bar{x}(0), \quad \tilde{y}'(0) = \bar{x}'(0), \quad \tilde{y}(T) = \bar{x}(T), \quad \tilde{y}'(T) = \bar{x}'(T), \quad \tilde{y}'' + a_1(t)\tilde{y}' + a_0(t)\tilde{y} \in \{0, \psi_2(t) - \psi_1(t)\} \text{ a.e.}, \]
\[ \forall t \in [0, T] \quad \tilde{y}(t) \leq \bar{x}(t). \]
It is now easy to check that the function $y$ defined by
\[ y(t) = \bar{y}(t) + \int_0^t h(t, s)\psi_1(s)\, ds + x'(0)t + x(0) \]
is a solution to our problem.

(ii) Assume first that the $\delta$ of Property (S) can be chosen in such a way that $I_\delta = [0, T]$. In this case, if we set
\[ \rho = x'' + a_1 x' + a_0 x \]
then by Proposition 2.1 we can write
\[ x(t) = \int_0^t h(t, s)\rho(s)\, ds, \]
where $h$ satisfies Property (S(1)) and, in addition,
\[ h(t, s) > 0 \text{ if } s < t, \quad h(t, s) < 0 \text{ if } t < s, \]
\[ \forall s, t \in [0, T] \, \frac{\partial h}{\partial t}(t, s) > 0, \]
\[ \forall t \in [0, T] \, s \mapsto h(t, s)/\frac{\partial h}{\partial t}(t, s) \text{ is decreasing on } [0, t]. \]

In particular, the functions $f$ and $g$ defined on $[0, T]$ by
\[ g(s) = h(T, s), \quad f(s) = \frac{\partial h}{\partial t}(T, s) \]
verify Hypothesis H with $k(\cdot) = h(T, \cdot)/\frac{\partial h}{\partial t}(T, \cdot)$.

By Proposition 2.1, each bang--bang solution $y$ such that $x(0) = x'(0) = 0$ is given by the formula $y(t) = \int_0^t h(t, s)\nu(s)\, ds$ for some measurable function $\nu$ with values in $\{0, \phi(t)\}$.

We are thus led to show that there exists such a $\nu$ satisfying
\[ \int_0^T h(T, s)\rho(s)\, ds = \int_0^T h(T, s)\nu(s)\, ds, \]
\[ \int_0^T \frac{\partial h}{\partial t}(T, s)\rho(s)\, ds = \int_0^T \frac{\partial h}{\partial t}(T, s)\nu(s)\, ds, \]
and for each $t$ in $[0, T]$,
\[ \int_0^t h(t, s)\rho(s)\, ds \geq \int_0^t h(t, s)\nu(s)\, ds. \]

(a) Assume $0 < \rho < \phi$ a.e.

By Proposition 2.2 there exist $\alpha, \beta \in [0, T]$ such that
\[ \int_0^T h(T, s)\rho(s)\, ds = \int_\alpha^\beta h(T, s)\phi(s)\, ds, \]
\[ \int_0^T \frac{\partial h}{\partial t}(T, s)\rho(s)\, ds = \int_\alpha^\beta \frac{\partial h}{\partial t}(T, s)\phi(s)\, ds. \]

It is clear that if we set
\[ \nu(s) = \phi(s)\chi_{[\alpha, \beta]}(s) \]
then (3.5) and (3.6) are satisfied. In order to prove (3.7) we first show that under our assumptions on \( \rho \) and \( \phi \) we have

\[
(3.11) \quad 0 < \alpha < \beta < T.
\]

Notice first that the equalities \( (\alpha, \beta) = (0, T) \) or \( \alpha = \beta \) cannot hold otherwise by (3.8), \( \rho = \phi \) or \( \rho = 0 \) a.e., a contradiction. Assume, for instance, \( \alpha = 0 \) and \( \beta < T \), the case \( \alpha > 0 \) and \( \beta = T \) being similar. Under this assumption, equalities (3.8) and (3.9) become

\[
(3.12) \quad \int_0^T h(T, s) \rho(s) \, ds = \int_0^\beta h(T, s) \phi(s) \, ds,
\]

\[
(3.13) \quad \int_0^T \frac{\partial h}{\partial t}(T, s) \rho(s) \, ds = \int_0^\beta \frac{\partial h}{\partial t}(T, s) \phi(s) \, ds.
\]

Property (3.4) and the assumption \( 0 < \rho < \phi \) a.e. allow us to apply Lemma 2.1, from which we deduce

\[
\int_0^T h(T, s) \rho(s) \, ds < \int_0^\beta h(T, s) \phi(s) \, ds,
\]

contradicting (3.12).

Set \( y(t) = \int_0^t h(t, s) \nu(s) \, ds \) so that (3.8) and (3.9) become \( y(T) = x(T) \) and \( y'(T) = x'(T) \).

The purpose of what follows is to show (3.7), i.e., that \( y(t) \leq x(t) \) for each \( t \). We consider the cases \( t \in [0, \alpha] \), \( t \in [\beta, T] \), \( t \in [\alpha, \beta] \) separately.

Inequality (3.7) is trivial if \( t \leq \alpha \); in fact we have

\[
y(t) = 0 \leq \int_0^t h(t, s) \rho(s) \, ds = x(t),
\]

the inequality being strict for \( t \in ]0, \alpha] \). In particular

\[
(3.14) \quad y(\alpha) < x(\alpha).
\]

Assume \( t \in [\beta, T] \).

Since, taking (3.2) into account, \( h(t, s) \leq 0 \) whenever \( s \geq t \), we have

\[
(3.15) \quad \forall t \geq \beta \quad \int_t^T h(t, s) \rho(s) \, ds \leq 0 = \int_t^T h(t, s) \nu(s) \, ds
\]

or, equivalently,

\[
(3.16) \quad \forall t \geq \beta \quad \int_0^T h(t, s) \rho(s) \, ds - \int_0^t h(t, s) \rho(s) \, ds \leq \int_0^T h(t, s) \nu(s) \, ds - \int_0^t h(t, s) \nu(s) \, ds.
\]

Therefore, in order to prove that \( y(t) \leq x(t) \) for \( t \in [\beta, T] \) it is enough to show that

\[
(3.17) \quad \forall t \in [\beta, T] \quad \int_0^T h(t, s) \rho(s) \, ds = \int_0^T h(t, s) \nu(s) \, ds.
\]
For this purpose, we use Property (S(1)). Equalities (3.8) and (3.9) become

\[
\begin{cases}
w_1(T) \int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds + w_2(T) \int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0, \\
 w_1'(T) \int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds + w_2'(T) \int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0.
\end{cases}
\]

The condition on the Wronskian of \(w_1, w_2\) at \(T\) implies

\[
\begin{align*}
(3.18) & \quad \int_0^T z_1(s)(\rho(s) - \nu(s)) \, ds = 0, \\
(3.19) & \quad \int_0^T z_2(s)(\rho(s) - \nu(s)) \, ds = 0.
\end{align*}
\]

Multiplying (3.18) by \(w_1(t)\), (3.19) by \(w_2(t)\), and adding the two equations we obtain

\[
\int_0^T (w_1(t)z_1(s) + w_2(t)z_2(s))\rho(s) \, ds = \int_0^T (w_1(t)z_1(s) + w_2(t)z_2(s))\nu(s) \, ds,
\]

which, together with Property (S(1)), gives (3.17). Moreover, note that since inequality (3.15) is strict for \(t \neq T\),

\[
(3.20) \quad y(\beta) < x(\beta).
\]

At this stage, we only need to prove that (3.7) holds for \(t \in [\alpha, \beta]\).

Assume by contradiction that there exists \(t \in [\alpha, \beta]\) such that \(x(t) = y(t)\). Let

\[
\tilde{t} = \sup\{t \in [\alpha, \beta] : x(t) = y(t)\}.
\]

Then \(\alpha < \tilde{t} < \beta\) and, by the very definition of \(\tilde{t}\), \(x(\tilde{t}) = y(\tilde{t})\) so that

\[
y'(\tilde{t}) - y'(-\tilde{t}) = \lim_{\tilde{t} \to \pm} \frac{y(t) - x(t)}{t - \tilde{t}} \leq 0.
\]

It follows that

\[
(3.21) \quad \int_\alpha^\tilde{t} h(\tilde{t}, s)\phi(s) \, ds = \int_0^\tilde{t} h(\tilde{t}, s)\rho(s) \, ds,
\]

\[
(3.22) \quad \int_\alpha^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s)\phi(s) \, ds \leq \int_0^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s)\rho(s) \, ds.
\]

For each \(s \in [0, \tilde{t}]\) let \(f(s) = h(\tilde{t}, s), g(s) = \frac{\partial h}{\partial t}(\tilde{t}, s)\), and \(k = g/f\) so that by (3.2)-(3.4) the function \(k\) is increasing and \(f > 0, g > 0\). If we replace \((a, b)\) by \((0, \tilde{t})\), Lemma 2.1 together with (3.21) implies that

\[
\int_\alpha^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s)\phi(s) \, ds > \int_0^\tilde{t} \frac{\partial h}{\partial t}(\tilde{t}, s)\rho(s) \, ds,
\]

thus contradicting (3.22).

(b) Assume, in general, \(0 \leq \rho \leq \phi\) a.e. and let \(\phi_n, \rho_n \in L^1([0, T])\) be such that

\[
0 < \rho_n < \phi_n \text{ a.e. and } \rho_n \to \rho, \phi_n \to \phi \text{ in } L^1([0, T])
\]

(for instance, \(\rho_n = \rho + \frac{1}{n}, \phi_n = \phi + \frac{2}{n}\)).
Corresponding to each \( n \), there exist \( \alpha_n, \beta_n \in [0, T] \) such that, if we set \( \nu_n = \phi_n \chi_{[\alpha_n, \beta_n]} \), we have

\[
(3.23) \quad \int_0^T h(T, s) \rho_n(s) \, ds = \int_0^T h(T, s) \nu_n(s) \, ds,
\]

and, for each \( t \in [0, T] \),

\[
(3.25) \quad \int_0^t h(t, s) \rho_n(s) \, ds \geq \int_0^t h(t, s) \nu_n(s) \, ds.
\]

Because the interval \([0, T]\) is compact, we may assume \( \alpha_n \to \alpha, \beta_n \to \beta \) for some \( \alpha \leq \beta \in [0, T] \).

Clearly \( \nu_n = \phi_n \chi_{[\alpha_n, \beta_n]} \) converges to \( \phi \chi_{[\alpha, \beta]} \) in \( L^1([0, T]) \); therefore, if we pass through the limit in (3.23), (3.24), and (3.25) and we set \( \nu = \phi \chi_{[\alpha, \beta]} \), we obtain (3.5), (3.6), and (3.7).

(iii) In the general case, using Property (S) and the compactness of \([a, b]\), there exists a subdivision \( a_0 = 0 < a_1 < \cdots < a_i < T = a_{i+1} \) of \([0, T]\) such that, if we put \( I_j = [a_j, a_{j+1}] \), we have

- \( \forall s, t \in I_j \quad h(t, s) > 0 \) if \( s < t \), \( h(t, s) < 0 \) if \( t < s \);
- \( \forall s, t \in I_j \quad \frac{\partial h}{\partial t}(t, s) > 0 \);
- \( \forall t \in I_j \quad s \mapsto h(t, s) / \frac{\partial h}{\partial t}(t, s) \) is decreasing on \( I_j \).

By (ii), on each interval \( I_j \) there exist \( \alpha_j, \beta_j \) such that the solution \( y_j \) to the problem

\[
y'' + a_1(t)y' + a_0(t)y = \phi_1(t)\chi_{[a_j, a_{j+1}]} + \phi_2(t)\chi_{[\beta_j, \beta_{j+1}]}(t) \quad \text{a.e. on } I_j
\]

with the initial conditions

\[
y_j(a_j) = x(a_j), \quad y_j'(a_j) = x'(a_j)
\]
satisfies the equalities

\[
y_j(a_{j+1}) = x(a_{j+1}), \quad y_j'(a_{j+1}) = x'(a_{j+1}),
\]

and, moreover, \( y_j(t) \leq x(t) \) for each \( t \in I_j \).

Clearly the function \( y \in W^{2,1}([0, T]) \) obtained by gluing together the functions \( y_j \) is a solution to our problem. \( \square \)

**Remark 3.1.** The proof of Theorem 3.1, part (ii(a)) shows in fact that when \( 0 < \rho < \phi \), we have \( y(t) < x(t) \) on \([0, T]\).

**Remark 3.2.** With the notations introduced in Proposition 2.1, the proof of Theorem 3.1, part (ii) shows that if \( T = \delta \) then, given a solution \( x \) to (P), there exists a bang-bang solution \( y \leq x \) satisfying

\[
y'' + a_1(t)y' + a_0(t)y = \min \Phi(t) \text{ on } [0, \alpha] \cup [\beta, T],
\]

\[
y'' + a_1(t)y' + a_0(t)y = \max \Phi(t) \text{ on } [\alpha, \beta].
\]

Because the number \( \delta \) depends only on the function \( h \), it can happen that \( \delta = +\infty \).
This is the case when $a_1$ and $a_0$ are constant and the equation $\lambda^2 + a_1\lambda + a_0 = 0$ admits two real roots $\lambda_1, \lambda_2$. In fact, under this assumption we have either

$$h(t, s) = \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)}) \text{ if } \lambda_1 \neq \lambda_2, \text{ or}$$

$$h(t, s) = (t-s)e^{\lambda(t-s)} \text{ if } \lambda_1 = \lambda_2 = \lambda.$$

4. Applications. Our first application concerns the reachable set of bang–bang constrained solutions. Let $c$ be an arbitrary function defined on $[0, T]$ and consider the reachable sets $X_T^c$ and $Y_T^c$ associated with (P) defined by

$$X_T^c = \{(y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \Phi(t), (y(0), y'(0)) = (x_0, v_0)\},$$

$$Y_T^c = \{(y(T), y'(T)) : y \leq c, y'' + a_1(t)y' + a_0(t)y \in \text{extr } \Phi(t), (y(0), y'(0)) = (x_0, v_0)\}.$$

Then Theorem 3.1 claims $X_T^c = Y_T^c$, whence $Y_T^c$ is convex.

Finally, we give an application to the calculus of variations.

THEOREM 4.1. Let $a_0, a_1 \in C([0, T])$, $\phi_1, \phi_2 \in L^1([0, T])$ verify $\phi_1(t) \leq \phi_2(t)$. Let $x_0, v_0, x_1, v_1$ be 4 fixed real numbers. Then there exists a dense subset $D$ of $C(\mathbb{R})$ for the uniform convergence such that for $g$ in $D$ the problem

$$\text{minimize } \left\{ \int_0^T g(x(t)) \, dt + \int_0^T h(\rho(t)) \, dt \right\}$$

on the subset of $W^{2,1}([0, T]) \times L^1([0, T])$ of those functions $(x, \rho)$ satisfying

$$(x(0), x'(0), x(T), x'(T)) = (x_0, v_0, x_1, v_1), \quad x'' + a_1(t)x' + a_0(t)x = \rho(t) \in [\phi_1(t), \phi_2(t)]$$

admits at least one solution for every lower semicontinuous function $h$ satisfying the growth condition $h(u) \geq c\psi(|u|)$, $\psi$ being lower semicontinuous and convex, $\lim_{r \to +\infty} \psi(r)/r = +\infty$.

Proof. With Theorem 3.1 and the preceding application, the proof is a direct adaptation of the proof given in [3].

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