Geometria Superiore Reference Cards

Homological Categories

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Additive Categories.

A category $\mathcal{C}$ is **additive** if the following conditions hold:

(i) for any objects X and Y, the morphisms set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ has a structure of abelian group;

(ii) the composition law is bilinear w.r.t. the group structures;

(iii) there exists a zero object 0 such that $\mathrm{Hom}_{\mathcal{C}}(0,0)=0$ (trivial group);

(iv) for any pair of objects, X and Y, there exists the direct sum $X\oplus Y$;

(v) for any pair of objects, X and Y, there exists the direct product $X\times Y$.

Notice that the conditions (i), (ii) and (iii) make equivalent (iv) and (v), and moreover make isomorphic $X\oplus Y$ and $X\times Y$. The condition (iv) (resp. (v)) is equivalent to the existence of a left (resp. right) adjoint to the diagonal inclusion of categories of $\mathcal{C}$ in $\mathcal{C}\times\mathcal{C}$.

A functor F between additive categories $\mathcal{C}$ and $\mathcal{D}$ is additive if for any X and Y objects of $\mathcal{C}$ the maps $\mathrm{Hom}_{\mathcal{C}}(X,Y)\to\mathrm{Hom}_{\mathcal{D}}(FX,FY)$, $f\mapsto F(f)$, are group morphisms. Notice that the functors $h_X=\mathrm{Hom}(X,-)$ and $h^{\circ}_X=\mathrm{Hom}(-,X)$ are additive.

Kernel and Cokernel: given a morphism $f:X\to Y$ in $\mathcal{C}$, we define the functor in $\mathcal{C}^{\vee}=\mathcal{F}unct(\mathcal{C}^{\circ},\mathbf{Set})$ by

$$W\mapsto\ker(f_*:\mathrm{Hom}_{\mathcal{C}}(W,X)\to\mathrm{Hom}_{\mathcal{C}}(W,Y))$$

where f_* is given by composition with f: $f_*g=f\circ g$. We define $\ker f$ object of $\mathcal{C}$ to be the representant of this functor, if it exists. There is a natural α: $\ker f\to X$ with $f\circ\alpha=0$ and such that the composition with α gives the bijection $\mathrm{Hom}_{\mathcal{C}}(W,\ker f)\longrightarrow\ker(f_*:\mathrm{Hom}_{\mathcal{C}}(W,X)\to\mathrm{Hom}_{\mathcal{C}}(W,Y))$, from which we have the following universal property:

$$\begin{tikzcd}
\ker f \arrow[r, "\alpha"] & X \arrow[r, "f"] & Y \\
 & \forall W \arrow[ul, "\exists!"] \arrow[u, "\forall g"] \arrow[ur, "0"'] &
\end{tikzcd}$$

A morphism is a monomorphism iff $\ker f=0$; in particular the morphism α is mono.

Dually we define a functor in $\mathcal{C}^{\wedge}=\mathcal{F}unct(\mathcal{C},\mathbf{Set})$ given by

$$W\mapsto\ker\left(f^*:\mathrm{Hom}_{\mathcal{C}}(Y,W)\to\mathrm{Hom}_{\mathcal{C}}(X,W)\right)$$

where f^* is given by composition with f: $f^*h=h\circ f$. We define $\mathrm{coker}f$ as the representant, if it exists, of this functor. There is a natural morphism γ: $Y\to\mathrm{coker}f$ for which $\gamma\circ f=0$ and such that the composition with γ gives the bijection $\mathrm{Hom}_{\mathcal{C}}(\mathrm{coker}f,W)\longrightarrow\ker(\mathrm{Hom}_{\mathcal{C}}(Y,W)\to\mathrm{Hom}_{\mathcal{C}}(X,W))$, from which we have the following universal property:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "0"'] & Y \arrow[r, "\gamma"] \arrow[d, "\forall g"] & \mathrm{coker}f \arrow[dl, "\exists!"] \\
 & \forall W &
\end{tikzcd}$$

Notice that f is epimorphism iff $\mathrm{coker}f=0$, and that γ is.

Image and Coimage: We put $\mathrm{im}f=\ker(\gamma:Y\to\mathrm{coker}f)$ and $\mathrm{coim}f=\mathrm{coker}(\alpha:\ker f\to X)$ and we call α', γ' the canonical morphisms $\alpha':\mathrm{im}f\hookrightarrow Y$ and $\gamma':X\twoheadrightarrow\mathrm{coim}f$. If the defined objects exists for f we have the following situation:

$$\begin{tikzcd}
\ker f \arrow[r, "\alpha"] & X \arrow[r, "f"] \arrow[d, "\gamma'"'] & Y \arrow[r, "\gamma"] & \mathrm{coker}f \\
 & \mathrm{coim}f & \mathrm{im}f \arrow[u, "\alpha'"'] &
\end{tikzcd}$$

There exists a canonical morphism $\bar{f}:\mathrm{coim}f\longrightarrow\mathrm{im}f$ whose construction can be given in two ways:

(i) $f\circ\alpha=0$, so $\exists!$ f': $\mathrm{coim}f\to Y$ s.t. $f'\circ\gamma'=f$; by composition $\gamma\circ f'\circ\gamma'=\gamma\circ f=0$, and γ is epi, so $f'\circ\gamma=0$, and finally $\exists!$ $\bar{f}$: $\mathrm{coim}f\to\mathrm{im}f$ s.t. $\alpha'\circ\bar{f}=f'$;

(i') or dually $\gamma\circ f=0$, so $\exists!$ f'': $X\to\mathrm{im}f$ s.t. $\alpha'\circ f''=f$; by composition $\alpha'\circ f''\circ\alpha=f\circ\alpha=0$ and α' is mono, so $f''\circ\alpha=0$ and finally $\exists!$ $\bar{f}$: $\mathrm{coim}f\to\mathrm{im}f$ s.t. $\bar{f}\circ\gamma'=f''$.

The morphism f is **strict** if it admits the canonical decomposition with f' an isomorphism. Notice that the morphisms α, γ, α', γ' are strict if they exist.

Abelian Categories.

(pA) An additive category $\mathcal{C}$ is called **pre-Abelian** if any morphism admits kernel and cokernel. This is equivalent to the existence of the right and left adjoint functors for the functor $\mathcal{C}$ in $\mathcal{F}(\mathcal{C})$ sending any object of $\mathcal{A}$ in its zero morphism.

Remark the existence of fiber products (pull-backs) and amalgamed sums (push-outs) realized by the formulas:

for $X\xrightarrow{f}Z\xleftarrow{g}Y$ we have $X\times_Z Y\cong\ker((f,-g):X\times Y\to Z)$;

for $X\xleftarrow{f}Z\xrightarrow{g}Y$ we have $X\oplus_Z Y\cong\mathrm{coker}\left(\binom{f}{-g}:Z\to X\times Y\right)$.

(qA) A pre-Abelian category $\mathcal{C}$ is called **quasi-Abelian** if the strict monomorphisms are co-universal and (dually) the strict epimorphisms are universal. Notice that for any morphism f in a quasi-abelian category, $\bar{f}$ is always a bijection (i.e. injective and surjective). Moreover the composition of strict monomorphisms (resp. epimorphisms) is a strict mono (resp. epi).

(A) A pre-Abelian category $\mathcal{A}$ is called **Abelian** if all morphisms are strict, i.e. if the canonical morphism $\bar{f}$ is an isomorphism for any f. If $\mathcal{A}$ is abelian, then a morphism is an isomorphism if and only if it is a bijection. An abelian category is quasi-abelian.

Remark the self-duality of the axioms, so that if $\mathcal{C}$ is an additive, pre-abelian, quasi-abelian, abelian category, then also $\mathcal{C}^{\circ}$ is.

Exactness of sequences. A zero sequence $X\xrightarrow{f}Y\xrightarrow{g}Z$ in an Abelian category (i.e. such that $g\circ f=0$) is an exact sequence in Y if the canonical morphism $\mathrm{im}f\simeq\mathrm{coim}f\to\ker g$ is an isomorphism. A sequence is exact if it is exact in any (non extremal) object. We have the following standard exact sequences:

$$0\longrightarrow\ker f\xrightarrow{\alpha}X\xrightarrow{\gamma'}\mathrm{coim}f\longrightarrow 0$$

$$0\longrightarrow\ker f\xrightarrow{\alpha}X\xrightarrow{f}Y\xrightarrow{\gamma'}\mathrm{coim}f\longrightarrow 0$$

$$0\longrightarrow\mathrm{im}f\xrightarrow{\alpha'}Y\xrightarrow{\gamma}\mathrm{coker}f\longrightarrow 0$$

A morphism $f:X\to Y$ is a monomorphism (resp. epimorphism, isomorphism) iff the sequence $0\to X\to Y$ (resp. $X\to Y\to 0$, $0\to X\to Y\to 0$) is exact.

A sequence $0\to X\xrightarrow{f}Y\xrightarrow{g}Z$ is exact in $\mathcal{A}$ if and only if the sequence

$$0\to\mathrm{Hom}_{\mathcal{A}}(W,X)\xrightarrow{f_*}\mathrm{Hom}_{\mathcal{A}}(W,Y)\xrightarrow{g_*}\mathrm{Hom}_{\mathcal{A}}(W,Z)$$

in **Ab** is exact for any $W\in\mathcal{A}$; dually, a sequence $X\xrightarrow{f}Y\xrightarrow{g}Z\to 0$ is exact in $\mathcal{A}$ if and only if the sequence

$$0\to\mathrm{Hom}_{\mathcal{A}}(Z,W)\xrightarrow{g^*}\mathrm{Hom}_{\mathcal{A}}(Y,W)\xrightarrow{f^*}\mathrm{Hom}_{\mathcal{A}}(X,W)$$

in **Ab** is exact for any $W\in\mathcal{A}$.

Exactness of functors. A (covariant) additive functor F: $\mathcal{A}\to\mathcal{A}'$ of abelian categories is:

(r) right exact if it sends right exact sequences $X\to Y\to Z\to 0$ in right exact sequences (so it respects epimorphisms and cokernels);

(l) left exact if it sends left exact sequences $0\to X\to Y\to Z$ in left exact sequences (so it respects monomorphisms and kernels);

(e) exact if it transforms exact sequences in exact sequences (in particular it is right and left exact).

We define the exactness of a contravariant functor G form $\mathcal{A}$ to $\mathcal{A}'$ as exactness of $G:\mathcal{A}^\circ\to\mathcal{A}'$ (i.e. we use the sides on the image category).

For an abelian category the functors $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ are left exact. Notice that a short sequence is left exact iff the image sequence by $\mathrm{Hom}(X,-)$ is left exact; dually a short sequence is right exact iff the image sequence by $\mathrm{Hom}(-,X)$ is left exact.

Injective and Projective Objects. An object X of an abelian category $\mathcal{A}$ is:

(p) projective if the covariant functor $\mathrm{Hom}_{\mathcal{A}}(X,-)$ is exact; it is enough that it respects the epimorphisms, i.e. that the property of the following diagram hold

$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & 0 \\ & X \arrow[ul, "\exists"] \arrow[u, "\forall"'] & \end{tikzcd}$$

for any exact sequence as the first line; remark that the direct sum of a family of objects is projective if and only if any object is;

(i) injective if the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,X)$ is exact; it is enough that it respects the monomorphisms, i.e. that the property of the following diagram hold

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\forall"'] & B \arrow[dl, "\exists"] \\ & X & \end{tikzcd}$$

for any exact sequence as the first line; remark that the product of a family of objects is injective if and only if any object is.

Split sequences. A zero sequence $0\to X\overset{i_X}{\to}Y\overset{p_Z}{\to}Z\to 0$ in an additive category $\mathcal{C}$ is split if the following conditions hold:

(i) there exists $p_X:Y\to X$ such that $p_X\circ i_X=\mathrm{id}_X$;

(i') there exists $i_Z:Z\to Y$ such that $p_Z\circ i_Z=\mathrm{id}_Z$;

(j) $p_X\circ i_Z=0$ and $i_X\circ p_X+i_Z\circ p_Z=\mathrm{id}_Y$.

The sequence is split iff Y is isomorphic to the direct sum of X and Z. Any additive functor respects the split sequences; in particular the Hom functors send split sequences in exact (and split) sequences of abelian groups.

Moreover, a sequence is split iff the sequence

$$0\to\mathrm{Hom}_{\mathcal{A}}(W,X)\to\mathrm{Hom}_{\mathcal{A}}(W,Y)\to\mathrm{Hom}_{\mathcal{A}}(W,Z)\to 0$$

or

$$0\to\mathrm{Hom}_{\mathcal{A}}(Z,W)\to\mathrm{Hom}_{\mathcal{A}}(Y,W)\to\mathrm{Hom}_{\mathcal{A}}(X,W)\to 0$$

is exact in **Ab** for any $W\in\mathcal{A}$.

If $\mathcal{A}$ is an Abelian category, a split sequence is exact, and a zero sequence is split iff one of the first two conditions holds. If $0\to A_1\to A_2\to A_3\to 0$ is an exact sequence, we have the following facts:

(i) if A_1 is injective then the sequence splits and A_2 is injective iff A_3 is;

(i') if A_3 is projective then the sequence splits and A_1 is projective iff A_2 is.

The five lemma: let

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"'] & A_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "\alpha_3"'] & A_4 \arrow[r, "f_4"] \arrow[d, "\alpha_4"'] & A_5 \arrow[d, "\alpha_5"'] \\ B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4 \arrow[r, "g_4"'] & B_5 \end{tikzcd}$$

be a commutative diagram with exact rows in an Abelian category; then we have the following implications:

(i) if α_2 and α_4 are monomorphisms and α_1 is an epimorphism then α_3 is a monomorphism;

(i') if α_2 and α_4 are epimorphisms and α_5 is a monomorphism then α_3 is an epimorphism;

(j) if α_2 and α_4 are isomorphisms, α_1 is an epimorphism and α_5 is a monomorphism then α_3 is an isomorphism.

Triangulated Categories.

An additive category $\mathcal{T}$ is triangulated if it is endowed with an automorphism $T:\mathcal{T}\to\mathcal{T}$ (the translation functor) and a family $\mathcal{T}r(\mathcal{T})$ of distinguished "triangles" $X\overset{u}{\to}Y\overset{v}{\to}Z\overset{w}{\to}T(X)$ written also as $\begin{tikzcd}[column sep=small] & Z \arrow[dl, "+1"'] & \\ X \arrow[rr, "u"'] & & Y \arrow[ul, "v"'] \end{tikzcd}$ or $X\to Y\to Z\overset{+1}{\to}$ or (X,Y,Z,u,v,w). We call morphism of triangles any system (f,g,h) making commutative

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] \arrow[d, "h"'] & TX \arrow[d, "Tf"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & TX' \end{tikzcd}$$

and we write also $(f,g,h):(X,Y,Z)\to(X',Y',Z')$; isomorphism if f,g,h are. The data are subject to the following axioms:

($TR0$) for any object X the triangle $X\overset{\mathrm{id}_X}{\to}X\overset{0}{\to}0\overset{0}{\to}TX$ is distinguished;

($TR1$) any morphism $u:X\to Y$ can be completed in a distinguished triangle $X\overset{u}{\to}Y\overset{v}{\to}Z\overset{+1}{\to}$;

($TR2$) (shift axiom) $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}T(X)$ is a distinguished triangle iff $Y\xrightarrow{v}Z\xrightarrow{w}T(X)\xrightarrow{-T(u)}T(Y)$ is a distinguished triangle;

($TR3$) any triangle isomorphic to a distinguished triangle is distinguished;

($TR4$) given two distinguished triangles and morphisms f,g such that

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"'] & Z \arrow[r, "w"] & TX \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & TX' \end{tikzcd}$$

commutes, there exists a morphism $h:Z\to Z'$ which completes the morphism of triangles (we don't require unicity);

($TR5$) (octahedral axiom) given three distinguished triangles on a commutative diagram:

$$\begin{tikzcd} Y' \arrow[dr, "+1"'] & & Z \arrow[ll] \arrow[rr] & & X' \arrow[dl, "+1"] \\ & X \arrow[ur, "v\circ u"] \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \arrow[dl] & \\ & & Z' \arrow[ul, "+1"] & & \end{tikzcd}$$

there exist f,g such that the "external" triangle:

$$\begin{tikzcd} Y' \arrow[rr, "g"] & & X' \arrow[dl, "+1"] \\ & Z' \arrow[ul, "f"] & \end{tikzcd}$$

(where the morphism $+1$ from X' to $T(Z')$ is that given by the composition) is a distinguished triangle and the (octahedral) diagram obtained by superposition has commutative (or distinguished) triangles; in other words, the diagram gives the following morphisms of triangles:

$$(Y,Z,X')\longrightarrow(Z',Y',X')\longrightarrow(Z',TX,TY)\ .$$

The last axiom can be written also in two other forms:

$(TR5')$

$$\begin{tikzcd}
 & Y \arrow[rr] \arrow[dr, "v"] & & Z' \arrow[r, "+1"] \arrow[dr] & {} & \\
X \arrow[ur, "u"] \arrow[rr, "v\circ u"'] & & Z \arrow[rr] \arrow[dr] & & Y' \arrow[r, "+1"] \arrow[dl] & {} \\
 & & & X' \arrow[dl, "+1"'] \arrow[dr, "+1"] & & \\
 & & {} & & {} &
\end{tikzcd}$$

Remarks that this formulation makes evident, in the case of derived categories, an easy interpretation of the axiom: it corresponds to the canonical isomorphism of abelian categories "$(Z/X)/(Y/X)\cong Z/Y$".

$(TR5'')$ given the "lower cup" of an octahedron, we can complete the "upper cup":

$$\begin{tikzcd}
Z \arrow[rr] & & X' \arrow[dd, "+1"] \arrow[dl, "+1"'] \\
 & Y \arrow[ul, "v"'] \arrow[dr] & \\
X \arrow[uu, "v\circ u"] \arrow[ur, "u"] & & Z' \arrow[ll, "+1"]
\end{tikzcd}
\quad \text{to} \quad
\begin{tikzcd}
Z \arrow[rr] \arrow[dr] & & X' \arrow[dd, "+1"] \\
 & Y' \arrow[ur] \arrow[dl, "+1"'] & \\
X \arrow[uu, "v\circ u"] & & Z' \arrow[ll, "+1"] \arrow[ul]
\end{tikzcd}$$

Remark that this formulation makes evident the selfduality and symmetries of the axiom.

(a) If we have a dist.tr. $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}T(X)$ then $v\circ u=0$, by completion of the diagram:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[r, "0"] \arrow[d, "u"'] & 0 \arrow[r, "0"] & TX \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & TX
\end{tikzcd}$$

(b) The shift property $(TR2)$ permits to extends the property $(TR4)$ to gives the completion of morphism of triangles given any two compatible arrows; and $(TR1)$ to insert a given morphism in any position of a distinguished triangle.

(c) If we have a dist.tr. $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}T(X)$ then also the triangle $TX\xrightarrow{Tu}TY\xrightarrow{-Tv}TZ\xrightarrow{Tw}T^2(X)$ is distinguished by the isomorphism

$$\begin{tikzcd}
TX \arrow[r, "-Tu"] \arrow[d, "\mathrm{id}"'] & TY \arrow[r, "-Tv"] \arrow[d, "-\mathrm{id}"'] & TZ \arrow[r, "-Tw"] \arrow[d, "-\mathrm{id}"'] & T^2X \arrow[d, "\mathrm{id}"'] \\
TX \arrow[r, "-Tu"'] & TY \arrow[r, "Tv"'] & TZ \arrow[r, "-Tw"'] & T^2X
\end{tikzcd}$$

(d) The direct sum (termwise) of two distinguished triangles is distinguished. In particular $X\xrightarrow{0}Y\xrightarrow{i}TX\oplus Y\xrightarrow{-p}TX$ is a dist.tr. for any X and Y.

Triangulated Functors. An additive functor between triangulated categories $F:(\mathcal{T},T)\longrightarrow(\mathcal{T}',T')$ is a functor of triangulated categories (or a triangulated functor, or an exact functor) if it commutes with the translation functors, i.e. $T'\circ F=F\circ T$, and sends dist.tr. of $\mathcal{T}$ in dist.tr. of $\mathcal{T}'$, i.e. $F(\mathcal{T}r(\mathcal{T}))\subseteq\mathcal{T}r(\mathcal{U})$. The family of the triangulated functors is denoted $\mathcal{TF}unct(\mathcal{T},\mathcal{T}')$. Same for contravariant functors.

Cohomological Functors. A functor $F:\mathcal{T}\longrightarrow\mathcal{A}$ from a triangulated category to an abelian category is cohomological if it sends dist.tr. in exact sequences, i.e. if for any $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}T(X)$ distinguished, the sequence $FX\xrightarrow{Fu}FY\xrightarrow{Fv}FZ$ is exact. In that case, written F^k for $F\circ T^k$ we have that a dist.tr. generates a long exact sequence

$$\cdots\longrightarrow F^{k-1}Z\longrightarrow F^kX\longrightarrow F^kY\longrightarrow F^kZ\longrightarrow F^{k+1}X\longrightarrow\cdots$$

by the shift property of triangles.

Notice that for any $W\in\mathcal{T}$ the functors $\mathrm{Hom}_{\mathcal{T}}(W,-)$ and $\mathrm{Hom}_{\mathcal{T}}(-,W)$ are cohomological.

Moreover if we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & TX \arrow[d, equal] \\
X \arrow[r] & Y \arrow[r] & Z' \arrow[r] & TX
\end{tikzcd}$$

where the first line is a distinguished triangle, then the second line is a distinguished triangle iff one of the following conditions hold:

(i) for any $P\in\mathcal{T}$ the sequence in **Ab** obtained by $\mathrm{Hom}_{\mathcal{T}}(P,\text{-})$ to the second line is exact;

(i') for any $Q\in\mathcal{T}$ the sequence in **Ab** obtained by $\mathrm{Hom}_{\mathcal{T}}(\text{-},Q)$ to the second line is exact.

(e) In the hypothesis of $(TR4)$ we have that if f and g are isomorphisms, then also h is.

(f) Let $f:X\to Y$ be a morphism in a triangulated category; then f is an isomorphism iff $X\to Y\to 0\to TX$ is a distinguished triangle.

For X object of a triangulated category we have: $X\cong 0$ iff $\mathrm{id}_X=0$ iff $X\xrightarrow{0}X\to 0\to TX$ is distinguished.

(g) In the situation of $(TR4)$ we have that the following facts are equivalent:

(i) $v'gu=0$;

(ii) there exists f with $u'f=gu$;

(iii) there exists h with $hv=v'g$;

(iv) there exists (f,g,h) morphism of triangles.

Moreover we have unicity of the morphism if $\mathrm{Hom}^{-1}(X,Z')\equiv\mathrm{Hom}(X,T^{-1}Z')=0$.

In particular: if $\mathrm{Hom}^{-1}(X,Z)=0$ then the dist.tr. with first morphism u are *canonically* isomorphic.

The nine diagram: if we have a commutative square $\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ X' & \to & Y' \end{array}$ in a triangulated category, then we can embed it into a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & TX \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & TX' \arrow[d] \\
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & TX'' \arrow[d] \\
TX \arrow[r] & TY \arrow[r] & TZ \arrow[r] & T^2X
\end{tikzcd}$$

where the first three rows and columns are dist.tr. and all the square are commutative, out of those signed $-$, which is anticommutative.

The result follows by three octahedral diagrams, applying the axiom to the two compositions of the original commutative square and to the composition $X''\to A\to Y''$, if A is an object which completes the dist.tr. of the diagonal of the square.

Opposite categories. The opposite category of a triangulated category $\mathcal{T}$ has a natural structure of triangulated category using T^{-1} as translation functor and $\mathcal{T}r(\mathcal{T})$ as family of distinguished triangles (i.e. (X,Y,Z,u,v,w) is distinguished in $\mathcal{T}^\circ$ if and only if (Z,Y,X,w,v,u) is distinguished in $\mathcal{T}$).

Anti-distinguished triangles and the anti-categories. A triangle of $\mathcal{T}$, says $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{w}TX$, is anti-distinguished if $X\xrightarrow{u}Y\xrightarrow{v}Z\xrightarrow{-w}TX$ is a distinguished triangle of $\mathcal{T}$. Let $\mathcal{T}r^a(\mathcal{T})$ be the family of anti-distinguished triangle of a triangulated category $\mathcal{T}$; then $(\mathcal{T},T)$ endowed with the family $\mathcal{T}r^a(\mathcal{T})$ is a triangulated category.

Truncation Categories.

A t-category is a triangulated category $\mathcal{T}$ endowed with a truncation structure (or t-structure), i.e. the data of $(\mathcal{T}_{\leqslant 0},\mathcal{T}_{\geqslant 0})$ two full triangulated subcategories of $\mathcal{T}$ satisfying the conditions below stated. Define $\mathcal{T}_{\leqslant n}=T^{-n}\mathcal{T}_{\leqslant 0}$ and $\mathcal{T}_{\geqslant n}=T^{-n}\mathcal{T}_{\geqslant 0}$ (we will use also $\mathcal{T}_{<n}=\mathcal{T}_{\leqslant n-1}$ and $\mathcal{T}_{>n}=\mathcal{T}_{\geqslant n+1}$).

$(t1)$ $\mathcal{T}_{<0}\subseteq\mathcal{T}_{\leqslant 0}$ and $\mathcal{T}_{>0}\subseteq\mathcal{T}_{\geqslant 0}$;

$(t2)$ $\mathrm{Hom}_{\mathcal{T}}(X,Y)=0$ if $X\in\mathcal{T}_{\leqslant 0}$ and $Y\in\mathcal{T}_{>0}$;

$(t3)$ for any $X\in\mathcal{T}$ there exists a distinguished triangle
$$X_0\longrightarrow X\longrightarrow X_1\longrightarrow TX_0$$
with $X_0\in\mathcal{T}_{\leqslant 0}$ and $X_1\in\mathcal{T}_{>0}$.

Truncation functors. The inclusion functor $\mathcal{T}_{\leqslant n}\to\mathcal{T}$ admits a right adjoint denoted $\tau_{\leqslant n}:\mathcal{T}\to\mathcal{T}_{\leqslant n}$ and $\mathcal{T}_{\geqslant n}\to\mathcal{T}$ admits a left adjoint denoted $\tau_{\geqslant n}:\mathcal{T}\to\mathcal{T}_{\geqslant n}$. In fact we have $\tau_{\leqslant n}=T^{-n}\circ\tau_{\leqslant 0}\circ T^n$ and $\tau_{\geqslant n}=T^{-n}\circ\tau_{\geqslant 0}\circ T^n$.

There exists a functor morphism $\tau_{>n}\to T\tau_{\leqslant n}$ such that the triangle
$$\tau_{\leqslant n}X\longrightarrow X\longrightarrow\tau_{>n}X\longrightarrow T\tau_{\leqslant n}X$$
is distinguished for any $X\in\mathcal{T}$.

We have the following rules for composite of truncations:

(i) $\tau_{\leqslant n}\circ T^m\cong T^m\circ\tau_{\leqslant n+m}$ and $\tau_{\geqslant n}\circ T^m\cong T^m\circ\tau_{\geqslant n+m}$;

(ii) $\tau_{\leqslant n}\circ\tau_{\leqslant m}=\tau_{\leqslant min(m,n)}=\tau_{\leqslant m}\circ\tau_{\leqslant n}$ and $\tau_{\geqslant n}\circ\tau_{\geqslant m}=\tau_{\geqslant max(m,n)}=\tau_{\geqslant m}\circ\tau_{\geqslant n}$;

(iii) if $m>n$ then $\tau_{\leqslant n}\circ\tau_{\geqslant m}=0=\tau_{\geqslant m}\circ\tau_{\leqslant n}$;

(iv) if $m>n$ then there exists a unique functorial isomorphism $\varphi:\tau_{\geqslant n}\circ\tau_{\leqslant m}\to\tau_{\leqslant m}\circ\tau_{\geqslant n}$ such that the following diagram
$$\begin{tikzcd}
\tau_{\leqslant m}X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geqslant n}X \arrow[d] \\
\tau_{\geqslant n}\circ\tau_{\leqslant m}X \arrow[rr, "\varphi"'] & & \tau_{\leqslant m}\circ\tau_{\geqslant n}X
\end{tikzcd}$$
commutes for any $X\in\mathcal{T}$.

Then we define the functors $\tau_{[n,m]}=\tau_{\geqslant n}\tau_{\leqslant m}\cong\tau_{\leqslant m}\tau_{\geqslant n}$; in particular we put $H^n=\tau_{[n,n]}$. Then applying $\tau_{\geqslant n}$ to the distinguished triangle
$$\tau_{\leqslant n}X\longrightarrow X\longrightarrow\tau_{>n}X\longrightarrow T\tau_{\leqslant n}X$$
we obtain a distinguished triangle
$$H^nX\longrightarrow\tau_{\geqslant n}X\longrightarrow\tau_{>n}X\longrightarrow TH^nX$$
and dually, applying $\tau_{\leqslant n}$ to the distinguished triangle
$$\tau_{<n}X\longrightarrow X\longrightarrow\tau_{\geqslant n}X\longrightarrow T\tau_{<n}X$$
we obtain a distinguished triangle
$$\tau_{<n}X\longrightarrow\tau_{\leqslant n}X\longrightarrow H^nX\longrightarrow T\tau_{<n}X\ .$$
We have that the following characterizations:
$X\in\mathcal{T}_{\leqslant n}$ iff $\tau_{\leqslant n}X\to X$ is an isomorphism, iff $\tau_{>n}X\cong 0$; dually $X\in\mathcal{T}_{\geqslant n}$ iff $X\to\tau_{\geqslant n}X$ is an isomorphism, iff $\tau_{<n}X\cong 0$.

The categories $\mathcal{T}_{\leqslant n}$ (resp. $\mathcal{T}_{\geqslant n}$) are middle closed, i.e. for any distinguished triangle
$$X'\longrightarrow X\longrightarrow X''\longrightarrow TX'$$
we have that $X',X''\in\mathcal{T}_{\leqslant n}$ (resp. $\mathcal{T}_{\geqslant n}$) implies $X\in\mathcal{T}_{\leqslant n}$ (resp. $\mathcal{T}_{\geqslant n}$)

The t-structure is non degenerated if $\bigcap_{n\in\mathbb{Z}}\mathcal{T}_{\leqslant n}=0=\bigcap_{n\in\mathbb{Z}}\mathcal{T}_{\geqslant n}$; in this case we have that $X\in\mathcal{T}_{\leqslant n}$ iff $H^iX=0$ for all $i>n$ and $X\in\mathcal{T}_{\geqslant n}$ iff $H^iX=0$ for all $i<n$.

We define also the subcategories $\mathcal{T}^*$ of $\mathcal{T}$ for $*\in\{+,-,b\}$ by the formula $\mathcal{T}^-=\bigcup_{n\in\mathbb{Z}}\mathcal{T}_{\leqslant n}$, $\mathcal{T}^+=\bigcup_{n\in\mathbb{Z}}\mathcal{T}_{\geqslant n}$, $\mathcal{T}^b=\bigcup_{m,n\in\mathbb{Z}}\mathcal{T}_{[m,n]}$ where $\mathcal{T}_{[m,n]}$ is the essential image of the functor $\tau_{[m,n]}$.

t-functors. A triangulated functor $F:\mathcal{T}\longrightarrow\mathcal{T}'$ of t-categories is a right t-functor if $F(\mathcal{T}_{\geqslant 0})\subseteq\mathcal{T}'_{\geqslant 0}$, is a left t-functor if $F(\mathcal{T}_{\leqslant 0})\subseteq\mathcal{T}'_{\leqslant 0}$, it is a t-functor if it is a right and left t-functor.

The Heart. We define the heart of $\mathcal{T}$ as $\mathcal{CT}=\mathcal{T}_{\leqslant 0}\cap\mathcal{T}_{\geqslant 0}=\mathcal{T}_{[0,0]}$. It is an abelian subcategory of $\mathcal{T}$. Moreover the functors
$$H^0=\tau_{\leqslant 0}\circ\tau_{\geqslant 0}=\tau_{[0,0]}:\mathcal{T}\longrightarrow\mathcal{CT}$$
and
$$H^n=H^0\circ T^n=T^n\circ\tau_{[n,n]}:\mathcal{T}\longrightarrow\mathcal{CT}$$
are cohomological functors.

Distinguished triangles and exact sequences. Let $X\xrightarrow{f}Y\to Z\to TX$ be a distinguished triangle with $X,Y\in\mathcal{CT}$; then we have that $H^0(Z)\cong\mathrm{coker}f$ and $H^0(T^{-1}Z)\cong\ker f$.

For any $0\to X\to Y\to Z\to 0$ exact sequence in $\mathcal{CT}$ there exists a unique morphism $Z\to TX$ such that $X\to Y\to Z\to TX$ is a distinguished triangle of $\mathcal{T}$.

Functoriality. Let $F:\mathcal{T}\longrightarrow\mathcal{T}'$ be a triangulated functor; then we have a natural functor $\mathcal{C}F:\mathcal{CT}\longrightarrow\mathcal{CT}'$ defined by $\mathcal{C}F=H'^0\circ F\circ\epsilon$ where $\epsilon:\mathcal{CT}\to\mathcal{T}$ is the inclusion functor. If F if a left t-functor, then for $X\in\mathcal{T}$ we have $\tau'_{\leqslant 0}F\tau_{\leqslant 0}X\cong\tau_{\leqslant 0}FX$, hence for $X\in\mathcal{T}_{\leqslant 0}$ we find $H'^0FX\cong\mathcal{C}F(H^0X)$. Dually for right t-functors; for t-functors we have $\mathcal{C}F(H^0X)=H'^0FX$ for $X\in\mathcal{T}$ and $\mathcal{C}F(X)=FX$ for $X\in\mathcal{CT}$.

If F is a right (resp. left, both sides) t-functor, then the functor $\mathcal{C}F$ is right exact (resp. left exact, exact).

If $G:\mathcal{T}'\longrightarrow\mathcal{T}$ is a left adjoint of $F:\mathcal{T}\longrightarrow\mathcal{T}'$, then F is right t-iff G is a left t-functor, and in that case we have that $\mathcal{C}G$ is a right adjoint of $\mathcal{C}F$.

Let $F:\mathcal{T}\longrightarrow\mathcal{T}'$ and $F:\mathcal{T}'\longrightarrow\mathcal{T}''$ be left t-functors; then also $F'\circ F$ is a left t-functor, and $\mathcal{C}(F'\circ F)=\mathcal{C}F'\circ\mathcal{C}F$.

Cohomological functors. Let $H:\mathcal{T}\longrightarrow\mathcal{A}$ be a cohomological functor from a triangulated to an abelian category. Suppose that $\mathcal{N}_H=0$ (or take $\mathcal{T}/\mathcal{N}_H$ as $\mathcal{T}$). Then $\mathcal{T}_{\leqslant 0}=\{X\in\mathcal{T}|\ H(T^nX)=0\ \forall n>0\}$ and $\mathcal{T}_{\geqslant 0}=\{X\in\mathcal{T}|\ H(T^nX)=0\ \forall n<0\}$ define a natural t-structure in $\mathcal{T}$ if the inclusion functors have the required adjoints. In that case we have that $H_{|\mathcal{CT}}$ is a faithful functor.

Way-out functors. Let $F:\mathcal{T}\longrightarrow\mathcal{T}'$ be a triangulated functor between t-categories. It is right way-out (resp. left way-out, way-out) if for any $n\in\mathbb{Z}$ there exists $m\in\mathbb{Z}$ such that $F(\mathcal{T}_{\geqslant m})\subseteq\mathcal{T}'_{\geqslant n}$ (resp. $F(\mathcal{T}_{\leqslant m})\subseteq\mathcal{T}'_{\leqslant n}$, way-out in both directions). For contravariant functors F , as usual we apply the definition to ${}^\circ F:\mathcal{T}^\circ\longrightarrow\mathcal{T}'$, so that it is right way-out (resp. left way-out, way-out) if for any $n\in\mathbb{Z}$ there exists $m\in\mathbb{Z}$ such that $F(\mathcal{T}_{\leqslant m})\subseteq\mathcal{T}'_{\geqslant n}$ (resp. $F(\mathcal{T}_{\geqslant m})\subseteq\mathcal{T}'_{\leqslant n}$, way-out in both directions).

Since triangulated functors commute with the translations, we have that a functor is right way-out (resp. left way-out) iff there exists $m_0\in\mathbb{Z}$ such that $F(\mathcal{T}_{\geqslant m_0})\subseteq\mathcal{T}'_{\geqslant 0}$ (resp. $F(\mathcal{T}_{\leqslant m_0})\subseteq\mathcal{T}'_{\leqslant 0}$) and also iff there exists $n_0\in\mathbb{Z}$ such that $F(\mathcal{T}_{\geqslant 0})\subseteq\mathcal{T}'_{\geqslant n_0}$ (resp. $F(\mathcal{T}_{\leqslant 0})\subseteq\mathcal{T}'_{\leqslant n_0}$). Then the notion of way-out functor is a generalization of the notion of t-functor.

Suppose now that $\mathcal{N}_{H^0}=0$, and take an abelian middle closed subcategory $\mathcal{A}$ of $\mathcal{CT}$; we define $\mathcal{T}_{\mathcal{A}}=\{X\in\mathcal{T}|\ H^nX\in\mathcal{A}\ \forall n\in\mathbb{Z}\}$. Suppose $\eta:F\to G$ be a morphism of triangulated functors; then we have the following facts:

(*i*) if $\eta(Y)$ is iso for any $Y\in\mathcal{A}$ then $\eta(X)$ is iso for any $X\in\mathcal{T}^b_{\mathcal{A}}$;

(*ii*) if $\eta(Y)$ is iso for any $Y\in\mathcal{A}$ and F,G are right way-out, then $\eta(X)$ is iso for any $X\in\mathcal{T}^+_{\mathcal{A}}$;

(*iii*) if $\eta(Y)$ is iso for any $Y\in\mathcal{A}$ and F,G are way-out, then $\eta(X)$ is iso for any $X\in\mathcal{T}_{\mathcal{A}}$.

Let $\mathcal{T},\mathcal{T}'$ be t-categories; suppose $\mathcal{A}$ (resp. $\mathcal{A}'$) an abelian middle closed subcategory of $\mathcal{CT}$ (resp. $\mathcal{CT}'$), and let $F:\mathcal{T}_{\mathcal{A}}\longrightarrow\mathcal{T}'$ be a triangulated functor; then we have the following facts:

(*i*) if $F(Y)\in\mathcal{T}'_{\mathcal{A}'}$ for any $Y\in\mathcal{A}$, then $F(X)\in\mathcal{T}'_{\mathcal{A}'}$ for any $X\in\mathcal{T}^b_{\mathcal{A}}$;

(*ii*) if $F(Y)\in\mathcal{T}'_{\mathcal{A}'}$ for any $Y\in\mathcal{A}$ and F,G are right way-out, then $F(X)\in\mathcal{T}'_{\mathcal{A}'}$ for any $X\in\mathcal{T}^+_{\mathcal{A}}$;

(*iii*) if $F(Y)\in\mathcal{T}'_{\mathcal{A}'}$ for any $Y\in\mathcal{A}$ and F,G are way-out, then $F(X)\in\mathcal{T}'_{\mathcal{A}'}$ for any $X\in\mathcal{T}_{\mathcal{A}}$.

Localized Categories and Functors.

Let $\mathcal{C}$ be a category; a multiplicative system in $\mathcal{C}$ is a family S of morphism of $\mathcal{C}$ such that:

($S1$) $\mathrm{id}_X\in S$ for any X in $\mathcal{C}$;

($S2$) if $f,g\in S$ then $g\circ f\in S$ if it exists;

($S3$) we may complete any diagram with $s\in S$

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad\text{in}\quad \begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative with $t\in S$; and dually with the arrows reversed;

($S4$) for any two morphisms $f,g:X\to Y$ in $\mathcal{C}$ the following conditions are equivalent:

(*i*) there exists $z\in S$, $z:Y\to Z$ such that $z\circ f=z\circ g$

(*i'*) there exists $w\in S$, $w:W\to X$ such that $f\circ w=g\circ w$.

A multiplicative system S is quasi-saturated if the following condition holds: if $g\circ f\in S$ then $f\in S$ if and only if $g\in S$.

Let S be a multiplicative system in $\mathcal{C}$; we define the localized category of $\mathcal{C}$ w.r.t. S, denoted $\mathcal{C}_S$ or $\mathcal{C}[S^{-1}]$, a category endowed with a functor $Q:\mathcal{C}\to\mathcal{C}_S$ such that for any $s\in S$ the image $Q(s)$ is an isomorphism, and it is universal for this property: for any category $\mathcal{D}$ with a functor $F:\mathcal{C}\to\mathcal{D}$ such that $F(s)$ is isomorphism for any $s\in S$, then there exists a unique factorization:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "F"'] & \mathcal{C}_S \arrow[d, "\exists!"] \\ & \mathcal{D} \end{tikzcd} .$$

More generally we may solve the same universal problem for any family of morphisms of a category, but the localized category has a too complicated construction.

If S is a multiplicative system in $\mathcal{C}$, then the category $\mathcal{C}_S$ is constructed in the following way: the objects of $\mathcal{C}_S$ are the objects of $\mathcal{C}$; the morphisms are

$$\mathrm{Hom}_{\mathcal{C}_S}(X,Y)=\varinjlim_{(X',s')\in S/X}\mathrm{Hom}_{\mathcal{C}}(X',Y)$$

where S/X is the full subcategory of $\mathcal{C}/X$ given by the objects $s':X'\to X$, with $s'\in S$. The functor Q is defined as the identity on the objects and for the morphisms: $Q(f:X\to Y)=X\overset{\mathrm{id}_X}{\leftarrow}X\overset{f}{\to}Y$. For $s\in S$, $s:X\to Y$, the inverse in $\mathcal{C}_S$ of $Q(s)$ is $Y\overset{s}{\leftarrow}X\overset{\mathrm{id}_X}{\to}X$.

The dual construction for $\mathcal{C}_S$ also works:

$$\mathrm{Hom}_{\mathcal{C}_S}(X,Y)=\varinjlim_{(Y',t')\in Y\backslash S}\mathrm{Hom}_{\mathcal{C}}(X,Y')$$

where $Y\backslash S$ is the full subcategory of $Y\backslash\mathcal{C}$ given by the objects $t':Y\to Y'$ with $t'\in S$. This corresponds to $(\mathcal{C}^\circ)_S=(\mathcal{C}_S)^\circ$.

(1) For any multiplicative system S we may define another class of morphisms given by all the morphisms of $\mathcal{C}$ whose image in $\mathcal{C}_S$ is invertible; the S^{qs} is a quasi-saturated multiplicative system containing S, and the canonical functor $\mathcal{C}_S\longrightarrow\mathcal{C}_{S^{qs}}$ is an equivalence of categories.

(2) Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$, S be a multiplicative system in $\mathcal{C}$ such that $T=S\cap\mathcal{F}(\mathcal{D})$ is a multiplicative system in $\mathcal{D}$ and the following property: if $s:X\to Y$ is in S with $Y\in\mathcal{D}$ then there exist $X'\in\mathcal{D}$ and a morphism $g:X'\to X$ with $g\circ s$ in S, or the dual property (if $s:X\to Y$ is in S with $X\in\mathcal{D}$ then there exist $Y'\in\mathcal{D}$ and a morphism $f:Y\to Y'$ with $s\circ f$ in S) holds. Then the canonical functor $\mathcal{D}_T\longrightarrow\mathcal{C}_S$ is fully faithful.

(3) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and S be a multiplicative system in $\mathcal{C}$; then there is a bijection

$$\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C}_S,\mathcal{D})}(F,G)\cong\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C},\mathcal{D})}(F\circ Q,G\circ Q)\ .$$

(4) Let $f:X\to Y$ be a morphism in $\mathcal{C}$; then $Q(f)$ is an isomorphism in $\mathcal{C}_S$ iff there exist $g:W\to X$ and $h:Y\to Z$ in $\mathcal{C}$ such that $f\circ g$ and $h\circ f$ belong to S; if f is a monomorphism (resp. epimorphism) in $\mathcal{C}$ then $Q(f)$ is in $\mathcal{C}_S$.

(5) If $\mathcal{C}$ is an additive category, then $\mathcal{C}_S$ can be canonically endowed with a structure of additive category.

Let $f:X\to Y$ be a morphism in $\mathcal{C}$ (additive); then $Q(f)=0$ iff there exists $s:W\to X$ morphism in S with $f\circ s=0$, iff there exists $t:Y\to Z$ morphism in S with $t\circ f=0$.

Let X be an object of $\mathcal{C}$; then $X\cong 0$ in $\mathcal{C}_S$ iff there exists Y such that the zero morphism $0:Y\to X$ belongs to S, iff there exists Z such that the zero morphism $0:X\to Z$ belongs to S.

Localization of Functors. Let $F:\mathcal{C}\longrightarrow\mathcal{C}'$ be a functor, and let S, S' be multiplicative systems in $\mathcal{C}$, $\mathcal{C}'$ respectively. Then we have the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C}' \arrow[d, "Q'"] \\ \mathcal{C}_S & \mathcal{C}'_{S'} \end{tikzcd} .$$

There exists a functor $F:\mathcal{C}_S\longrightarrow\mathcal{C}'_{S'}$ making commutative the square if and only if $F(S)\subseteq\cong(\mathcal{C}'_{S'})$, i.e. F sends morphisms in S to invertible morphisms of $\mathcal{C}'_{S'}$.

Definitions. In general we can propose the following universal problem: find a functor $RF:\mathcal{C}_S\longrightarrow\mathcal{C}'_{S'}$ endowed with a morphism of functors $\vartheta:Q'\circ F\longrightarrow RF\circ Q$. Universal in the category of functors from $\mathcal{C}_S$ to $\mathcal{C}'_{S'}$ means: for any other functors $G:\mathcal{C}_S\longrightarrow\mathcal{C}'_{S'}$ with $\eta:Q'\circ F\longrightarrow G\circ Q$ there exists a unique $\zeta:RF\to G$ such that $\eta=(Q\bullet\zeta)\circ\vartheta$. The functor RF is called the right localized functor of F. It is the representative of the functor

$$\mathcal{F}unct(\mathcal{C}_S,\mathcal{C}'_{S'})\longrightarrow\mathbf{Set}$$

sending G to $\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C},\mathcal{C}'_{S'})}(Q'\circ F,G\circ Q)$, and ϑ realizes the bijection

$$\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C}_S,\mathcal{C}'_{S'})}(RF,G)\longrightarrow\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C},\mathcal{C}'_{S'})}(Q'\circ F,G\circ Q)\ .$$

The dual problem of representing the functor sending G to $\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C},\mathcal{C}'_{S'})}(G\circ Q,Q'\circ F)$ defines the left localized functor LF of F. It is a functor $LF:\mathcal{C}_S\longrightarrow\mathcal{C}'_{S'}$ endowed with a morphism of functors $\vartheta:LF\circ Q\longrightarrow Q'\circ F$ universal for this property, i.e. realizing the bijection

$$\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C}_S,\mathcal{C}'_{S'})}(G.LF)\longrightarrow\mathrm{Hom}_{\mathcal{F}unct(\mathcal{C},\mathcal{C}'_{S'})}(G\circ Q,Q'\circ F)\ .$$

The case of contravariant functors $F:\mathcal{C}\longrightarrow\mathcal{C}'$ is solved by the following remark: we may consider the functor ${}^{\circ}F:\mathcal{C}^{\circ}\longrightarrow\mathcal{C}'$ which is covariant, and, denoted ${}^{\circ}:\mathcal{C}\longrightarrow\mathcal{C}^{\circ}$ the canonical contravariant functor, we have $RF=R({}^{\circ}F)\circ{}^{\circ}$, $LF=L({}^{\circ}F)\circ{}^{\circ}$. If we consider the covariant functor $F^{\circ}:\mathcal{C}\longrightarrow\mathcal{C}'^{\circ}$ we have $RF={}^{\circ}\circ L(F^{\circ})$ and $LF={}^{\circ}\circ R(F^{\circ})$.

Special subcategories. Let $\mathcal{B}$ a subcategory of $\mathcal{C}$ such that $T:=S\cap\mathcal{B}$ is a multiplicative system. It is a localizing subcategory if the natural functor $\mathcal{B}_T\to\mathcal{C}_S$ is fully faithful.

It is right (resp. left) sufficient if for any $X\in\mathcal{C}$ there exists $Y\in\mathcal{B}$ and a morphism $X\to Y$ (resp. $Y\to X$) in $\mathcal{C}$ whose image in $\mathcal{C}_S$ is invertible. In that case the functor $\mathcal{B}_T\to\mathcal{C}_S$ is essentially surjective.

It is F-inert if any element of $F(T)$ becomes invertible in $\mathcal{C}'_{S'}$.

Existence. Suppose that $\mathcal{B}$ is a localizing, right (resp. left) sufficient and F-inert subcategory of $\mathcal{C}$; then RF (resp. LF) exists and the morphism $\vartheta(Y):Q'\circ F(Y)\longrightarrow RF\circ Q(Y)$ (resp. $\vartheta(Y):LF\circ Q(Y)\longrightarrow Q'\circ F(Y)$) is an isomorphism for any $Y\in\mathcal{B}$. We have the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dd, "Q"'] & & \mathcal{C}' \arrow[dd, "Q'"] \\ & \mathcal{B} \arrow[ul] \arrow[ur] \arrow[d] & \\ & \mathcal{B}_{S\cap\mathcal{B}} \arrow[dl, shift left] \arrow[dr] & \\ \mathcal{C}_S \arrow[ur, shift left] \arrow[rr, "RF\ (LF)"'] & & \mathcal{C}'_{S'} \end{tikzcd}$$

Localization of bifunctors. Suppose given the following situation:

$$\begin{array}{ccc} \mathcal{C}\times\mathcal{C}' & \xrightarrow{F} & \mathcal{C}'' \\ {\scriptstyle Q\times Q'}\downarrow & & \downarrow{\scriptstyle Q''} \\ \mathcal{C}_S\times\mathcal{C}'_{S'} & & \mathcal{C}''_{S''} \ . \end{array}$$

(1) Suppose there exists $\mathcal{B}$ full subcategory of $\mathcal{C}$ which is right sufficient and $F(-,Y)$-acyclic for any $Y\in\mathcal{C}'$; then $R_{I}F(-,Y):\mathcal{C}_S\to\mathcal{C}''_{S''}$ exists. Suppose moreover that $R_{I}F(\mathcal{B},S')$ has image in the class of isomorphisms of $\mathcal{C}''_{S''}$ (i.e. $F(b,s')$ is invertible in $\mathcal{C}''_{S''}$); then the bifunctor $R_{I}F:\mathcal{C}_S\times\mathcal{C}'\to\mathcal{C}''_{S''}$ induces a triangulated functor $R_{I}F:\mathcal{C}_S\times\mathcal{C}'_{S'}\to\mathcal{C}''_{S''}$, having the universal property of localized bifunctor of F.

(1′) Reversing the side of the hypothesis we can find a localized functor $R_{II}F:\mathcal{C}_S\times\mathcal{C}'_{S'}\to\mathcal{C}''_{S''}$, if $\mathcal{B}'$ is a right sufficient and $F(X,-)$-acyclic subcategory of $\mathcal{C}'$ such that $F(S,\mathcal{B}')$ are invertible in $\mathcal{C}''_{S''}$.

(2) If either $R_{I}F$ and $R_{II}F$ exist, they are canonically isomorphic.

Functoriality. Suppose that $F,G:\mathcal{C}\longrightarrow\mathcal{C}'$ admit right localized functors $RF,RG:\mathcal{C}_S\longrightarrow\mathcal{C}'_{S'}$; if $\varphi:F\to G$ is a morphism of functors, then there exists a unique morphism $R\varphi:RF\to FG$ such that the following diagram commutes

$$\begin{array}{ccc} Q'\circ F & \xrightarrow{Q'\bullet\varphi} & Q'\circ G \\ {\scriptstyle\vartheta_F}\downarrow & & \downarrow{\scriptstyle\vartheta_G} \\ RF\circ Q & \xrightarrow[R\varphi\bullet Q]{} & RG\circ Q \ . \end{array}$$

Moreover for $\psi:G\to H$, where $H:\mathcal{C}\longrightarrow\mathcal{C}'$ admits RH, then we have $R(\psi\circ\varphi)=R\psi\circ R\varphi$.

Localization of generalized adjunctions. Let

$$\varphi(X,Y,Z):F_1(G_1(X,Y),Z)\longrightarrow F_2(X,G_2(Y,Z))$$

be a morphism of functors, where

$$\begin{array}{llll} G_1: \ \mathcal{C}\times\mathcal{C}' & \longrightarrow \mathcal{D} & \quad F_1:\mathcal{D}\times\mathcal{C}'' & \longrightarrow \mathcal{C}''' \\ G_2: \mathcal{C}'\times\mathcal{C}'' & \longrightarrow \mathcal{D}' & \quad F_2: \mathcal{C}\times\mathcal{D}' & \longrightarrow \mathcal{C}''' \end{array}$$

i.e. we have a morphism $\varphi:F_1\circ(G_1\times\mathcal{C}'')\longrightarrow F_2\circ(\mathcal{C}\times G_2)$ between functors

$$F_1\circ(G_1\times\mathcal{C}''),F_2\circ(\mathcal{C}\times G_2):\mathcal{C}\times\mathcal{C}'\times\mathcal{C}''\longrightarrow\mathcal{C}'''$$

Suppose there exist $\mathcal{U}$ a localizing, right sufficient, G_1- and F_2-inert subcategory of $\mathcal{C}$, $\mathcal{U}''$ a localizing, right sufficient, G_2- and F_1-inert subcategory of $\mathcal{C}''$. Suppose moreover that we have the condition for the localization of bifunctors; then there exists a unique morphism of functors

$$\tilde{\varphi}:RF_1(RG_1(X,Y),Z)\longrightarrow RF_2(X,RG_2(Y,Z))$$

making commutative the diagram

$$\begin{array}{ccc} Q'''\circ F_1\circ(G_1\times\mathcal{C}'') & \xrightarrow{Q'''\bullet\varphi} & Q'''\circ F_2\circ(\mathcal{C}\times G_2) \\ {\scriptstyle\vartheta_{F_1}\bullet(G_1\times\mathcal{C}'')}\downarrow & & \downarrow{\scriptstyle\vartheta_{F_2}\bullet(\mathcal{C}\times G_2)} \\ RF_1\circ(\mathcal{C}\times\mathcal{C}'\times Q'')\circ(G_1\times\mathcal{C}'') & & RF_2\circ(Q\times\mathcal{C}'\times\mathcal{C}'')\circ(\mathcal{C}\times G_2) \\ {\scriptstyle RF_1\bullet(\vartheta_{G_1}\times\mathcal{C}'')}\downarrow & & \downarrow{\scriptstyle RF_2\bullet(\mathcal{C}\times\vartheta_{G_2})} \\ RF_1\circ(RG_1\times\mathcal{C}'')\circ\bar{Q} & \xrightarrow[\tilde{\varphi}\bullet\bar{Q}]{} & RF_2\circ(\mathcal{C}\times RG_2)\circ\bar{Q} \end{array}$$

where $\bar{Q}=Q\times Q'\times Q''$; $\tilde{\varphi}$ is an isomorphism if φ is.

Composition of localized functors. Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ be three categories, S, S', S'' multiplicative systems, and $F:\mathcal{C}\to\mathcal{C}'$ and $G:\mathcal{C}'\to\mathcal{C}''$ functors. Suppose there exists the right localized functors RF, RG, $R(G\circ F)$ and $R(RG\circ Q'\circ F)$. Then, by the universal property of $R(G\circ F)$, there exists a canonical morphism $\zeta_{G,F}:R(G\circ F)\to R(RG\circ Q'\circ F)$ making commutative the following diagram

$$\begin{array}{ccc} Q''\circ G\circ F & \xrightarrow{\vartheta_G\bullet F} & RG\circ Q'\circ F \\ {\scriptstyle Q_{G\circ F}}\downarrow & & \downarrow{\scriptstyle\vartheta_{RG\circ Q'\circ F}} \\ R(G\circ F)\circ Q & \xrightarrow[\zeta_{G,F}\bullet Q]{} & R(RG\circ Q'\circ F)\circ Q \ . \end{array}$$

Moreover, by the universal property of $R(RG\circ Q'\circ F)$, there exists a canonical morphism $\eta_{G,F}:R(RG\circ Q'\circ F)\to RG\circ RF$ making commutative the diagram

$$\begin{array}{ccc} RG\circ Q'\circ F & \xrightarrow{=} & RG\circ Q'\circ F \\ {\scriptstyle\vartheta_{RG\circ Q'\circ F}}\downarrow & & \downarrow{\scriptstyle RG\bullet\vartheta_F} \\ R(RG\circ Q'\circ F)\circ Q & \xrightarrow[\eta_{G,F}\bullet Q]{} & RG\circ RF\circ Q \ . \end{array}$$

Remark that $\eta_{G,F}$ is an isomorphism. The composition $\eta_{G,F}\circ\zeta_{G,F}$ is the canonical morphism $\xi_{G,F}:R(G\circ F)\to RG\circ RF$, induced by the universal property of $R(G\circ F)$, such that the diagram

$$\begin{array}{ccc} Q''\circ G\circ F & \xrightarrow{\vartheta_G\bullet F} & RG\circ Q'\circ F \\ {\scriptstyle Q_{G\circ F}}\downarrow & & \downarrow{\scriptstyle RG\bullet\vartheta_F} \\ R(G\circ F)\circ Q & \xrightarrow[\xi_{G,F}\bullet Q]{} & RG\circ RF\circ Q \end{array}$$

commutes. In particular we have the commutative triangle of morphisms of functors

$$\begin{tikzcd} R(G\circ F) \arrow[rr, "\xi_{G,F}"] \arrow[dr, "\zeta_{G,F}"'] & & RG\circ RF \\ & R(RG\circ F) \arrow[ur, "\eta_{G,F}"'] & \end{tikzcd}$$

and $\xi_{G,F}$ is an isomorphism iff $\zeta_{G,F}$ is.

If we have three composable right localized functors, then the following diagram of canonical morphisms

$$\begin{array}{ccc} R(H\circ G\circ F) & \xrightarrow{\xi_{H,GF}} & RH\circ R(G\circ F) \\ \xi_{HG,F}\downarrow & & \downarrow RH\bullet\xi_{G,F} \\ R(H\circ G)\circ RF & \xrightarrow[\xi_{H,G}\bullet RF]{} & RH\circ RG\circ RF \end{array}$$

commutes.

Suppose $\mathcal{B}$ a localizing, right sufficient and F-inert subcategory of $\mathcal{C}$, $\mathcal{B}'$ a localizing, right sufficient and G-inert subcategory of $\mathcal{C}'$ and $F(\mathcal{B})\subseteq\mathcal{B}'$. Then $\mathcal{B}$ is also $G\circ F$-inert and we have $R(G\circ F)\cong RG\circ RF$.

Localization of Triangulated Categories.

Let $(\mathcal{T},T)$ be a triangulated category; a null system $\mathcal{N}$ of $\mathcal{T}$ is a family of objects of $\mathcal{T}$ such that:

$(N1)$ $0\in\mathcal{N}$,

$(N2)$ $N\in\mathcal{N}$ if and only if $TN\in\mathcal{N}$,

$(N3)$ if $X\to Y\to Z\to TX$ is a distinguished triangle and $X,Y\in\mathcal{N}$, then $Z\in\mathcal{N}$.

If $H:\mathcal{T}\to\mathcal{A}$ is a cohomological functor, then the class $\mathcal{N}_H=\{X\in\mathcal{T}|H(T^nX)=0\ \forall n\in\mathbb{Z}\}$ of H-acyclic objects is a null system.

Notice that the shift properties $(TR2)$ and $(N2)$ permit us to extend $(N3)$: if two vertex of a distinguished triangle are in $\mathcal{N}$, the third also is.

In the triangulated category we can localize w.r.t. a null system; in fact the family of morphisms

$$S(\mathcal{N})=\left\{f:X\to Y|\exists\ \text{dist.tr.}X\xrightarrow{f}Y\to N\to TX\ \text{with}\ N\in\mathcal{N}\right\}$$

is a quasi-saturated multiplicative system in $\mathcal{T}$:

$(S1)$ by $(TR0)$ and $(N1)$;

$(S2)$ by $(TR1,2,5)$ and $(N2,3)$ (this is an important application of the octahedral axiom);

$(S3)$ by $(TR1,2,4)$, because we complete the morphism of triangles

$$\begin{array}{ccccccc} Z & \xrightarrow{s} & Y & \longrightarrow & N & \longrightarrow & TZ \\ & & f\uparrow & & \| & & \\ W & \xrightarrow[t]{} & X & \longrightarrow & N & \longrightarrow & TW \end{array}$$

$(S4)$ by $(TR1,2,4)$, because if $fs=0$ then we complete the morphism of triangles

$$\begin{array}{ccccccc} X' & \xrightarrow{s} & X & \longrightarrow & N & \longrightarrow & TX' \\ \downarrow & & f\downarrow & & & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y & \longrightarrow & 0 \\ & & & & t\downarrow & & \\ & & & & Y' & & \\ & & & & +1\downarrow & & \end{array}$$

so that $tf=0$.

The quasi-saturation follows with the same argument as for $(S2)$. Moreover $S(\mathcal{N})$ satisfies the following two properties:

$(ST1)$ $s\in S(\mathcal{N})$ if and only if $T(s)\in S(\mathcal{N})$, by $(TR2)$ and $(N2)$;

$(ST2)$ in the hypothesis of $(TR4)$, if $f,g\in S(\mathcal{N})$ then $h\in S(\mathcal{N})$. This is an application of the nine diagram.

Remark that $S(\mathcal{N}_H)=\{f|H(T^nf)\ \text{iso}\ \forall n\in\mathbb{Z}\}$.

(1) Put $\mathcal{T}/\mathcal{N}=\mathcal{T}_{S(\mathcal{N})}$ and let $Q:\mathcal{T}\longrightarrow\mathcal{T}/\mathcal{N}$ be the canonical functor; then $\mathcal{T}/\mathcal{N}$ is canonically a triangulated category; $Q(N)\cong 0$ for any $N\in\mathcal{N}$ and $\mathcal{T}/\mathcal{N}$ is universal w.r.t. this property in the category of triangulated category.

(2) Let X be an object in $\mathcal{T}$; then $Q(X)$ is isomorphic to the zero object in $\mathcal{T}/\mathcal{N}$ iff there exists Y in $\mathcal{C}$ such that $X\oplus Y$ is isomorphic to an object of $\mathcal{N}$; we can use $Y=T(X)$. In the particular case of $\mathcal{N}_H$, we have $Q(X)\cong 0$ iff $X\in\mathcal{N}_H$; moreover for a morphism f we have that $Q(f)$ is isomorphism iff $f\in S(\mathcal{N}_H)$.

(3) Let $\mathcal{T}$ be a triangulated category, $\mathcal{N}$ be a null system in $\mathcal{T}$ and $\mathcal{U}$ a full subcategory of $\mathcal{T}$; if any dist. triangle of $\mathcal{T}$ with the first two objects in $\mathcal{U}$ is a dist. triangle of $\mathcal{U}$, then $\mathcal{N}\cap\mathcal{U}$ is a null system of U. Moreover if any morphism $X\longrightarrow Y$ in $\mathcal{T}$ with $X\in\mathcal{U}$ and $Y\in\mathcal{N}$ factorizes trough an object $N\in\mathcal{N}\cap\mathcal{U}$, then $\mathcal{U}/(\mathcal{N}\cap\mathcal{U})$ is a full subcategory of $\mathcal{T}/\mathcal{N}$.

Localization of Triangulated Functors. Let $F:\mathcal{T}\longrightarrow\mathcal{T}'$ be a triangulated functor, and let $\mathcal{N}$, $\mathcal{N}'$ be null systems in $\mathcal{N}$, $\mathcal{N}'$ respectively. Then we have the following diagram:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & \mathcal{T}' \\ Q\downarrow & & \downarrow Q' \\ \mathcal{T}/\mathcal{N} & & \mathcal{T}'/\mathcal{N}' \end{array}.$$

There exists a functor $F:\mathcal{T}/\mathcal{N}\longrightarrow\mathcal{T}'/\mathcal{N}'$ making commutative the square if and only if $Q'\circ F(\mathcal{N})\cong 0$, i.e. F sends objects in $\mathcal{N}$ to objects isomorphic to 0 in $\mathcal{T}'/\mathcal{N}'$.

Definitions. In general we can propose the following universal problem: find a triangulated functor $RF:\mathcal{T}/\mathcal{N}\longrightarrow\mathcal{T}'/\mathcal{N}'$ endowed with a morphism of functors $\vartheta:Q'\circ F\longrightarrow RF\circ Q$. The functor RF is called the right localized functor of F.

The dual problem define the left localized functor LF of F. It is a triangulated functor $LF:\mathcal{T}/\mathcal{N}\longrightarrow\mathcal{T}'/\mathcal{N}'$ endowed with a morphism of functors $\vartheta:LF\circ Q\longrightarrow Q'\circ F$ (universal for this property).

Special subcategories. Let $\mathcal{U}$ be a triangulated subcategory of $\mathcal{T}$ such that $\mathcal{N}\cap\mathcal{U}$ is a null system. It is a localizing subcategory if the natural functor $\mathcal{U}/\mathcal{N}\cap\mathcal{U}\to\mathcal{T}/\mathcal{N}$ is fully faithful.

It is F-inert if $F(U)\cong 0\in\mathcal{T}'/\mathcal{N}'$ for any $U\in\mathcal{U}$; inert if $\mathcal{U}\cap\mathcal{N}=0$.

If $H':\mathcal{T}'\to\mathcal{A}$ is a cohomological functor, we can use $\mathcal{N}'=\mathcal{N}_{H'}$ and $\mathcal{N}=\mathcal{N}_H$ where $H=H'F$. Then a right (resp. left) sufficient subcategory is also localizing. Moreover a subcategory $\mathcal{U}$ of $\mathcal{T}$ is called F-acyclic w.r.t. H if $F(\mathcal{U}\cap\mathcal{N}_H)\subseteq\mathcal{N}_{H'}$. If $H:\mathcal{T}\to\mathcal{A}$ is a cohomological functor, a subcategory $\mathcal{U}$ of $\mathcal{T}$ is called acyclic w.r.t. H if $\mathcal{U}\cap\mathcal{N}_H=0$.

Existence. Suppose that $\mathcal{U}$ is a localizing, right (resp. left) sufficient and F-inert subcategory of $\mathcal{C}$; then RF (resp. LF) exists and the morphism $\vartheta(Y):Q'\circ F(Y)\longrightarrow RF\circ Q(Y)$ (resp. $\vartheta(Y):LF\circ Q(Y)\longrightarrow Q'\circ F(Y)$) is an isomorphism for any $Y\in\mathcal{U}$.

Suppose $\mathcal{U}$ a localizing, right sufficient and F-inert subcategory of $\mathcal{T}$, $\mathcal{U}'$ a localizing, right sufficient and G-inert subcategory of $\mathcal{T}'$ and $F(\mathcal{U})\subseteq\mathcal{U}'$. Then $\mathcal{U}$ is also $G\circ F$-inert and we have $R(G\circ F)\cong RG\circ RF$.

Localization of bifunctors. Suppose given the following situation:

$$\begin{array}{ccc} \mathcal{T}\times\mathcal{T}' & \xrightarrow{F} & \mathcal{T}'' \\ Q\times Q'\downarrow & & \downarrow Q'' \\ \mathcal{T}/\mathcal{N}\times\mathcal{T}'/\mathcal{N}' & & \mathcal{T}''/\mathcal{N}'' \end{array}.$$

(1) Suppose there exists $\mathcal{U}$ triangulated subcategory of $\mathcal{T}$ which is right sufficient and $F(-,Y)$-acyclic for any $Y\in\mathcal{T}'$; then $R_IF(-,Y):\mathcal{T}/\mathcal{N}\to\mathcal{T}''/\mathcal{N}''$ exists. Suppose moreover that

$R_I F(\mathcal{U},\mathcal{N}')=0$ (i.e. $F(\mathcal{U},\mathcal{N}')=0$ in $\mathcal{T}''/\mathcal{N}''$); then the bifunctor $R_I F:\mathcal{T}/\mathcal{N}\times\mathcal{T}'\to\mathcal{T}''/\mathcal{N}''$ induces a triangulated functor $R_I F:\mathcal{T}/\mathcal{N}\times\mathcal{T}'/\mathcal{N}'\to\mathcal{T}''/\mathcal{N}''$, having the universal property of localized bifunctor of F.

(1′) Reversing the side of the hypothesis we can find a localized functor $R_{II}F:\mathcal{T}'/\mathcal{N}'\times\mathcal{T}'/\mathcal{N}'\to\mathcal{T}''/\mathcal{N}''$, if $\mathcal{U}'$ is a right sufficient and $F(X,-)$-acyclic subcategory of $\mathcal{T}'$ such that $F(\mathcal{N},\mathcal{U}')=0$ in $\mathcal{T}''/\mathcal{N}''$.

(2) If either $R_I F$ and $R_{II}F$ exist, they are canonically isomorphic.

Objects of finite-F-dimension. Let G be a bifunctor between t-categories. For $X\in\mathcal{T}^+$, we define $F=G(X,\text{-})$; then the following conditions are equivalent

(i) there exists $X\to U$ iso in $\mathcal{T}/\mathcal{N}$ with $U\in\mathcal{T}^b\cap\mathcal{U}$;

(ii) the functor $RF=RG(X,\text{-})$ is right way-out;

(iii) there exists n_0 such that $RF(X,Y)\in\mathcal{T}''_{\leqslant n_0}$ for any $Y\in\mathcal{CT}'$.

In that case we say that X has finite-F-dimension and write $X\in\mathcal{T}_{\mathrm{fFd}}$. Remark that $\mathcal{T}_{\mathrm{fFd}}$ is a triangulated localizing subcategory of $\mathcal{T}$.

Localization of generalized adjunctions. Let

$$\varphi(X,Y,Z):F_1(G_1(X,Y),Z)\longrightarrow F_2(X,G_2(Y,Z))$$

be a morphism of functors, where

$$\begin{array}{llll} G_1: \mathcal{T}\times\mathcal{T}' \longrightarrow \mathcal{V} & & F_1: \mathcal{V}\times\mathcal{T}'' \longrightarrow \mathcal{T}''' \\ G_2: \mathcal{T}'\times\mathcal{T}'' \longrightarrow \mathcal{V}' & & F_2: \mathcal{T}\times\mathcal{V}' \longrightarrow \mathcal{T}''' \end{array}$$

i.e. we have a morphism

$$\varphi:F_1\circ(G_1\times\mathcal{T}'')\longrightarrow F_2\circ(\mathcal{T}\times G_2)\ .$$

Suppose there exist $\mathcal{U}$ a localizing, right sufficient, G_1- and F_2-inert subcategory of $\mathcal{T}$, $\mathcal{U}''$ a localizing, right sufficient, G_2- and F_1-inert subcategory of $\mathcal{T}''$. Suppose moreover that we have the conditions for the derivation of bifunctors; then there exists a morphism of functors

$$\tilde\varphi:RF_1(RG_1(X,Y),Z)\longrightarrow RF_2(X,RG_2(Y,Z))$$

which is an isomorphism if φ is.

Localization of t-categories. Let $\mathcal{T}$ be a t-category, $\mathcal{N}$ a null system of $\mathcal{T}$; denote by $(\mathcal{T}_{\leqslant 0}/\mathcal{N},\mathcal{T}_{\geqslant 0}/\mathcal{N})$ the image by $Q:\mathcal{T}\longrightarrow\mathcal{T}/\mathcal{N}$ of the t-structure $(\mathcal{T}_{\leqslant 0},\mathcal{T}_{\geqslant 0})$ of $\mathcal{T}$. Then $(\mathcal{T}_{\leqslant 0}/\mathcal{N},\mathcal{T}_{\geqslant 0}/\mathcal{N})$ gives a t-structure for $\mathcal{T}/\mathcal{N}$ if and only if the following condition hold: for any distinguished triangle in $\mathcal{T}$ $X_1\longrightarrow X_0\longrightarrow N\longrightarrow TX_1$ with $X_1\in\mathcal{T}_{>0}$, $X_0\in\mathcal{T}_{\leqslant 0}$ and $N\in\mathcal{N}$, then $X_0,X_1\in\mathcal{N}$. In that case we write $((\mathcal{T}/\mathcal{N})_{\leqslant 0},(\mathcal{T}/\mathcal{N})_{\geqslant 0})$ for the t-structure.

Categories of Complexes.

Let $\mathcal{C}$ be an additive category; we define the category $\mathbf{C}(\mathcal{C})$ of complexes of $\mathcal{C}$ by: objects of $\mathbf{C}(\mathcal{C})$ are the data $\big((X^n)_{n\in\mathbb{Z}},(d^n_X)_{n\in\mathbb{Z}}\big)$ where $X^n\in\mathcal{C}$, $d^n_X:X^n\to X^{n+1}$ and $d^n_X\circ d^{n+1}_X=0$ for $\forall n\in\mathbb{Z}$; morphisms between $(X^\cdot,d^\cdot_X)$ and $(Y^\cdot,d^\cdot_Y)$ are the data $((f^n)_{n\in\mathbb{Z}})$ with $f^n:X^n\to Y^n$ commuting with the ds, i.e. $d^n_Y\circ f^n=f^{n+1}\circ d^n_X$. The composition of morphisms is defined using the composition of $\mathcal{C}$.

We represent the data by zero sequences and diagrams as

$$\begin{array}{ccccccccc} \cdots\longrightarrow & X^{n-1} & \xrightarrow{d_{n-1}} & X^n & \xrightarrow{d_n} & X^{n+1} & \xrightarrow{d_{n+1}} & X^{n+2} & \longrightarrow\cdots \\ \cdots & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \downarrow{\scriptstyle f^{n+2}} & \cdots \\ \cdots\longrightarrow & Y^{n-1} & \xrightarrow{d_{n-1}} & Y^n & \xrightarrow{d_n} & Y^{n+1} & \xrightarrow{d_{n+1}} & Y^{n+2} & \longrightarrow\cdots \end{array}$$

We may define a category $\widetilde{\mathbb{Z}}$ by: objects are the elements of $\mathbb{Z}$ (integer numbers); morphisms are

$$\mathrm{Hom}_{\widetilde{\mathbb{Z}}}(m,n)=\begin{cases}\mathbb{Z} & \text{if } n=m\\ \mathbb{Z} & \text{if } n=m+1\\ \mathbb{Z}/\mathbb{Z} & \text{otherwise}\end{cases}$$

with the composition given by the product in $\mathbb{Z}$; moreover the sum in $\mathbb{Z}$ gives a structure of abelian groups to the homomorphisms compatible (bilinear) with the composition. Then we may define $\mathbf{C}(\mathcal{C})=\mathcal{F}unct(\widetilde{\mathbb{Z}},\mathcal{A})$ (functors respecting the group structures of morphisms).

In case $\mathcal{C}=R$-Mod with R a ring, then the category $\mathbf{C}(R\text{-Mod})$ of complexes of R-modules is (isomorphic to) the category of $\mathbb{Z}$-graded R-modules endowed with a 1-degree homomorphism d with $d^2=0$. Moreover, let $R[\epsilon]=R[X]/(X^2)$ with graduation $R[\epsilon]=R\oplus R\epsilon$ (in degree 0 and 1); then $\mathbf{C}(R\text{-Mod})$ is the category of $\mathbb{Z}$-graded $R[\epsilon]$-modules (the action of ϵ gives the differential).

We call below limited (resp. upper limited, limited) complexes and we denote by $\mathbf{C}^+(\mathcal{C})$ (resp. $\mathbf{C}^-(\mathcal{C})$, $\mathbf{C}^b(\mathcal{C})$) the full subcategory of $\mathbf{C}(\mathcal{C})$ given by the objects for which $X^n=0$ for $n\ll 0$ (resp. $n\gg 0$, $|n|\gg 0$).

For $*\in\{\varnothing,+,-,b\}$ we remark that if $\mathcal{A}$ is an abelian category, then $\mathbf{C}^*(\mathcal{A})$ is abelian. An additive functor F between additive categories gives a functor $\mathbf{C}^*(F)$ (but noted also F) between the categories of complexes (i.e. $\mathbf{C}^*$ is an endofunctor of the category of additive categories, respecting pre-abelian and abelian categories).

We may embed the category $\mathcal{C}$ in its category of complexes $\mathbf{C}^*(\mathcal{C})$ sending any object X in the complex having X in degree zero, and zero otherwise (and with zero differentials).

For any $k\in\mathbb{Z}$ we define a translation functor $[k]$ on $\mathbf{C}^*(\mathcal{C})$ by:

$$\begin{array}{rcll} [k]:\ \mathbf{C}^*(\mathcal{C}) & \longrightarrow & \mathbf{C}^*(\mathcal{C}) & \\ (X^\cdot,d^\cdot_X) & \longmapsto & (X[k]^\cdot,d[k]^\cdot_X) & \begin{cases} X[k]^n=X^{n+k}\\ d[k]^n_X=(-)^k d^{n+k}_X\end{cases} \\ (f^\cdot) & \longmapsto & (f[k]^\cdot) & \quad f[k]^n=f^{n+k} \end{array}$$

Notice the sign change. The translation functors are isomorphisms.

Cohomology functors. Given an pre-abelian category $\mathcal{A}$ we define for any $n\in\mathbb{Z}$ the functors: $\mathbf{C}^*(\mathcal{A})\longrightarrow\mathcal{A}$ by

$$(X^\cdot,d^\cdot_X)\longmapsto\begin{cases} Z^n(X^\cdot,d^\cdot_X):=\ker(d^n_X)\\ B^n(X^\cdot,d^\cdot_X):=\mathrm{im}(d^{n-1}_X)\\ H^n(X^\cdot,d^\cdot_X):=\mathrm{coker}(\mathrm{im}(d^{n-1}_X)\to\ker(d^n_X))\\ \qquad\qquad\quad =Z^n(X^\cdot,d^\cdot_X)\big/B^n(X^\cdot,d^\cdot_X)\end{cases}$$

(the actions on morphisms are induced by those given by the functors ker, im, coker). The object $H^n(X^\cdot,d^\cdot_X)$ (respectively $Z^n(X^\cdot,d^\cdot_X)$, $B^n(X^\cdot,d^\cdot_X)$) is called the n-th object of cohomology (resp. cocycle, cobord) of the complex $(X^\cdot,d^\cdot_X)$. Remark that $H^n(X^\cdot)=H^0(X[n]^\cdot)$.

Let $\mathcal{A}$ be abelian. We have the exact sequences:

$$\begin{array}{cccccccccc} & X^{k-1} & \longrightarrow & Z^k(X) & \longrightarrow & H^k(X) & \longrightarrow & 0 & & \\ 0\longrightarrow & H^k(X) & \longrightarrow & \mathrm{coker}d^{k-1} & \longrightarrow & X^{k+1} & & & & \\ 0\longrightarrow & Z^{k-1}(X) & \longrightarrow & X^{k-1} & \longrightarrow & B^k(X) & \longrightarrow & 0 & & \\ 0\longrightarrow & B^k(X) & \longrightarrow & X^k & \longrightarrow & \mathrm{coker}d^{k-1} & \longrightarrow & 0 & & \\ 0\longrightarrow & H^k(X) & \longrightarrow & \mathrm{coker}d^{k-1} & \longrightarrow & Z^{k+1}(X) & \longrightarrow & H^{k+1}(X) & \longrightarrow & 0 \end{array}$$

The snake lemma: given a short exact sequence in $\mathbf{C}(\mathcal{A})$

$$0\longrightarrow X^\cdot\xrightarrow{f^\cdot}Y^\cdot\xrightarrow{g^\cdot}Z^\cdot\longrightarrow 0$$

we have a long exact sequence in $\mathcal{A}$

$$\cdots \xrightarrow{\delta^{n-1}} H^n(X^\cdot) \xrightarrow{H^n(f^\cdot)} H^n(Y^\cdot) \xrightarrow{H^n(g^\cdot)} H^n(Z^\cdot) \xrightarrow{\delta^n} H^{n+1}(X^\cdot) \xrightarrow{H^{n+1}(f^\cdot)} \cdots$$

where $\delta^\cdot$ is functorially defined (i.e. commutes with the morphism induced in the cohomology by morphism of short exact sequences in $\mathbf{C}(\mathcal{A})$), and it is given by the following construction: given a commutative diagram of exact rows

$$\begin{array}{ccccccc} & & X_1 & \longrightarrow & Y_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\ & & \xi\downarrow & & \eta\downarrow & & \zeta\downarrow & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & Y_2 & \longrightarrow & Z_2 & & \end{array}$$

there exists a canonical (functorial) morphism $\delta:\ \ker\zeta \to \operatorname{coker}\xi$ making exact the sequence

$$\ker\xi \to \ker\eta \to \ker\zeta \xrightarrow{\delta} \operatorname{coker}\xi \to \operatorname{coker}\eta \to \operatorname{coker}\zeta .$$

The construction is applied to the diagram:

$$\begin{array}{ccccccccc} & & \operatorname{coker} d_X^{n-1} & \longrightarrow & \operatorname{coker} d_Y^{n-1} & \longrightarrow & \operatorname{coker} d_Z^{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{n+1}(X) & \longrightarrow & Z^{n+1}(Y) & \longrightarrow & Z^{n+1}(Z) & & . \end{array}$$

The nine lemma: suppose the following diagram is commutative with exact columns and the central row is a zero sequence:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

then if two rows are exact, also the remaining row is.

Truncation functors:

(σ) If $\mathcal{C}$ is an additive category, we have the following (naif) truncation functors:

$$\sigma_{\geqslant n}:\mathbf{C}(\mathcal{C})\longrightarrow\mathbf{C}^+(\mathcal{C}) \qquad \sigma_{\geqslant n}(X)^m = \begin{cases} 0 & \text{if } m<n \\ X^m & \text{if } m\geqslant n \end{cases}$$

and

$$\sigma_{\leqslant n}:\mathbf{C}(\mathcal{C})\longrightarrow\mathbf{C}^-(\mathcal{C}) \qquad \sigma_{\leqslant n}(X)^m = \begin{cases} X^m & \text{if } m\leqslant n \\ 0 & \text{if } m<n \end{cases}$$

Remark that the cohomology at level n of the truncated complexes are not in general those of the original one (if exists).

We have canonical morphisms making exact the following diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_{\geqslant n}(X) & \longrightarrow & X & \longrightarrow & \sigma_{\leqslant n+1}(X) & \longrightarrow & 0 \\ 0 & \longrightarrow & X^n[-n] & \longrightarrow & \sigma_{\leqslant n}(X) & \longrightarrow & \sigma_{\leqslant n-1}(X) & \longrightarrow & 0 \\ 0 & \longrightarrow & \sigma_{\geqslant n+1}(X) & \longrightarrow & \sigma_{\geqslant n}(X) & \longrightarrow & X^n[-n] & \longrightarrow & 0 \end{array}$$

(τ) If $\mathcal{C}$ is a pre-abelian category, we can define the canonical truncation functors, in order to preserve the coholomogy just to the truncation index. More precisely we define four functors

$$\tau_{\geqslant n},\tau_{>n}:\mathbf{C}(\mathcal{C})\longrightarrow\mathbf{C}^+(\mathcal{C}) \quad \text{and} \quad \tau_{\leqslant n},\tau_{<n}:\mathbf{C}(\mathcal{C})\longrightarrow\mathbf{C}^-(\mathcal{C})$$

respectively by

$$\begin{array}{llllll} \tau_{\geqslant n}(X) = \cdots \to & 0 & \to \operatorname{coker}(d^{n-1}) & \to X^{n+1} & \to \cdots \\ \tau_{>n}(X) = \cdots \to & 0 & \to \operatorname{im}(d^n) & \to X^{n+1} & \to \cdots \\ \tau_{\leqslant n}(X) = \cdots \to & X^{n-1} & \to \ker(d^n) & \to 0 & \to \cdots \\ \tau_{<n}(X) = \cdots \to & X^{n-1} & \to \operatorname{coim}(d^{n-1}) & \to 0 & \to \cdots \end{array}$$

Remark that the cohomology of the complex X is preserved at the truncation index, i.e. $H^n(X) = H^n(\tau_{\geqslant n}(X)) = H^n(\tau_{\leqslant n}(X))$. Moreover we have canonical morphisms making exact the following diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau_{\leqslant n}(X) & \longrightarrow & X & \longrightarrow & \tau_{>n}(X) & \longrightarrow & 0 \\ 0 & \longrightarrow & \tau_{<n}(X) & \longrightarrow & X & \longrightarrow & \tau_{\geqslant n}(X) & \longrightarrow & 0 \\ 0 & \longrightarrow & \tau_{<n}(X) & \longrightarrow & \tau_{\leqslant n}(X) & \longrightarrow & H^n(X)[-n] & \longrightarrow & 0 \\ 0 & \longrightarrow & H^n(X)[-n] & \longrightarrow & \tau_{\geqslant n}(X) & \longrightarrow & \tau_{>n}(X) & \longrightarrow & 0 \end{array}$$

Passage to the opposite category. We have a canonical functor $\mathbf{C}(\mathcal{A})^\circ \longrightarrow \mathbf{C}(\mathcal{A}^\circ)$ defined sending X to the complex X° given by $X^{\circ n} = X^{-n}$ and $d^n_{X^\circ} = d_X^{-n-1}$. It is an equivalence of categories, exchanging the subcategories $\mathbf{C}^\pm(\mathcal{A})$ with $\mathbf{C}^\mp(\mathcal{A})$ and the truncation functors.

Categories of Homotopic Complexes.

A morphism of complexes $\varphi^\cdot:\ X^\cdot \longrightarrow Y^\cdot$ in $\mathbf{C}(\mathcal{C})$ is homotopic to zero ($\varphi \sim 0$) if there exists a family $(s^n)_{n\in\mathbb{Z}}$ of morphisms of $\mathcal{C}$, $s^n: X^n \to Y^{n-1}$, such that $\varphi^n = s^{n+1}\circ d^n_X + d_Y^{n-1}\circ s^n$ (notice that we don't ask the commutation between homotopies and differentials). We may see the situation in

$$\begin{array}{ccccc} X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} \\ & {\scriptstyle s^n}\swarrow & \downarrow{\scriptstyle\varphi_n} & \swarrow{\scriptstyle s^{n+1}} & \\ Y^{n-1} & \xrightarrow[d_Y^{n-1}]{} & Y^n & \xrightarrow[d_Y^n]{} & Y^{n+1} \end{array}$$

Two morphisms $f^\cdot, g^\cdot:\ X^\cdot \longrightarrow Y^\cdot$ are homotopic if the difference is homotopic to zero.

We denote $Ht(X^\cdot,Y^\cdot) = \left\{\varphi \in \operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(X^\cdot,Y^\cdot) \mid \varphi\sim 0\right\}$.

With respect to the composition of morphisms in $\mathbf{C}(\mathcal{C})$ we have that

$$Ht(X^\cdot,Y^\cdot)\times\operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(Y^\cdot,Z^\cdot) \xrightarrow{\circ} Ht(X^\cdot,Z^\cdot)$$
$$\operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(X^\cdot,Y^\cdot)\times Ht(Y^\cdot,Z^\cdot) \xrightarrow{\circ} Ht(X^\cdot,Z^\cdot)$$

If $\mathcal{C}$ is pre-Abelian, a morphism homotopic to zero induces the zero morphism in any degree of the cohomolgy of the complexes.

We define the category $\mathbf{K}(\mathcal{C})$ of complexes up to homotopy as

$$\operatorname{ob}(\mathbf{K}(\mathcal{C})) = \operatorname{ob}(\mathbf{C}(\mathcal{C}))$$
$$\operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(X^\cdot,Y^\cdot) = \operatorname{Hom}_{\mathbf{C}(\mathcal{C})}(X^\cdot,Y^\cdot)\big/ Ht(X^\cdot,Y^\cdot)$$

and similarly for $\mathbf{K}^*(\mathcal{C})$ where $*\in\{+,-,b\}$.

A functor F between additive categories induces a functor $\mathbf{K}^*(F)$ from the resp. categories of complexes up to homotopy, because morphism homotopic to zero are stable by $\mathbf{C}^*(F)$. In particular the cohomological functors $H^n:\mathbf{C}^*(\mathcal{A})\to\mathcal{A}$ induce functors $\mathbf{K}^*(\mathcal{A})\to\mathcal{A}$.

Triangulated structure. In general, if $\mathcal{A}$ is an Abelian category, $\mathbf{K}^*(\mathcal{A})$ is not (more precisely $\mathbf{K}^*(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple, i.e. every short exact sequence splits); it is an additive category endowed with a structure of triangulated category. We have already the translation functor [1] (the shift of complexes); we define the distinguished triangle by the notion of mapping cone.

Given a morphism $u^\cdot:\ X^\cdot \longrightarrow Y^\cdot$ in $\mathbf{C}(\mathcal{C})$ with $\mathcal{C}$ an additive category, we define the mapping cone $M(u)^\cdot$ of u the complex

$$M(u)^\cdot = X^\cdot[1]\oplus Y^\cdot \qquad \text{and} \qquad d_{M(f)} = \begin{pmatrix} d_{X[1]} & 0 \\ u[1] & d_Y \end{pmatrix}$$

i.e.

$$M(u)^n = X^{n+1}\oplus Y^n \qquad \text{and} \qquad d^n_{M(u)} = \begin{pmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_Y^n \end{pmatrix}$$

and we define a triangle by

$$X^\cdot \xrightarrow{u^\cdot} Y^\cdot \xrightarrow{\alpha(u)^\cdot} M(u)^\cdot \xrightarrow{\beta(u)^\cdot} X[1]^\cdot$$

where $\alpha(u)^\cdot = \begin{pmatrix}0\\ 1_{Y^\cdot}\end{pmatrix}$ and $\beta(u)^\cdot = \begin{pmatrix}-1_{X^\cdot[1]} & 0\end{pmatrix}$

Remark that if two morphisms $u^\cdot, v^\cdot : X^\cdot \longrightarrow Y^\cdot$ are homotopic (say by $k^\cdot$), then $M(u)^\cdot \simeq M(v)^\cdot$ by the isomorphisms of complexes $\theta^\cdot = \begin{pmatrix}1_{X^\cdot[1]} & 0\\ k^\cdot[1] & 1_{Y^\cdot}\end{pmatrix}$ and $\eta^\cdot = \begin{pmatrix}1_{X^\cdot[1]} & 0\\ -k^\cdot[1] & 1_{Y^\cdot}\end{pmatrix}$ one inverse of the other and making commutative the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(u)} & X[1]\\
\| & & \| & & \Updownarrow & & \|\\
X & \xrightarrow[v]{} & Y & \xrightarrow[\alpha(v)]{} & M(v) & \xrightarrow[\beta(v)]{} & X[1]
\end{array}$$

in particular the two triangles are isomorphic in $\mathbf{K}(\mathcal{C})$.

We define the family of distinguished triangle of $\mathbf{K}(\mathcal{C})$ as the triangles isomorphic to a mapping cone over some morphism of $\mathbf{C}(\mathcal{C})$. Then $\mathbf{K}(\mathcal{C})$ is a triangulated category:

($TR0$) the mapping cone on $0\xrightarrow{0}X$ gives the triangle $0\xrightarrow{0}X\xrightarrow{1_X}X\xrightarrow{0}0[1]$, then use ($TR2$); remark directly that $M(X)=M(\mathrm{id}_X)$ is homotopic to 0, i.e. that $\mathrm{id}_{M(X)}\sim 0$ using the homotopy $\begin{pmatrix}0 & 0\\ 0 & 1_{X[1]}\end{pmatrix}$;

($TR1$) any morphism embeds in its mapping cone;

($TR2$) is based on the following result: for any morphism $u^\cdot : X^\cdot \longrightarrow Y^\cdot$ in $\mathbf{C}(\mathcal{C})$ there exists an isomorphism $\varphi^\cdot : X[1]^\cdot \longrightarrow M(\alpha(u))^\cdot$ in $\mathbf{K}(\mathcal{C})$ such that the diagram of triangles

$$\begin{array}{ccccccc}
Y & \xrightarrow{\alpha(u)} & M(u) & \xrightarrow{\beta(u)} & X[1] & \xrightarrow{-u[1]} & Y[1]\\
\| & & \| & & \varphi\downarrow & & \|\\
Y & \xrightarrow[\alpha(u)]{} & M(u) & \xrightarrow[\alpha(\alpha(u))]{} & M(\alpha(u)) & \xrightarrow[\beta(\alpha(u))]{} & Y[1]
\end{array}$$

commutes in $\mathbf{K}(\mathcal{C})$. In fact we take $\varphi^\cdot = \begin{pmatrix}-u^\cdot[1]\\ 1_{X^\cdot[1]}\\ 0\end{pmatrix}$ with inverse $\psi^\cdot = \begin{pmatrix}0 & 1_{X^\cdot[1]} & 0\end{pmatrix}$. We have

$$\beta(\alpha(u))\circ\varphi = -u[1] \qquad \text{and} \qquad \varphi\circ\beta(u)\sim\alpha(\alpha(u))$$

the homotopy given by $(t^n)_{n\in\mathbb{Z}}$:

$$t^\cdot = \begin{pmatrix}0 & 1_{Y^\cdot}\\ 0 & 0\\ 0 & 0\end{pmatrix} : M(u)^\cdot \longrightarrow M(\alpha(u))^\cdot[-1]$$

then we have $\psi\circ\varphi = 1_{X[1]}$ and $\varphi\circ\psi\sim 1_{M(\alpha(u))}$ using the homotopy $(s^n)_{n\in\mathbb{Z}}$:

$$s^\cdot = \begin{pmatrix}0 & 0 & 1_{Y^\cdot}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} : M(\alpha(u))^\cdot \longrightarrow M(\alpha(u))^\cdot[-1] .$$

($TR3$) by definition;

($TR4$) starting with mapping cones

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{\alpha u} & M(u) & \xrightarrow{\beta u} & X[1]\\
f\downarrow & & g\downarrow & & & & \\
X' & \xrightarrow[u']{} & Y' & \xrightarrow[\alpha u']{} & M(u') & \xrightarrow[\beta u']{} & X'[1]
\end{array}$$

the commutativity in $\mathbf{K}(\mathcal{C})$ of the square means that $g\circ u - u'\circ f\sim 0$, i.e. there exists a family $(s^n)_{n\in\mathbb{Z}}$, $s^n : X^n \to Y'^{n-1}$ for which

$$g^n\circ u^n - u'^n\circ f^n = s^{n+1}\circ d_X^n + d_{Y'}^{n-1}\circ s^n$$

so we define h as

$$h = \begin{pmatrix}f[1] & 0\\ s[1] & g\end{pmatrix} : M(u) = X[1]\oplus Y \longrightarrow M(u') = X'[1]\oplus Y'$$

and we may verify that

$$h\circ\alpha(u) = \alpha(u')\circ g \qquad \text{and} \qquad f[1]\circ\beta(u) = \beta(u')\circ h .$$

($TR5$) we have the mapping cones on u, v and $v\circ u$ We define the required morphisms f and g as

$$f = \begin{pmatrix}1_{X[1]} & 0\\ 0 & v\end{pmatrix} : M(u) = X[1]\oplus Y \longrightarrow M(v\circ u) = X[1]\oplus Z$$

and

$$g = \begin{pmatrix}u[1] & 0\\ 0 & 1_Z\end{pmatrix} : M(v\circ u) = X[1]\oplus Z \longrightarrow M(v) = Y[1]\oplus Z$$

The diagram is then commutative, so we have to verify that the triangle

$$M(u)\xrightarrow{f} M(v\circ u)\xrightarrow{g} M(v)\xrightarrow{h} M(u)[1]$$

where $h = \begin{pmatrix}0 & 0\\ 0 & 1_{Y[1]}\end{pmatrix}$ is distinguished; we show that it is isomorphic to the triangle of the mapping cone of its first morphism f. Notice that

$$M(f) = M(u)[1]\oplus M(v\circ u) = x[2]\oplus Y[1]\oplus X[1]\oplus Z$$

and define two morphisms

$$\Phi = \begin{pmatrix}0 & 1_{Y[1]} & u[1] & 0\\ 0 & 0 & 0 & 1_Z\end{pmatrix} : M(f)\longrightarrow M(v)$$

$$\Psi = \begin{pmatrix}0 & 0\\ 1_{Y[1]} & 0\\ 0 & 0\\ 0 & 1_{X[1]}\end{pmatrix} : M(v)\longrightarrow M(f)$$

for which we verify that they are morphisms of complexes, making commutative the squares in the diagram of triangle

$$\begin{array}{ccccccc}
M(u) & \xrightarrow{f} & M(v\circ u) & \xrightarrow{g} & M(v) & \xrightarrow{h} & M(u)[1]\\
\| & & \| & & \Updownarrow & & \|\\
M(u) & \xrightarrow[f]{} & M(v\circ u) & \xrightarrow[\alpha f]{} & M(f) & \xrightarrow[\beta f]{} & M(u)[1]
\end{array}$$

then that $\Phi\circ\Psi = 1_{M(v)}$ and $\Psi\circ\Phi\sim 1_{M(f)}$ using the homotopy

$$t = \begin{pmatrix}0 & 0 & 1_{X[1]} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} : M(f)\longrightarrow M(f)[-1] .$$

We conclude that Φ and Ψ are isomorphisms in $\mathbf{K}(\mathcal{C})$, one inverse of the other.

(a) Remark that in a distinguished triangle of $\mathbf{K}(\mathcal{C})$ the composition of two morphisms is zero, but only in $\mathbf{K}(\mathcal{C})$, i.e. homotopic to zero in $\mathbf{C}(\mathcal{C})$. In the case of the mapping cone on $f : X\to Y$ we have $\alpha(u)\circ u = \begin{pmatrix}0\\ 1_Y\end{pmatrix}\circ u = \begin{pmatrix}0\\ u\end{pmatrix}$ which is not zero in $\mathbf{C}(\mathcal{C})$ but homotopic to zero by $\begin{pmatrix}1_X\\ 0\end{pmatrix} : X\to M(u)[-1]$.

(b) Let $\mathcal{A}$ be an Abelian category; then $H^0 : \mathbf{K}(\mathcal{A})\longrightarrow\mathcal{A}$ is a cohomological functor; in fact the sequence

$$0\longrightarrow Y\xrightarrow{\alpha(u)} M(u)\xrightarrow{\beta(u)} X[1]\longrightarrow 0$$

associated to the triangle on $f : X\to Y$ is exact and split in $\mathbf{C}(\mathcal{A})$.

(c) For an object $X\in\mathbf{C}(\mathcal{A})$ we have that $Q(X)\in\mathbf{K}(\mathcal{A})$ is isomorphic to zero iff the sequence $\mathrm{Hom}_{\mathcal{A}}(Y,X)$ is exact in $\mathbf{Ab}$ for any $Y\in\mathcal{A}$, or iff the sequence $\mathrm{Hom}_{\mathcal{A}}(X,Z)$ is exact in $\mathbf{Ab}$ for any $Z\in\mathcal{A}$ (i.e. by definition iff X is a split complex).

The mapping cone as defect of quasi-isomorphism of f. In particular we have the long exact sequence

$$\cdots\to H^{n-1}(X[1])\xrightarrow{\delta} H^n(Y)\to H^n(M(f))\to H^n(X[1])\xrightarrow{\delta} H^{n+1}(Y)\to\cdots$$

where $\delta = -H(f)$. Then we have: f is a quasi-isomorphism (i.e. $H^n(f)$ for $\forall n$) if and only if $M(f)$ is acyclic (i.e. exact, i.e. $H^n(M(f)) = 0$ for $\forall n$).

Triangles of $\mathbf{K}(\mathcal{A})$ as semi-split exact sequences of $\mathbf{C}(\mathcal{A})$. Let $E_{ss}(\mathbf{C}(\mathcal{A}))$ be the category of short exact sequences of $\mathbf{C}(\mathcal{A})$

which are semi-split, i.e. which splits "degree by degree", i.e. which are splits when the complexes are endowed with the null differentials. Then we have a functor:

$$E_{ss}(\mathbf{C}(\mathcal{A})) \longrightarrow \mathcal{T}r(\mathbf{K}(\mathcal{A}))$$

which is essentially surjective. In fact, chosen a (semi-)splitting $s: Z \to Y$ of the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we can write a semi-split exact sequence as

$$0 \longrightarrow X \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow 0$$

where the differential of $X \oplus Z$ is given by $\begin{pmatrix} d_X & \varphi \\ 0 & d_Y \end{pmatrix}$ with $\varphi: Z \to X[1]$ a morphism of complexes. Now we can see that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\varphi} X[1] \xrightarrow{+1}$ is distinguished.

The essential surjectivity follows from the fact that every morphism in $\mathbf{K}(\mathcal{A})$ admits a representative in $\mathbf{C}(\mathcal{A})$ which is a split mono; in fact it is enough to add $M(X) \sim 0$ to Y.

Canonical t-structures. A sequence $X \to Y \to Z$ in an additive category $\mathcal{A}$ is **split** (resp. **cosplit**) if for any $W \in \mathcal{A}$ the sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{A}}(W,X) \to \mathrm{Hom}_{\mathcal{A}}(W,Y) \to \mathrm{Hom}_{\mathcal{A}}(W,Z)$$

(resp. $\mathrm{Hom}_{\mathcal{A}}(Z,W) \to \mathrm{Hom}_{\mathcal{A}}(Y,W) \to \mathrm{Hom}_{\mathcal{A}}(X,W)$) is exact. A complex is (co)split if it is (co)split in any degree; then a complex is split iff it is cosplit iff it is homotopically equivalent to zero.

Remark that:

- (i) $0 \to X \to Y$ is split (resp. cosplit) iff the morphism is mono (resp. split mono);
- (i') $X \to Y \to 0$ is split (resp. cosplit) iff the morphism is split epi (resp. epi);
- (ii) $0 \to X \to Y \to Z \to 0$ is split iff it is cosplit iff it splits in the usual sense ($Y \cong X \oplus Z$);
- (iii) suppose $\mathcal{A}$ pre-abelian; then $X \xrightarrow{f} Y \xrightarrow{g} Z$ is split (resp. cosplit) iff the sequence $0 \to \ker f \to X \to \ker g \to 0$ (resp. $0 \to \mathrm{coker} f \to Z \to \mathrm{coker} g \to 0$) splits.

Suppose now that $\mathcal{A}$ is a pre-abelian category. Let $\mathbf{K}^{\leqslant 0}(\mathcal{A})$ (resp. $\mathbf{K}^{\geqslant 0}(\mathcal{A})$) be the full subcategory of $\mathbf{K}(\mathcal{A})$ given by complexes which are split in any positive (resp. negative) degree. Then $(\mathbf{K}^{\leqslant 0}(\mathcal{A}), \mathbf{K}^{\geqslant 0}(\mathcal{A}))$ define the left t-structure on $\mathbf{K}(\mathcal{A})$.

The analogous definitions using cosplited complexes give rise to the right t-structure on $\mathbf{K}(\mathcal{A})$; the two structures are exchanged by passage to the opposite category.

For $X \in \mathcal{A}$ the distinguished triangle by the left t-structure $X^{\leqslant 0} \to X \to X^{>0} \to$ is given by

$$X^{\leqslant 0} = \cdots \to X^{-2} \to X^{-1} \to \ker d^0 \to 0 \to \cdots$$
$$X^{>0} = \cdots \to 0 \to \ker d^0 \to X^0 \to X^1 \to \cdots \;;$$

while for the right t-structure it is $X^{<1} \to X \to X^{\geqslant 1} \to$ given by

$$X^{<1} = \cdots \to X^0 \to X^1 \to \mathrm{coker} d^0 \to 0 \to \cdots$$
$$X^{\leqslant 1} = \cdots \to 0 \to \mathrm{coker} d^0 \to X^2 \to X^3 \to \cdots \;.$$

The left truncation functors are given by

$$\tau^{\leqslant n} X = \cdots \to X^{n-2} \to X^{n-1} \to \ker d^n \to 0 \to \cdots$$
$$\tau^{\geqslant n} X = \cdots \to 0 \to \ker d^{n-1} \to X^{n-1} \to X^n \to \cdots \;;$$

the right truncation functors are

$$\tau^{\leqslant n} X = \cdots \to X^n \to X^{n+1} \to \mathrm{coker} d^n \to 0 \to \cdots$$
$$\tau^{\geqslant n} X = \cdots \to 0 \to \mathrm{coker} d^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \;.$$

Therefore the left heart $\mathcal{LA}$ is given by complexes of the form $0 \to \ker f \to X \to Y \to 0$ (Y in degree zero) and the right heart $\mathcal{RA}$ is given by complexes of the form $0 \to X \to Y \to \mathrm{coker} f \to 0$ (X in degree zero) for a morphism $f: X \to Y$.

Derived Categories.

Let $\mathcal{A}$ be an Abelian category; a morphism $u: X \to Y$ in $\mathbf{K}(\mathcal{A})$ is a quasi-isomorphism if for any $n \in \mathbb{Z}$ the induced morphism $H^n(u)$ is an isomorphism. We denote $\mathbf{Qis}$ the family of quasi-isomorphisms of $\mathbf{K}(\mathcal{A})$. Notice that $u \in \mathbf{Qis}$ iff $H^n(M(u)) = 0$ for any $n \in \mathbb{Z}$.

We define a null system in $\mathbf{K}(\mathcal{A})$ by

$$\mathcal{N} = \{X \in \mathbf{K}(\mathcal{A}) | \;\; H^n(X) = 0 \;\; \forall n \in \mathbb{Z}\} \;.$$

Then we have that $S(\mathcal{N}) = \mathbf{Qis}$.

Define the derived category of $\mathcal{A}$ as $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A}) \big/ \mathcal{N} = \mathbf{K}(\mathcal{A})_{\mathbf{Qis}}$. It is a triangulated category. For $* \in \{+,-,b\}$ put also $\mathbf{D}^*(\mathcal{A}) = \mathbf{K}^*(\mathcal{A}) \big/ (\mathcal{N} \cap \mathbf{K}^*(\mathcal{A}))$.

The canonical functor $Q: \mathbf{K}^*(\mathcal{A}) \longrightarrow \mathbf{D}^*(\mathcal{A})$ factorizes the cohomology functors:

$$\begin{tikzcd} \mathbf{K}^*(\mathcal{A}) \arrow[rr, "H^n"] \arrow[dr, "Q"'] & & \mathcal{A} \\ & \mathbf{D}^*(\mathcal{A}) \arrow[ur, "H^n"'] & \end{tikzcd}$$

because they kill the objects in $\mathcal{N}$.

We have that $\mathbf{D}^*(\mathcal{A})$ for $* = +,-,b$ is equivalent to the full subcategory of $\mathbf{D}(\mathcal{A})$ defined by the objects X such that $H^n(X) = 0$ resp. for $n \ll 0$, $n \gg 0$, $|n| \gg 0$; moreover $\mathcal{A}$ is equivalent to the full subcategory of objects of $\mathbf{D}(\mathcal{A})$ such that $H^n(X) = 0$ for any $n \neq 0$.

Triangles of $\mathbf{D}(\mathcal{A})$ *and exact sequences of* $\mathbf{C}(\mathcal{A})$. Given a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathbf{C}(\mathcal{A})$ there exists a quasi isomorphism $\varphi = (0 \;\; g): M(f) \to Z$ in $\mathbf{K}(\mathcal{A})$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\alpha f"] \arrow[d, equal] & M(f) \arrow[d, "\varphi"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

commutative. In fact from the exact sequence of $\mathbf{C}(\mathcal{A})$

$$0 \longrightarrow M(1_X) \xrightarrow{\gamma} M(f) \xrightarrow{\varphi} Z \longrightarrow 0$$

where $\gamma = \begin{pmatrix} 1_{X[1]} & 0 \\ 0 & f \end{pmatrix}$ we deduce, since $M(1_X) \cong 0$ in $\mathbf{K}(\mathcal{A})$ so that $H^n(M(1_X)) = 0 \; \forall n \in \mathbb{Z}$, that $M(f) \underset{qis}{\simeq} Z$ by the snake lemma.

The dist.tr. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ where $h = \beta(f) \circ \varphi^{-1}$ is say to be associated to the short exact sequence; in the long exact sequence associated we find $H^n(h) = \delta$ (the cobord morphism of the original sequence: this is the main motivation for the change of sign in the projection of the mapping cone).

This construction gives an essentially surjective functor

$$E(\mathbf{C}(\mathcal{A})) \longrightarrow \mathcal{T}r(\mathbf{D}(\mathcal{A}))$$

so that the notion of distinguished triangles in $\mathbf{D}(\mathcal{A})$ replaces the notion of short exact sequences of $\mathbf{C}(\mathcal{A})$.

Remark that for the truncation functors we have the quasi-isomorphisms $\tau_{\geqslant n}(X) \to \tau_{>n}(X)$ and $\tau_{<n}(X) \to \tau_{\geqslant n}(X)$, so that we have the distinguished triangles

$$\begin{array}{ccccccc} \tau_{\leqslant n}(X) & \longrightarrow & X & \longrightarrow & \tau_{\geqslant n}(X) & \xrightarrow{+1} \\ \tau_{\leqslant n-1}(X) & \longrightarrow & \tau_{\leqslant n}(X) & \longrightarrow & H^n(X)[-n] & \xrightarrow{+1} \\ H^n(X)[-n] & \longrightarrow & \tau_{\geqslant n}(X) & \longrightarrow & \tau_{\geqslant n+1}(X) & \xrightarrow{+1} \end{array}$$

Special Subcategories. Let $\mathcal{S}$ be an abelian subcategory of $\mathcal{A}$;

(i) it is right (resp. left) sufficient if the following condition holds: for any object A of $\mathcal{A}$ there exists an object S of $\mathcal{S}$ and a short exact sequence $0\to A\to S$, i.e. any object of $\mathcal{A}$ embeds in one of $\mathcal{S}$ (resp. a short exact sequence $S\to A\to 0$, i.e. any object of $\mathcal{A}$ is a quotient of one in $\mathcal{S}$).

(ii) it is right (resp. left) closed if for any exact sequence $0\to A'\to A\to A''\to 0$ with $A',A\in\mathcal{S}$ we have $A''\in\mathcal{S}$ (resp. $A,A''\in\mathcal{S}$ implies $A'\in\mathcal{S}$).

(iii) it is middle (or extension) closed if for any short exact sequence as before, $A',A''\in\mathcal{S}$ implies $A\in\mathcal{S}$.

(iv) it is acyclic if $\mathcal{S}\cap\mathcal{N}=0$, F-acyclic if $F(\mathcal{S}\cap\mathcal{N})\subseteq\mathcal{N}$.

(v) it is right (resp. left) limiting if there exists $n\in\mathbb{N}$ such that for any sequence

$$0\to A_0\to A_1\to\cdots\to A_{n-1}\to A_n\to 0$$

we have that $A_i\in\mathcal{S}$ for $i<n$ implies $A_n\in\mathcal{S}$ (resp. for $i>0$ implies $A_0\in\mathcal{S}$).

If $\mathcal{S}$ is a right sufficient subcategory of $\mathcal{A}$, then $\mathbf{K}^+(\mathcal{S})$ is a right sufficient (and localizing) subcategory of $\mathbf{K}^+(\mathcal{A})$:

(i) for any X in $\mathbf{K}^+(\mathcal{A})$ there exists an object Y in $\mathbf{K}^+(\mathcal{S})$ and a quasi-isomorphism $X\xrightarrow{f}Y$;

(ii) the canonical functor $\mathbf{K}^+(\mathcal{S})\big/(\mathcal{N}\cap\mathbf{K}^+(\mathcal{S}))\to\mathbf{D}^+(\mathcal{A})$ is an equivalence of categories.

We have the analogous results for left sufficient subcategory, using upper bounded complexes.

If $\mathcal{S}$ is an acyclic subcategory of $\mathcal{A}$, then the natural functor $\mathbf{K}^*(\mathcal{S})\longrightarrow\mathbf{D}^*(\mathcal{S})$ is an equivalence.

If $\mathcal{S}$ is right sufficient, right and middle closed, right limiting then we have that $\mathbf{K}^*(\mathcal{S})\longrightarrow\mathbf{D}^*(\mathcal{A})$ is essentially surjective for $*\in\{\varnothing,+,b\}$. We have the dual results for the left properties.

Injective and Projective. An abelian category $\mathcal{A}$ has enough injective if every object can be embedded into an injective one. Let $\mathcal{I}$ be the subcategory of injective objects of $\mathcal{A}$ (remark that it is middle and right closed); then it is acyclic, i.e. $\mathcal{N}\cap\mathbf{K}^+(\mathcal{I})=0$. Hence, if there are enough injective then $\mathbf{K}^+(\mathcal{I})$ is equivalent to $\mathbf{D}^+(\mathcal{A})$. In fact, take I an object of $\mathbf{K}^+(\mathcal{I})$ and suppose that $I\in\mathcal{N}$; we have to show that it is (homotopic to) zero, i.e. that its identity is homotopic to the zero morphism. Put $Z^n=\ker d^n$; the we have the exact sequences

$$0\longrightarrow Z^n\xrightarrow{i^n}I^n\xrightarrow{j^n}Z^{n+1}\longrightarrow 0$$

from which by induction we find that every Z^n is injective. So we can split the sequences to have

$$Z^n\xleftarrow{u^n}I^n\xleftarrow{v^n}Z^{n+1}$$

such that $u^n\circ i^n=1_{Z^n}$, $j^n\circ v^n=1_{Z^{n+1}}$ and $i^n\circ u^n+v^n\circ j^n=1_{I^n}$. We put $s^n=v^{n-1}\circ u^n:\ I^n\to I^{n-1}$ so that

$$\begin{aligned}d^{n-1}\circ s^n+s^{n+1}\circ d^n&=d^{n-1}\circ v^{n-1}\circ u^n+v^n\circ u^{n+1}\circ d^n\\&=i^n\circ j^{n-1}\circ v^{n-1}\circ u^n+v^n\circ u^{n+1}\circ i^{n+1}\circ j^n\\&=i^n\circ u^n+v^n\circ j^n=1_{I^n}\end{aligned}$$

as we need.

We have also that $\mathbf{Qis}\cap\mathbf{K}^+(\mathcal{I})$ are the isomorphisms of $\mathbf{K}^+(\mathcal{I})$ (i.e. in $\mathbf{K}^+(\mathcal{I})$ quasi-isomorphism implies the existence of a homotopic inverse).

Moreover if the category has enough injectives (i.e. $\mathcal{I}$ is right sufficient as abelian subcategory of $\mathcal{A}$, so $\mathbf{K}^+(\mathcal{I})$ is right sufficient as triangulated subcategory of $\mathbf{K}^+(\mathcal{A})$) then we have, for $X\in\mathbf{K}(\mathcal{A})$ and $I\in\mathbf{K}^+(\mathcal{I})$, that $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,I)=\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,I)$, and it is zero if X or I is acyclic.

We have the dual results for projective objects: the subcategory $\mathcal{P}$ of projectives is left and middle closed, acyclic, i.e. $\mathcal{N}\cap\mathbf{K}^-(\mathcal{P})=0$, and if there are enough projectives then $\mathbf{K}^-(\mathcal{P})$ is equivalent to $\mathbf{D}^-(\mathcal{A})$. $\mathbf{Qis}\cap\mathbf{K}^-(\mathcal{P})$ are the isomorphisms of $\mathbf{K}^-(\mathcal{P})$. Moreover if the category has enough projectives, then for $P\in\mathbf{K}^-(\mathcal{P})$ and $Y\in\mathbf{K}(\mathcal{A})$, we have that $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P,Y)=\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P,Y)$, and it is zero if P or Y is acyclic.

Prescribed cohomology. A full subcategory $\mathcal{S}$ of $\mathcal{A}$ is thick if for any exact sequence $Y'\to Y\to X\to Z\to Z'$ with the four external object in $\mathcal{S}$, then also X belongs to $\mathcal{S}$. If $\mathcal{S}$ is a middle closed subcategories of $\mathbf{K}^*(\mathcal{A})$, we define the category $\mathbf{K}^*_{\mathcal{S}}(\mathcal{A})$ as the full subcategory of $\mathbf{K}^*(\mathcal{A})$ whose cohomology objects belong to $\mathcal{S}$. We have that $\mathbf{D}^*_{\mathcal{S}}(\mathcal{A})=\mathbf{K}^*_{\mathcal{S}}(\mathcal{A})\big/\mathcal{N}\cap\mathbf{K}^*_{\mathcal{S}}(\mathcal{A})$ is equivalent to the subcategory of $\mathbf{D}^*(\mathcal{A})$ of objects whose cohomology takes values in $\mathcal{S}$.

There exists a natural functor $\mathbf{D}^+(\mathcal{S})\to\mathbf{D}^+_{\mathcal{S}}(\mathcal{A})$. It is an equivalence of categories under one of the following hypothesis:

(i) $\mathcal{S}$ is thick subcategory of $\mathcal{A}$ and for any monomorphism $i: S\to A$ with $S\in\mathcal{S}$ there exists a morphism $j:A\to S'$ with $S'\in\mathcal{S}$ e $j\circ i$ monomorphism;

(ii) $\mathcal{S}$ is a middle closed subcategory of $\mathcal{A}$ having enough $\mathcal{A}$-injective objects.

We have the dual result for $\mathbf{D}^-(\mathcal{S})$ with the dual hypothesis on epimorphisms, or projective objects.

Canonical t-structure. Let $\mathbf{D}^{\leqslant 0}(\mathcal{A})$ (resp. $\mathbf{D}^{\geqslant 0}(\mathcal{A})$) be the full subcategory of $\mathbf{D}(\mathcal{A})$ of complexes X with $X^n(X)=0$ for all $n>0$ (resp. $n<0$). Then $(\mathbf{D}^{\leqslant 0}(\mathcal{A}),\mathbf{D}^{\geqslant 0}(\mathcal{A}))$ gives the canonical t-structure of $\mathbf{D}(\mathcal{A})$. We have the usual truncation functors, and the canonical functor from $\mathcal{A}$ to the heart $\mathcal{C}(\mathbf{D}(\mathcal{A}))$ is an equivalence.

Derived Functors.

Let $\mathbf{K}^\dagger(\mathcal{A})$ a localizing subcategory of $\mathbf{K}(\mathcal{A})$; this means that $\mathcal{N}\cap\mathbf{K}^\dagger(\mathcal{A})$ is a null system and that $\mathbf{K}^\dagger(\mathcal{A})/\mathcal{N}\cap\mathbf{K}^\dagger(\mathcal{A})$ is a full subcategory of $\mathbf{D}(\mathcal{A})$, denoted $\mathbf{D}^\dagger(\mathcal{A})$.

Examples: $\mathbf{K}^*(\mathcal{A})$ for $*\in\{+,-,b\}$; $\mathbf{K}^*_{\mathcal{A}'}(\mathcal{A})$ where $\mathcal{A}'$ is a middle closed subcategory of $\mathcal{A}$.

Let $F:\mathbf{K}^\dagger(\mathcal{A})\longrightarrow\mathbf{K}(\mathcal{B})$ a triangulated functor; the right derived functor is the localized functor $R^\dagger F:\mathbf{D}^\dagger(\mathcal{A})\longrightarrow\mathbf{D}(\mathcal{B})$, i.e. a functor endowed with a morphism $\vartheta:Q_{\mathcal{B}}\circ F\longrightarrow R^\dagger F\circ Q_{\mathcal{A}}$ realizing for every $G:\mathbf{D}^\dagger(\mathcal{A})\longrightarrow\mathbf{D}(\mathcal{B})$ the bijection

$$\begin{array}{c}\mathrm{Hom}_{\mathcal{TF}unct(\mathbf{D}^\dagger(\mathcal{A}),\mathbf{D}(\mathcal{B}))}(R^\dagger F,G)\\ \downarrow\\ \mathrm{Hom}_{\mathcal{TF}unct(\mathbf{K}^\dagger(\mathcal{A}),\mathbf{D}(\mathcal{B}))}(Q_{\mathcal{B}}\circ F,G\circ Q_{\mathcal{A}})\end{array}$$

sending η to $(\eta\bullet 1_{Q_{\mathcal{A}}})\circ\vartheta$.

It is the representative of the (covariant) functor

$$\mathcal{TF}unct\left(\mathbf{D}^\dagger(\mathcal{A}),\mathbf{D}(\mathcal{B})\right)\longrightarrow\mathbf{Set}$$

sending G to $\mathrm{Hom}_{\mathcal{TF}unct(\mathbf{K}^\dagger(\mathcal{A}),\mathbf{D}(\mathcal{B}))}(Q_{\mathcal{B}}\circ F,G\circ Q_{\mathcal{A}})$.

The dual construction defines the left derived functor $L^\dagger F: \mathbf{D}^\dagger(\mathcal{A})\longrightarrow\mathbf{D}(\mathcal{B})$, representative of the (contravariant) functor sending G to $\mathrm{Hom}_{\mathcal{TF}unct(\mathbf{K}^\dagger(\mathcal{A}),\mathbf{D}(\mathcal{B}))}(G\circ Q_{\mathcal{A}},Q_{\mathcal{B}}\circ F)$.

Let $\mathcal{U}$ be a full triangulated subcategory of $\mathbf{K}^\dagger(\mathcal{A})$ which is right sufficient (i.e. for any X in $\mathbf{K}^\dagger(\mathcal{A})$ there exists $U\in\mathcal{U}$ and a qis $X\to U$) and F-acyclic (i.e. $H^i(FU)=0$ for any $U\in\mathcal{N}\cap\mathcal{U}$). Then $R^\dagger F$ exists and $\vartheta(Y):Q_{\mathcal{B}}\circ F(Y)\longrightarrow R^\dagger F\circ Q_{\mathcal{A}}(Y)$ is an isomorphism for any Y in $\mathcal{U}$.

Moreover if $\mathbf{K}^\ddagger(\mathcal{A})$ is a subcategory of $\mathbf{K}^\dagger(\mathcal{A})$ with the same properties as $\mathbf{K}^\dagger(\mathcal{A})$ respect to $\mathcal{U}\cap\mathbf{K}^\ddagger(\mathcal{A})$, then we have the compatibility $R^\ddagger(F_{|\mathbf{K}^\ddagger(\mathcal{A})})=R^\dagger F_{|\mathbf{D}^\ddagger(\mathcal{A})}$.

In particular, if $\mathcal{A}$ has enough injectives, then $\mathbf{K}^+(\mathcal{I})$ permits to calculate the right derived functors R^+F for any $F: \mathbf{K}^+(\mathcal{A})\to\mathbf{K}(\mathcal{B})$. Dually if if $\mathcal{A}$ has enough projectives, then $\mathbf{K}^+(\mathcal{P})$ permits to calculate the left derived functors R^-F for any $F:\mathbf{K}^-(\mathcal{A})\to\mathbf{K}(\mathcal{B})$.

Let now $F:\mathcal{A}\to\mathcal{B}$ an additive functor; an F-injective subcategory of $\mathcal{A}$ is a full abelian subcategory $\mathcal{J}$ with the following properties:

(i) right sufficient and right closed;

(ii) F-acyclic: if $0\to X'\to X\to X''\to 0$ is an exact sequence in $\mathcal{J}$, then the image by F is exact (if F is left (resp. right) exact, then we obtain an equivalent condition with the weaker hypothesis $X'\in\mathcal{J}$ (resp. $X''\in\mathcal{J}$)).

Let suppose that there exists an F-injective subcategory $\mathcal{J}$ of $\mathcal{A}$; then $\mathbf{K}^+(\mathcal{J})$ is a right sufficient, F-acyclic subcategory of $\mathbf{K}^+(\mathcal{A})$ and the right derived functor $R^+F:\mathbf{D}^+(\mathcal{A})\to\mathbf{D}^+(\mathcal{B})$ exists, because we have a canonical factorization

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{J}) \arrow[rr] \arrow[dr] & & \mathbf{D}^+(\mathcal{B}) \\
& \mathbf{K}^+(\mathcal{J})\big/\mathcal{N}\cap\mathbf{K}^+(\mathcal{J}) \arrow[ur, "R(F)"'] &
\end{tikzcd} .$$

We can see the situation in the diagram

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{A}) \arrow[rr, "\mathbf{K}^+(F)"] \arrow[ddd, "Q_{\mathcal{A}}"'] & & \mathbf{K}^+(\mathcal{B}) \arrow[ddd, "Q_{\mathcal{B}}"] \\
& \mathbf{K}^+(\mathcal{J}) \arrow[ul] \arrow[ur] \arrow[d] & \\
& \mathbf{K}^+(\mathcal{J})_{\mathbf{Qis}(\mathcal{J})} \arrow[dl, shift left] \arrow[dr] & \\
\mathbf{D}^+(\mathcal{A}) \arrow[rr, "R(F)"'] \arrow[ur, shift left] & & \mathbf{D}^+(\mathcal{B})
\end{tikzcd}$$

If F has finite cohomological dimension, then also RF exists, and $RF_{\mathbf{D}^+(\mathcal{A})}=R^+F$.

For an object $A\in\mathcal{A}$ we have the usual calculus: $R^+F(A)=F(I)$ where $I\in\mathbf{K}^+(\mathcal{J})$ is a complex quasi isomorphic to A, i.e. $0\to A\to I$ is an exact sequence (a right resolution of A with objects in $\mathcal{J}$).

Dually for left derived functors on objects $A\in\mathcal{A}$ we have $L^-F(A)=F(I)$ where $I\in\mathbf{K}^-(\mathcal{J})$ is a complex quasi isomorphic to A, i.e. $I\to A\to 0$ is an exact sequence (a left resolution of A with objects in $\mathcal{J}$).

Let $F:\mathcal{A}\longrightarrow\mathcal{B}$ be a left exact functor, and $\mathcal{A}$ having enough injectives; then the canonical morphism $F\to R^+F$ induce an isomorphism $F\cong R^0F=H^0\circ R^+F$. Moreover if we have an exact sequence $0\to A'\to A\to A''\to 0$ in $\mathcal{A}$ then we obtain a long exact sequence of derived functors

$$0\to FA'\to FA\to FA''\xrightarrow{\delta} R^1FA'\to R^1FA\to R^1FA''\xrightarrow{\delta} R^2FA'\to\cdots$$

In fact we can take an injective resolution $0\to A\xrightarrow{\varphi} I^0\to I^1\to\cdots$ i.e. a quasi-isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

and the calculus gives

$$R^0F(A)=H^0\left(F(I^\cdot)\right)=\ker(FI^0\to FI^1)=\mathrm{im}F(\varphi)\simeq F(A).$$

The long exact sequence is given by the snake lemma applied to the short exact sequence of complexes, resolutions of the three objects of $\mathcal{A}$.

We have the dual results for right exact functors; if $\mathcal{A}$ has enough projectives, then L^-F exists, it is calculated using projective resolutions $\cdots\to P_1\to P_0\to A\to 0$, and we have $L^-F(A)=F(P.)$, $L_iF(A)=H_iF(P.)$, $L_0F\cong F$ and the long exact sequence

$$\cdots\to L_2FA''\to L_1FA'\to L_1FA\to L_1FA''\to FA'\to FA\to FA''\to 0\ .$$

The case of contravariant functors can be recovered by dualizing the categories. We give a survey of the four cases for variance and exactness.

(i) $F:\mathcal{A}\to\mathcal{B}$ left exact covariant:

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{A}) \arrow[r, "\mathbf{K}^+(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathbf{K}^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
\mathbf{D}^+(\mathcal{A}) \arrow[r, "RF"'] & \mathbf{D}^+(\mathcal{B})
\end{tikzcd}$$

calculus for $A\in\mathcal{A}$: take a (left) injective resolution $0\to A\to I^\cdot$, then $RF(A)=F(I^\cdot)$ and $R^iF(A)=H^i(F(I^\cdot))$. From the exact sequence $0\to A\to B\to C\to 0$ of $\mathcal{A}$ one has the long exact sequence

$$0\to FA\to FB\to FC\to R^1F(A)\to R^1F(B)\to R^1F(C)\to R^2F(A)\to\cdots$$

(ii) $F:\mathcal{A}\to\mathcal{B}$ right exact covariant: ${}^\circ F^\circ:\ \mathcal{A}^\circ\to\mathcal{B}^\circ$ is left exact covariant

$$\begin{tikzcd}
\mathbf{K}^-(\mathcal{A})^\circ \arrow[r, "-"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathbf{K}^+(\mathcal{A}^\circ) \arrow[r, "\mathbf{K}^+({}^\circ F^\circ)"] \arrow[d, "Q_{\mathcal{A}^\circ}"'] & \mathbf{K}^+(\mathcal{B}^\circ) \arrow[r, "-"] \arrow[d, "Q_{\mathcal{B}^\circ}"'] & \mathbf{K}^-(\mathcal{B})^\circ \arrow[d, "Q_{\mathcal{B}}"] \\
\mathbf{D}^-(\mathcal{A})^\circ \arrow[r, "-"'] & \mathbf{D}^+(\mathcal{A}^\circ) \arrow[r, "R({}^\circ F^\circ)"'] & \mathbf{D}^+(\mathcal{B}^\circ) \arrow[r, "-"'] & \mathbf{D}^-(\mathcal{B})^\circ
\end{tikzcd}$$

one sees $LF=-\circ R({}^\circ F^\circ)\circ -$; calculus for $A\in\mathcal{A}$: take $P.\to A\to 0$ a (right) projective resolution, then $LF(A)=F(P.)$ and $L_iF(A)=H_i(F(P.))$. From the exact sequence $0\to A\to B\to C\to 0$ of $\mathcal{A}$ one has the long exact sequence

$$\cdots\to L_2F(C)\to L_1F(A)\to L_1F(B)\to L_1F(C)\to FA\to FB\to FC\to 0$$

(iii) $F:\mathcal{A}\to\mathcal{B}$ left exact contravariant: ${}^\circ F:\ \mathcal{A}^\circ\to\mathcal{B}$ is left exact covariant

$$\begin{tikzcd}
\mathbf{K}^-(\mathcal{A})^\circ \arrow[r, "-"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathbf{K}^+(\mathcal{A}^\circ) \arrow[r, "\mathbf{K}^+({}^\circ F)"] \arrow[d, "Q_{\mathcal{A}^\circ}"'] & \mathbf{K}^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\
\mathbf{D}^-(\mathcal{A})^\circ \arrow[r, "-"'] & \mathbf{D}^+(\mathcal{A}^\circ) \arrow[r, "R({}^\circ F)"'] & \mathbf{D}^+(\mathcal{B})
\end{tikzcd}$$

one sees $RF=R({}^\circ F)\circ -$; calculus for $A\in\mathcal{A}$: take $P.\to A\to 0$ a (right) projective resolution, then $RF(A)=F(P.)$ and $R^iF(A)=H_i(F(P.))$. From the exact sequence $0\to A\to B\to C\to 0$ of $\mathcal{A}$ one has the long exact sequence

$$0\to FC\to FB\to FA\to R^1F(C)\to R^1F(B)\to R^1F(A)\to R^2F(C)\to\cdots$$

(iv) $F:\mathcal{A}\to\mathcal{B}$ right exact contravariant: then $F^\circ:\ \mathcal{A}\to\mathcal{B}^\circ$ is left exact covariant

$$\begin{tikzcd}
\mathbf{K}^+(\mathcal{A}) \arrow[r, "\mathbf{K}^+(F^\circ)"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathbf{K}^+(\mathcal{B}^\circ) \arrow[r, "-"] \arrow[d, "Q_{\mathcal{B}^\circ}"'] & \mathbf{K}^-(\mathcal{B})^\circ \arrow[d, "Q_{\mathcal{B}}"] \\
\mathbf{D}^+(\mathcal{A}) \arrow[r, "R(F^\circ)"'] & \mathbf{D}^+(\mathcal{B}^\circ) \arrow[r, "-"'] & \mathbf{D}^-(\mathcal{B})^\circ
\end{tikzcd}$$

one sees $LF=-\circ R(F^\circ)$; calculus for $A\in\mathcal{A}$: take a (left) injective resolution $0\to A\to I^\cdot$, then $LF(A)=F(I^\cdot)$ and $L_iF(A)=H^i(F(I^{-\cdot}))=H^{-i}(F(I^\cdot))$. From the exact sequence $0\to A\to B\to C\to 0$ of $\mathcal{A}$ one has the long exact sequence

$$\cdots\to L_2F(A)\to L_1F(C)\to L_1F(B)\to L_1F(A)\to FC\to FB\to FA\to 0$$

Composition of derived functors. Let $F:\mathbf{K}^{\dagger}(\mathcal{A})\longrightarrow\mathbf{K}(\mathcal{B})$ and $G:\mathbf{K}^{\ddagger}(\mathcal{B})\longrightarrow\mathbf{K}(\mathcal{C})$ be triangulated functors with $F(\mathbf{K}^{\dagger}(\mathcal{A}))\subseteq\mathbf{K}^{\ddagger}(\mathcal{B})$; suppose that the right derived functors $R^{\dagger}F$, $R^{\ddagger}G$ and $R^{\dagger}(G\circ F)$ exist. Then we have a canonical morphism $\xi_{G,F}:R^{\dagger}(G\circ F)\to R^{\ddagger}G\circ R^{\dagger}F$ such that the diagram

$$\begin{tikzcd} Q_{\mathcal{C}}\circ G\circ F \arrow[r, "\vartheta_G\bullet F"] \arrow[d, "\vartheta_{G\circ F}"'] & R^{\ddagger}G\circ Q_{\mathcal{B}}\circ F \arrow[d, "R^{\ddagger}G\bullet\vartheta_F"] \\ R^{\dagger}(G\circ F)\circ Q_{\mathcal{A}} \arrow[r, "\xi_{G,F}\bullet Q_{\mathcal{A}}"'] & R^{\ddagger}G\circ R^{\dagger}F\circ Q_{\mathcal{A}} \end{tikzcd}$$

commutes.

Suppose that $\mathcal{U}$ is a right sufficient and F-acyclic subcategory of $\mathbf{K}^{\dagger}(\mathcal{A})$, $\mathcal{V}$ is a right sufficient and F-acyclic subcategory of $\mathbf{K}^{\ddagger}(\mathcal{B})$ and $F(\mathcal{U})\subseteq\mathcal{V}$. Then $\mathcal{U}$ is also $G\circ F$-acyclic, the three right derived functors $R^{\dagger}F$, $R^{\ddagger}G$, $R^{\dagger}(G\circ F)$ exist and the canonical morphism $\xi_{G,F}:R^{\dagger}(G\circ F)\to R^{\ddagger}G\circ R^{\dagger}F$ is an isomorphism.

In particular, for functors $F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{C}$ suppose that there exist an F-injective subcategory $\mathcal{I}$ of $\mathcal{A}$, a G-injective subcategory $\mathcal{J}$ of $\mathcal{B}$ and that $F(\mathcal{I})\subseteq\mathcal{J}$. Then $\mathcal{I}$ is also $G\circ F$-injective, the right derived functors R^+F, R^+G, $R^+(G\circ F)$ exist and we have the canonical isomorphism $R^+(G\circ F)\cong R^+G\circ R^+F$.

We have the dual results for left derived functors.

The derived functor of Hom. We define a bifunctor

$$\mathrm{Hom}^{\cdot}_{\mathcal{A}}:\mathbf{C}(\mathcal{A})^{\circ}\times\mathbf{C}(\mathcal{A})\longrightarrow\mathbf{C}(\mathbf{Ab})$$

by $\mathrm{Hom}^n_{\mathcal{A}}(X^{\cdot},Y^{\cdot})=\prod_{p\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{A}}(X^p,Y^{p+n})$ with differential given by $d^n(\varphi)=d_Y\circ\varphi+(-)^{n+1}\varphi\circ d^X$.

We have that $\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X^{\cdot},Y^{\cdot})[k]=\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X^{\cdot},Y^{\cdot}[k])$, but remark that the differential of $\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X^{\cdot}[-k],Y^{\cdot})$ does not coincides with the others.

We have a canonical morphism

$$\lambda(X,Y):H(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y))\longrightarrow\mathrm{Hom}^{\cdot}_{\mathcal{A}}(H(X),H(Y))$$

which is an isomorphism in degree zero if $X\in\mathbf{C}^-(\mathcal{A})$ and $Y\in\mathbf{C}^+(\mathcal{A})$ (and in that case we have $\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y)\in\mathbf{C}^+(\mathbf{Ab})$).

Remark that

$$\begin{aligned} Z^0(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y)) &= \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X,Y) \\ B^0(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y)) &= Ht(X,Y) \\ H^0(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y)) &= \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y) \end{aligned}$$

and we have also $H^i(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y))=\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,Y[i])$.

Let $\mathcal{A}$ an abelian category with enough injectives. Suppose I is a complex of injective objects; then $\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,I)$ is acyclic if X or I are acyclic. Dually, if $\mathcal{A}$ has enough projectives, if P a complex of projective objects; then $\mathrm{Hom}^{\cdot}_{\mathcal{A}}(P,Y)$ is acyclic if P or Y are.

As usual the functor $\mathrm{Hom}^{\cdot}_{\mathcal{A}}$ induces a functor

$$\mathrm{Hom}^{\cdot}_{\mathcal{A}}:\mathbf{K}(\mathcal{A})^{\circ}\times\mathbf{K}(\mathcal{A})\longrightarrow\mathbf{K}(\mathbf{Ab})$$

and if the category has enough injective and projective objects, then we are in the conditions in order to define the derived functors

$$\begin{aligned} R_{I}\mathrm{Hom}^{\cdot}_{\mathcal{A}}:&\ \mathbf{D}^-(\mathcal{A})^{\circ}\times\mathbf{D}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \\ R_{II}\mathrm{Hom}^{\cdot}_{\mathcal{A}}:&\ \mathbf{D}(\mathcal{A})^{\circ}\times\mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \\ R\mathrm{Hom}^{\cdot}_{\mathcal{A}}:&\ \mathbf{D}^-(\mathcal{A})^{\circ}\times\mathbf{D}^+(\mathcal{A}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \end{aligned}$$

where the third functor is the restriction of the first two. Moreover the calculus gives quasi isomorphisms

$$\begin{tikzcd} \mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,I) \arrow[r] \arrow[d, equal] & \mathrm{Hom}^{\cdot}_{\mathcal{A}}(P,I) & \mathrm{Hom}^{\cdot}_{\mathcal{A}}(P,Y) \arrow[l] \arrow[d, equal] \\ R_{II}\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y) & & R_{I}\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y) \end{tikzcd}$$

as we can see also with an easy argument of spectral sequences.

As usual we define $\mathrm{Ext}^i(X,Y)=H^i(R\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y))$ and the chain of isomorphisms

$$\begin{aligned} H^i(R\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,Y)) &= H^i(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,I)) \\ &= H^0(\mathrm{Hom}^{\cdot}_{\mathcal{A}}(X,I[i])) \\ &= \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X,I[i]) \\ &= \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y[i]) \end{aligned}$$

permits us to identify $\mathrm{Ext}^i(X,Y)$ with $\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y[i])\equiv\mathrm{Hom}^i_{\mathbf{D}(\mathcal{A})}(X,Y)$.

Since the Hom is a left exact functor we have for $X,Y\in\mathcal{A}$ that $\mathrm{Ext}^0(X,Y)\cong\mathrm{Hom}_{\mathcal{A}}(X,Y)$; moreover if $0\to X'\to X\to X''\to 0$ is an exact sequence in $\mathcal{A}$, then one has the long exact sequence

$$\begin{aligned} &0\to\mathrm{Hom}_{\mathcal{A}}(X'',Y)\to\mathrm{Hom}_{\mathcal{A}}(X,Y)\to\mathrm{Hom}_{\mathcal{A}}(X',Y)\to \\ &\to\mathrm{Ext}^1_{\mathcal{A}}(X'',Y)\to\mathrm{Ext}^1_{\mathcal{A}}(X,Y)\to\mathrm{Ext}^1_{\mathcal{A}}(X',Y)\to\mathrm{Ext}^2_{\mathcal{A}}(X'',Y)\to\cdots \end{aligned}$$

and for $0\to Y'\to Y\to Y''\to 0$ is an exact sequence in $\mathcal{A}$, then one has the long exact sequence

$$\begin{aligned} &0\to\mathrm{Hom}_{\mathcal{A}}(X,Y')\to\mathrm{Hom}_{\mathcal{A}}(X,Y)\to\mathrm{Hom}_{\mathcal{A}}(X,Y'')\to \\ &\to\mathrm{Ext}^1_{\mathcal{A}}(X,Y')\to\mathrm{Ext}^1_{\mathcal{A}}(X,Y)\to\mathrm{Ext}^1_{\mathcal{A}}(X',Y'')\to\mathrm{Ext}^2_{\mathcal{A}}(X,Y'')\to\cdots \end{aligned}$$

by the usual arguments.

In particular when X and Y are objects of $\mathcal{A}$ we have the isomorphism $\mathrm{Ext}^i(X,Y)\cong\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y[i])$, and in this case we have also the interpretation in terms of n-th extensions of the Ext groups. Let $\mathrm{Ex}^i(X,Y)$ be the set of "morphism classes" of n-extension of X by Y, i.e. of exact sequences $0\to Y\to E.\to X\to 0$ of the form

$$0\to Y\to E_1\to\cdots\to E_n\to X\to 0$$

up to the equivalence generated by the morphisms of exact sequences over the identities of X and Y. We put the group structure defined as follows.

For $u:Y\to Y'$ and $\mathcal{E}\in\mathrm{Ex}^i(X,Y)$ define $u*\mathcal{E}\in\mathrm{Ex}^i(X,Y')$ as the class of

$$0\to Y'\to Y'\oplus_Y E_1\to\cdots\to E_n\to X\to 0\ .$$

For $v:X'\to X$ and $\mathcal{E}\in\mathrm{Ex}^i(X,Y)$ define $\mathcal{E}*v\in\mathrm{Ex}^i(X',Y)$ as the class of

$$0\to Y\to E_1\to\cdots\to E_n\times_X X'\to X'\to 0\ .$$

Then for $\mathcal{E},\mathcal{E}'\in\mathrm{Ex}^i(X,Y)$ we define $\mathcal{E}+\mathcal{E}'=\nabla_Y*(\mathcal{E}\oplus\mathcal{E}')*\Delta_X$ where $\nabla_Y:Y\times Y\to Y$ is given by $(1_Y\ 1_Y)$ and $\Delta_X:X\to X\times X$ is the diagonal $\binom{1_X}{1_X}$. So the sum $\mathcal{E}+\mathcal{E}'$ is

$$0\to Y\to G\to E_2\oplus E_2'\to\cdots\to E_{n-1}\oplus E_{n-1}'\to F\to X\to 0$$

with $F=(E_n\oplus E_n')\times_{X\oplus X}X$ and $G=Y\oplus_{Y\oplus Y}(E_1\oplus E_1')$ if $n>1$, otherwise, if $n=1$, the sum is the exact sequence

$$0\to Y\to F\to X\to 0$$

where $F=Y\oplus_{Y\oplus Y}((E\oplus E')\times_{X\oplus X}X)$.

Then we have isomorphisms of groups

$$\mathrm{Ex}^i(X,Y)\longrightarrow\mathrm{Ext}^i(X,Y)\cong\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X,Y[i])$$

which are canonically defined up to a sign $(-)^n$. In fact the i-extension $0\to Y\to E.\to X\to 0$ define either a morphism

$$X\xrightarrow{\beta}(0\to E.\to X\to 0)\xleftarrow{\alpha}Y[i]$$

with α a quasi-isomorphism, either a morphism

$$X\xleftarrow{\alpha'}(0\to Y\to E.\to 0)\xrightarrow{\beta'}Y[i]$$

with α' a quasi-isomorphism.

Finite injective (projective) dimension. For $X\in\mathbf{K}^+(\mathcal{A})$, for $\mathcal{A}$ having enough injectives, the following conditions are equivalent:

(i) there exists $X \to I$ quasi-iso with $I \in \mathbf{D}^b(\mathcal{I})$;

(ii) the functor $R\mathrm{Hom}(\text{-},X)$ is left way-out (so it is way-out);

(iii) there exists n_0 such that $\mathrm{Ext}^i(Y,X)=0$ for any $Y \in \mathcal{A}$.

In that case we say that X has finite injective dimension and write $X \in \mathbf{K}^+(\mathcal{A})_{\mathrm{fid}}$. Remark that $\mathbf{K}^+(\mathcal{A})_{\mathrm{fid}}$ is a triangulated localizing subcategory of $\mathbf{K}^+(\mathcal{A})$.

We have the dual results for $X \in \mathbf{K}^-(\mathcal{A})$ with the notion of finite projective dimension.

The derived functor of $\otimes$. Let R be a ring. Then we have a bifunctor

$$\otimes^{\cdot}_R : \mathbf{C}(\text{mod-}R) \times \mathbf{C}(R\text{-mod}) \longrightarrow \mathbf{C}(\mathbf{Ab})$$

by $(X^{\cdot} \otimes^{\cdot}_R Y^{\cdot})^n = \bigoplus_{p+q=n} X^p \otimes_R Y^q$ with differential given by $d^n(x \otimes y) = d_X(x) \otimes y + (-)^p x \otimes d_Y(y)$ if $x \in X^p$ and $y \in Y^q$.

Remark that as target category can be taken the category of complexes of modules over the center of R. Moreover, if R° denotes the opposed ring, then the position $x \otimes y \longmapsto (-)^{p+q} y \otimes x$ ($x \in X^p$ and $y \in Y^q$) gives an isomorphism of abelian groups $X^{\cdot} \otimes^{\cdot}_R Y^{\cdot} \longrightarrow Y^{\cdot} \otimes^{\cdot}_{R^{\circ}} X^{\cdot}$. The associativity property is true if R is commutative. However we have the associativity

$$(X^{\cdot} \otimes^{\cdot}_R Y^{\cdot}) \otimes^{\cdot}_S Z^{\cdot} \cong X^{\cdot} \otimes^{\cdot}_R (Y^{\cdot} \otimes^{\cdot}_S Z^{\cdot})$$

for $X^{\cdot} \in \mathbf{C}(\text{mod-}R)$, $Y^{\cdot} \in \mathbf{C}(R\text{-mod-}S)$ and $Z^{\cdot} \in \mathbf{C}(S\text{-mod})$.

We have that $(X^{\cdot} \otimes^{\cdot}_R Y^{\cdot})[k] = X^{\cdot}[k] \otimes^{\cdot}_R Y^{\cdot}$, but remark that the differential of $X^{\cdot} \otimes^{\cdot}_R Y^{\cdot}[k]$ does not coincides with the others.

We have a canonical morphism

$$\gamma(X,Y) : H(X) \otimes^{\cdot}_R H(Y) \longrightarrow H(X \otimes^{\cdot}_R Y)$$

which is an isomorphism in degree zero if $X,Y \in \mathbf{C}^-(\mathcal{A})$ (and in that case we have $X \otimes^{\cdot}_R Y \in \mathbf{C}^-(\mathbf{Ab})$).

Suppose now P a complex of plat objects; then $P \otimes^{\cdot}_R Y$ is acyclic if P or Y are acyclic, and dually $X \otimes^{\cdot}_R P$ is acyclic if X or P are. As usual the functor $\otimes^{\cdot}_R$ induces a functor

$$\otimes^{\cdot}_R : \mathbf{K}(\text{mod-}R) \times \mathbf{K}(R\text{-mod}) \longrightarrow \mathbf{K}(\mathbf{Ab})$$

and if the category has enough plat objects, then we are in the conditions in order to define the derived functors

$$\begin{aligned} L_{I}\otimes^{\cdot}_R :\ & \mathbf{D}^-(\text{mod-}R) \times \mathbf{D}(R\text{-mod}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \\ L_{II}\otimes^{\cdot}_R :\ & \mathbf{D}(\text{mod-}R) \times \mathbf{D}^-(R\text{-mod}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \\ L\otimes^{\cdot}_R :\ & \mathbf{D}^-(\text{mod-}R) \times \mathbf{D}^-(R\text{-mod}) \longrightarrow \mathbf{D}(\mathbf{Ab}) \end{aligned}$$

where the third functor is the restriction of the first two. Moreover the calculus gives quasi isomorphisms

$$L_{I} \otimes^{\cdot}_R (X,Y) = P \otimes^{\cdot}_R Y \longleftarrow P \otimes^{\cdot}_R Q \longrightarrow X \otimes^{\cdot}_R Q = L_{II} \otimes^{\cdot}_R (X,Y)$$

as we can see also with an easy argument of spectral sequences.

As usual we define $\mathrm{Tor}^R_i(X,Y) = H^i(L \otimes^{\cdot}_R (X,Y))$.

Since the tensor product is a right exact functor we have for $X \in$ mod-R and $Y \in R$-mod that $\mathrm{Tor}^R_0(X,Y) \cong X \otimes_R Y$; moreover if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in mod-R, then one has the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}^R_2(X'',Y) \to \mathrm{Tor}^R_1(X',Y) \to \mathrm{Tor}^R_1(X,Y) \to \mathrm{Tor}^R_1(X'',Y) \to \\ \to X' \otimes_R Y \to X \otimes_R Y \to X'' \otimes_R Y \to 0 \end{aligned}$$

and for $0 \to Y' \to Y \to Y'' \to 0$ an exact sequence in R-mod, then one has the long exact sequence

$$\begin{aligned} \cdots \to \mathrm{Tor}^R_2(X,Y'') \to \mathrm{Tor}^R_1(X,Y') \to \mathrm{Tor}^R_1(X,Y) \to \mathrm{Tor}^R_1(X,Y'') \to \\ \to X \otimes_R Y' \to X \otimes_R Y \to X \otimes_R Y'' \to 0 \end{aligned}$$

by the usual arguments.

Finite Tor dimension. For $X \in \mathbf{K}^-(R\text{-mod})$, for $\mathcal{A}$ having enough projective (or flat) objects, the following conditions are equivalent:

(i) there exists $P \to X$ quasi-iso with $P \in \mathbf{D}^b(\mathcal{P})$;

(ii) the functor $\text{-} \otimes^L_R X$ is right way-out (so it is way-out);

(iii) there exists n_0 such that $\mathrm{Tor}^i_R(Y,X) = 0$ for any $Y \in$ mod-R and $n > n_0$.

In that case we say that X has finite Tor dimension and write $X \in \mathbf{K}^-(R\text{-mod})_{\mathrm{fTd}}$. Remark that $\mathbf{K}^-(R\text{-mod})_{\mathrm{fTd}}$ is a triangulated localizing subcategory of $\mathbf{K}^-(R\text{-mod})$.

Derived generalized adjunctions.

(a) We have a canonical morphism

$$X \longrightarrow R\mathrm{Hom}(R\mathrm{Hom}(X,Y),Y)$$

for any $X \in \mathbf{D}(\mathcal{A})$ and $Y \in \mathbf{D}^+(\mathcal{A})$.

(b) There is a canonical isomorphism

$$X \otimes^L_R Y \cong Y \otimes^L_{R^{\circ}} X$$

for $X,Y \in \mathbf{D}^-(R\text{-mod})$.

(c) There is a canonical isomorphism

$$(X \otimes^L_R Y) \otimes^L_S Z \cong X \otimes^L_R (Y \otimes^L_S Z) ,$$

for $X \in \mathbf{D}^-(\text{mod-}R)$, $Y \in \mathbf{D}(R\text{-mod-}S)$ and $Z \in \mathbf{D}^-(S\text{-mod})$.

(d) There is a canonical isomorphism

$$R\mathrm{Hom}_R(X \otimes^L_S Y, Z) \cong R\mathrm{Hom}_S(X, R\mathrm{Hom}_R(Y,Z))$$

for $X \in \mathbf{D}(\text{mod-}S)$, $Y \in \mathbf{D}^-(S\text{-mod-}R)$ and $Z \in \mathbf{D}^+(\text{mod-}R)$.

(e) There exists a canonical morphism

$$R\mathrm{Hom}_R(X,Y) \otimes^L_S Z \longrightarrow R\mathrm{Hom}_R(X, Y \otimes^L_S Z)$$

for $X \in \mathbf{D}^-(R\text{-mod})$, $Y \in \mathbf{D}(R\text{-mod-}S)$ and $Z \in \mathbf{D}^-(S\text{-mod})$. It is an isomorphism if $Y \in \mathbf{D}^b(R\text{-mod-}S)$.

(f) There exists a canonical morphism

$$R\mathrm{Hom}_R(X,Y) \otimes^L_S Z \longrightarrow R\mathrm{Hom}_R(R\mathrm{Hom}_S(Z,X),Y)$$

for $X \in \mathbf{D}(S\text{-mod-}R)$, $Y \in \mathbf{D}^+(\text{mod-}R)$ and $Z \in \mathbf{D}^-(S\text{-mod})$. It is an isomorphism if $X \in \mathbf{D}^b(S\text{-mod-}R)$.

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