Geometria Superiore Reference Cards

Logarithmic Geometry

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Logarithmic Structures.

Pre-Log Structures. Let $X = (X, \mathscr{O}_X)$ be a ringed space; a pre-log structure on X is a pair $(\mathscr{P}, \alpha)$ where $\mathscr{P}$ is a sheaf of monoids (or Monoid) and $\alpha : \mathscr{P} \to \mathscr{O}_X$ is a morphism of sheaves of monoids, where $\mathscr{O}_X$ is considered a monoid under its multiplication.

A morphism between two pre-log structures $(\mathscr{P}, \alpha)$ and $(\mathscr{Q}, \beta)$ on X is given by a morphism of Monoids $\varphi : \mathscr{P} \to \mathscr{Q}$ compatible with the morphisms of pre-log structures: $\beta \circ \varphi = \alpha$.

A pre-log structure $(\mathscr{P}, \alpha)$ on X is idealized if it is endowed with a sheaf of ideals (or Ideal) $\mathscr{K}$ of the Monoid $\mathscr{P}$ with the property that $\alpha(\mathscr{K}) = \{0\}$. A morphism of id. pre-log structures $(\mathscr{P}, \alpha, \mathscr{K}) \to (\mathscr{Q}, \beta, \mathscr{H})$ is a morphism of pre-log structures $\varphi : \mathscr{P} \to \mathscr{Q}$ compatible with the Ideals: $\varphi(\mathscr{K}) \subseteq \mathscr{H}$.

We use $\mathcal{PreLog}(X)$ and $\mathcal{IdPreLog}(X)$ to indicate the categories of pre-log and id. pre-log structures on X. Any pre log structure can be endowed with the trivial Ideal $\varnothing$ (the empty Ideal) so that it becomes an id. pre-log structure; this association gives the left adjoint to the (Ideal-)forget functor from $\mathcal{IdPreLog}(X)$ to $\mathcal{PreLog}(X)$.

Log Structures. For a pre-log structure $(\mathscr{M}, \alpha)$ on X we can consider the natural morphism $\alpha^{\leftarrow}(\mathscr{O}_X^\times) \xhookrightarrow{\alpha} \mathscr{O}_X^\times$. We say that $(\mathscr{M}, \alpha)$ is a logarithimic structure if this induced map is an isomorphism, i.e. if $\mathscr{M}$ contains $\mathscr{O}_X^\times$ via α.

A morphism between two (id.) log structures is a morphism as (id.) pre-log structures.

We also define the characteristic monoid sheaf as $\mathscr{C} := \mathscr{M} \big/ \mathscr{O}_X^\times$ and the characteristic abelian sheaf as $\mathscr{C}^{\mathrm{gp}} := (\mathscr{M} \big/ \mathscr{O}_X^\times)^{\mathrm{gp}}$.

We note that a prelog structure $\mathscr{M}$ on X is a log structure if and only if the the following conditions are satisfied: $\mathscr{O}_X^\times \subseteq \mathscr{M}$ as a submonoid; $\alpha_{|\mathscr{O}_X^\times}$ is the identity of $\mathscr{O}_X^\times$; $\alpha^{-1}(1) = \{1\}$. The first two conditions give the surjectivity of $\alpha_{|\mathscr{O}_X^\times}$, the third the injectivity.

Moreover, if $\mathscr{M}$ is a log str. on X, then $(\mathscr{M}/\mathscr{O}_X^\times)^\times = \{1\}$.

The canonical forgetful functor from $\mathcal{Log}(X)$ to $\mathcal{PreLog}(X)$ sending $(\mathscr{M}, \alpha)$ to itself admits a left adjoint a from $\mathcal{PreLog}(X)$ to $\mathcal{Log}(X)$ defined by $\mathscr{P}^a := \mathscr{P} \oplus_{\alpha_X^{\leftarrow}(\mathscr{O}_X^\times)} \mathscr{O}_X^\times$. Here the structural morphism α^a is obtained using the cocartesian diagram

$$\begin{tikzcd}
\alpha_X^{-1}(\mathscr{O}_X^\times) \arrow[r, hook] \arrow[d, "\alpha"'] & \mathscr{P} \arrow[d, "i_1"] \arrow[dr, "\alpha"] & \\
\mathscr{O}_X^\times \arrow[r, "i_2"'] & \mathscr{P}^a \arrow[r, "\alpha^a"'] & \mathscr{O}_X
\end{tikzcd}$$

in the category of sheaves of monoids on X, where the lower composition is the inclusion.

We can extend the adjunction to *idealized* pre-log structure, by simply endowing $(\mathscr{P}^a, \alpha^a)$ with the Ideal $\mathscr{K}^a$ generated in $\mathscr{P}^a$ by the image of $\mathscr{K}$ by i_1, so that $(\mathscr{P}^a, \alpha^a, \mathscr{K}^a)$ is an id. log structure and i_1 a morphism of that structure.

The (id.) log structure $\mathscr{P}^a$ on X is the associated (id.) log structure of $\mathscr{P}$.

For any ringed space X the canonical log structure $\mathscr{O}_X^\times$ (with the inclusion as morphism) is called trivial, and it is the initial object in the category $\mathcal{Log}(X)$. In this category we also have a final object given by $\mathscr{O}_X$ itself, with the identity as morphism.

Direct and inverse images. Given a morphism of ringed spaces $f : X \longrightarrow Y$ we can define the inverse and direct images of (id.) log structures by f.

For any pre-log structure $\mathscr{P}$ on X we define the direct image $f_*^{\log}(\mathscr{P})$ as the prelog structure on Y given by $f_*(\mathscr{P}) \times_{f_*(\mathscr{O}_X)} \mathscr{O}_Y$ with the second projection as structural morphism of monoids.

If $\mathscr{M}$ is already a log structure, then $f_*^{\log}(\mathscr{M})$ is as well.

If $\mathscr{P}$ is idealized by an ideal $\mathscr{K}$, we can idealize the direct image $f_*^{\log}(\mathscr{P})$ by the Ideal given by the inverse image (for the canonical projection) of $f_*(\mathscr{K})$ in $f_*^{\log}(\mathscr{P})$, intersected with the inverse image of zero for the projection to $\mathscr{O}_Y$.

For any pre-log structure $\mathscr{P}$ on Y we define the inverse image as the pre-log structure on X given by $f^{-1}(\mathscr{P})$ endowed with the composite morphism β:

$$f^{-1}(\mathscr{P}) \xrightarrow{f^{-1}(\alpha)} f^{-1}(\mathscr{O}_Y) \xrightarrow{f^\sharp} \mathscr{O}_X .$$

If $\mathscr{N}$ is a log structure, then $f^{-1}(\mathscr{N})$ is not in general already a log structure, and so we define the inverse image of the log structure $f^*(\mathscr{N})$ as the log structure associated to $f^{-1}(\mathscr{N})$. Explicitly: $f^*(\mathscr{N}) = f^{-1}(\mathscr{N}) \oplus_{\beta^{\leftarrow}(\mathscr{O}_X^\times)} \mathscr{O}_X^\times$.

If we start with an id. pre-log structure, we can endow $f^{-1}(\mathscr{P})$ with an Ideal, namely the image of $f^{-1}(\mathscr{K})$; in this way $f^*(\mathscr{N})$ too will become an id. log structure.

There is the usual adjunction property between direct and inverse images of log structures:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{Log}(Y)}(N_Y, f_*^{\log} M_X) \cong \mathrm{Hom}_{\mathcal{Log}(X)}(f^* N_Y, M_X) \cong \\ \cong \mathrm{Hom}_{\mathcal{PreLog}(X)}(f^{-1} N_Y, M_X)\end{aligned}$$

for any log structures M_X on X and N_Y on Y; similarly for prelog structures, and for idealized ones.

We collect some useful results on inverse images. Let $X \xrightarrow{f} Y$ be a morphism of ringed spaces,

(i) The inverse image of a trivial log str. is trivial: $f^* \mathscr{O}_Y^\times \cong \mathscr{O}_X^\times$.

(ii) $f^{-1}\left(\mathscr{M}_Y / \mathscr{O}_Y^\times\right) \cong f^* \mathscr{M}_Y \big/ \mathscr{O}_X^\times$

(iii) If P_Y is a prelog structure on Y such that $\mathscr{P}_Y^a \cong \mathscr{M}_Y$ then $f^* \mathscr{M}_Y$ is isomorphic to $\left(f^{-1} P_Y\right)^a$.

Log Schemes.

A (pre-)logarithmic scheme is a scheme $(X, \mathscr{O}_X)$ endowed with a (pre-)logarithmic structure (P_X, α_X). We write $(X, \mathscr{P}_X \xrightarrow{\alpha_X} \mathscr{O}_X)$ for a prelog schemes and we suppress the index X if no confusion can arise. We write also $(X, \mathscr{P}_X, \alpha_X)$ and $(X, \mathscr{P}_X)$ without mention of the map α_X. Similarly for the idealized case.

A morphism between two (id.) (pre-)log schemes $(X, \mathscr{P}_X)$ and $(Y, \mathscr{P}_Y)$ is a pair (f, φ) where $f : X \to Y$ is a morphism of schemes, $\varphi : f^{-1}(\mathscr{P}_Y) \to \mathscr{P}_X$ is a morphism of (idealized) Monoids and the natural diagram:

$$\begin{tikzcd}
f^{-1}(\mathscr{P}_Y) \arrow[r, "\varphi"] \arrow[d, "f^{-1}(\alpha_Y)"'] & \mathscr{P}_X \arrow[d, "\alpha_X"] \\
f^{-1}(\mathscr{O}_Y) \arrow[r, "f^\sharp"'] & \mathscr{O}_X
\end{tikzcd}$$

commutes (equivalently, φ is a morphism of (id.) pre log structures on X between the inverse image by f of $\mathscr{P}_Y$ and $\mathscr{P}_X$).

We write these categories as $\mathcal{PreLogSch}$, $\mathcal{IdPreLogSch}$, $\mathcal{LogSch}$, $\mathcal{IdLogSch}$.

The canonical forgetful functors $\mathcal{I}d\mathcal{L}og\mathcal{S}ch \to \mathcal{L}og\mathcal{S}ch \to \mathcal{S}ch$ sending $(X, \mathscr{M}_X, \mathscr{K}_X)$ to $(X, \mathscr{M}_X)$ and to X admit right adjoints $\mathcal{S}ch \to \mathcal{L}og\mathcal{S}ch \to \mathcal{I}d\mathcal{L}og\mathcal{S}ch$ defined resp. by the trivial log structure on the scheme X by $\mathscr{O}_X^\times \hookrightarrow \mathscr{O}_X$, and by the trivial idealized structure on the log scheme $(X, \mathscr{M}_X)$ by the empty Ideal. Moreover we have for any log scheme $(X, \mathscr{M}_X, \alpha_X)$ a canonical morphism of log schemes $(X, \mathscr{M}_X, \alpha_X) \longrightarrow (X, \mathscr{O}_X^\times)$ given by $(\mathrm{id}_X, \alpha_X^{\times\,-1})$; and for any id. log scheme $(X, \mathscr{M}_X, \mathscr{K}_X)$ a canonical morphism $(X, \mathscr{M}_X, \mathscr{K}_X) \longrightarrow (X, \mathscr{M}_X, \varnothing)$.

The canonical forgetful functor $\mathcal{L}og\mathcal{S}ch \to \mathcal{P}re\mathcal{L}og\mathcal{S}ch$ sending $(X, \mathscr{M}_X)$ to itself admits a left adjoint $\mathcal{P}re\mathcal{L}og\mathcal{S}ch \xrightarrow{a} \mathcal{L}og\mathcal{S}ch$ defined by $(X, \mathscr{P}_X)^a := (X, \mathscr{P}_X^a)$ where $\mathscr{P}_X^a$ is the associated log structure. $(X, \mathscr{P}_X^a)$ is the associated log scheme of $(X, \mathscr{P}_X)$.

Strict Morphisms. Given a morphism between log schemes $(X, \mathscr{M}_X) \xrightarrow{(f,\varphi)} (Y, \mathscr{M}_Y)$ the morphism $\varphi : f^{-1}\mathscr{M}_Y \longrightarrow \mathscr{M}_X$ induces a morphism $\varphi^a : f^*\mathscr{M}_Y \longrightarrow \mathscr{M}_X$. We say that (f, φ) is *strict* if φ^a is an isomorphism.

Note that the composite of two strict morphisms is strict.

For every morphism of log schemes as before we have a decomposition:

$$(X, \mathscr{M}_X) \xrightarrow{(\mathrm{id}_X, \varphi^a)} (X, f^*\mathscr{M}_Y) \xrightarrow{(f,\mathrm{id})} (Y, \mathscr{M}_Y)$$

where the first map uses the identity of the scheme X, and the second is strict.

We collect here some example.

Elementary Log Schemes. Given a ring k and a monoid P, we may construct the affine scheme of the monoidal algebra $k[P]$. Let X be the scheme $\mathrm{Spec}(k[P])$. The canonical morphism $P \longrightarrow k[P]$ defines a sheaf morphism $P_X \longrightarrow \mathscr{O}_X$ between the constant sheaf of value P and the structural sheaf. Then $(\mathrm{Spec}(k[P]), P_X \to \mathscr{O}_X)$ is a prelog scheme and we define the *elementary log scheme* as the associated log scheme

$$\Big(\mathrm{Spec}(k[P]), \mathscr{P}\Big) := \Big(\mathrm{Spec}(k[P]), (P_X \to \mathscr{O}_X)^a\Big).$$

In this way we have defined a (contravariant) functor from the category of monoids $\mathcal{M}on$ to the category $k{-}\mathcal{L}og\mathcal{S}ch$ of logarithmic schemes over k: for every map $P \xrightarrow{\varphi} Q$ of monoids we have an obvious extension $k[P] \xrightarrow{k[\varphi]} k[Q]$ making commutative the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ \downarrow & & \downarrow \\ k[P] & \xrightarrow[k[\varphi]]{} & k[Q] \end{array}$$

and so, finally, inducing the morphism

$$\Big(\mathrm{Spec}(k[\varphi]), k[\varphi]\Big) : \Big(\mathrm{Spec}(k[Q]), Q_Y\Big) \longrightarrow \Big(\mathrm{Spec}(k[P]), P_X\Big)$$

of prelog schemes and the corresponding morphism of log schemes

$$\Big(\mathrm{Spec}(k[\varphi]), k[\varphi]^a\Big) : \Big(\mathrm{Spec}(k[Q]), \mathscr{Q}\Big) \longrightarrow \Big(\mathrm{Spec}(k[P]), \mathscr{P}\Big).$$

The log structure $\mathscr{P}$ thus defined on $\mathrm{Spec}(k[P])$ is called the *canonical log structure* on $\mathrm{Spec}(k[P])$, and we write simply $\mathrm{Spec}(P \to k[P])$ or even $\mathrm{Spec}(k[P])$ instead of $\Big(\mathrm{Spec}(k[P]), \mathscr{P}\Big)$.

Affine Log Schemes. More generally, given a log ring (A, M), i.e. a monoid morphism from a monoid M to the multiplicative monoid structure of a ring A, we can associate a log scheme $\mathrm{Spec}(A, M)$ by taking $\mathrm{Spec}A$ as scheme, with the log structure induced by inverse image under the morphism $\mathrm{Spec}A \longrightarrow \mathrm{Spec}\mathbb{Z}[M]$ (with the canonical structure of $\mathrm{Spec}\mathbb{Z}[M]$).

This defines a contravariant functor from the category of log rings to the category of affine log schemes (endowed with a global chart, see below).

Schemes with Normal Crossing Divisors. Let X be a regular scheme with a normal crossing divisor D, union of smooth divisors D_i $(i = 1, \ldots, n)$. We take on X the log structure induced by direct image under the inclusion $X \smallsetminus D \hookrightarrow X$ of the trivial log structure on the open set $X \smallsetminus D$; that is $\mathscr{M}_X := \mathscr{O}_X \cap j_*\mathscr{O}_{X\smallsetminus D}^\times$. By the hypothesis on D we can see that locally, where the D_i have equations f_i, the log structure is defined as the inverse image of the canonical structure on $\mathrm{Spec}\mathbb{Z}[\mathbb{N}^n]$ under the morphism $X \longrightarrow \mathrm{Spec}\mathbb{Z}[\mathbb{N}^n]$ induced by the map $\mathbb{Z}[\mathbb{N}^n] \to \Gamma(X, \mathscr{O}_X)$ sending e_i to f_i.

Log Points. A log point is a log scheme whose scheme is the spectrum of a field. Thus its monoid sheaf is given by a monoid, and a log scheme is given by a morphism $P \to k$ from a monoid to the multiplicative structure of a field. It is obvious that P and $P/P^\times$ give the same log structure, which can expressed by $P \times k^\times$ with P sharp.

Charts. A very useful notion is that of charts for log schemes and morphisms of log schemes, as defined in [KK.89]. This uses elementary log schemes, and gives models for the log structures on a scheme:

Charts for Log Schemes. Let $(X, \mathscr{M})$ be a log scheme; a chart c of $(X, \mathscr{M})$ over the monoid P is an exact morphism of log schemes $(X, \mathscr{M}) \xrightarrow{c} \mathrm{Spec}(P \to \mathbb{Z}[P])$. Given a chart is equivalent to given a morphism of monoid sheaves $P_X \xrightarrow{\gamma} \mathscr{M}$ (or also a morphism of monoids $P \longrightarrow \Gamma(X, \mathscr{M})$) inducing an isomorphism of associated log structures. So if c is a chart, the scheme X is endowed with the log structure which is the inverse image of the canonical log structure on $\mathrm{Spec}(\mathbb{Z}[P])$.

If U is an étale scheme over X, a chart of $(X, \mathscr{M})$ on U over the monoid P is a chart of $(U, \mathscr{M}_U)$ over the monoid P.

Charts for Morphisms of Log Schemes. Let $(X, \mathscr{M}) \xrightarrow{(f,\varphi)} (Y, \mathscr{N})$ be a morphism of log schemes; a chart of (f, φ) over the morphism of monoids $Q \xrightarrow{u} P$ is a commutative diagram:

$$\begin{array}{ccc} (X, \mathscr{M}) & \xrightarrow{\quad (f,\varphi) \quad} & (Y, \mathscr{N}) \\ c\downarrow & & \downarrow d \\ \mathrm{Spec}(\mathbb{Z}[P]) & \xrightarrow[\mathrm{Spec}(\mathbb{Z}[\varphi])]{} & \mathrm{Spec}(\mathbb{Z}[Q]) \end{array}$$

where c and d are charts. The datum of a chart of a morphism over $Q \xrightarrow{u} P$ is equivalent to the data of two morphisms $P_X \xrightarrow{\gamma} \mathscr{M}$ and $Q_Y \xrightarrow{\delta} \mathscr{N}$ (inducing isomorphisms of associated log structures) and of a commutative diagram:

$$\begin{array}{ccc} Q_X & \xrightarrow{u_X} & P_X \\ f^{-1}\delta\downarrow & & \downarrow\gamma \\ f^{-1}\mathscr{N} & \xrightarrow[\varphi]{} & \mathscr{M} \end{array}$$

If U is an étale scheme over X, a chart of (f, φ) on U over $Q \xrightarrow{u} P$ is a chart of (f_U, φ_U) over $Q \xrightarrow{u} P$.

A chart $u : Q \to P$ of (f, φ) is neat at $x \in X$ if u^{gp} is injective and $\mathrm{coker}(u^{\mathrm{gp}}) \to \mathscr{C}_{X/Y,\bar{x}}^{\mathrm{gp}} = \mathrm{coker}(\mathscr{C}_{Y,\bar{y}}^{\mathrm{gp}} \to \mathscr{C}_{X,\bar{x}}^{\mathrm{gp}})$ is an isomorphism (morphisms of fine log schemes admit *fppf* locally neat charts).

Special Types of Log Structures. A log structure $\mathscr{M}$ on a scheme X is said to be a coherent, fine, "fine and saturated" (fs for short) log structure if locally there exist charts of $\mathscr{M}$ with monoids which are respectively finitely generated, f.g. and integral, f.g., integral and saturated. We use the same terminology for schemes with this type of log structures, and we indicated these categories as $\mathcal{L}og\mathcal{S}ch$, $\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$, $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$.

In the definition we have not specified the topology for which we use the word "locally"; there are at least two interesting choices: the Zariski and the étale topology. In fact in the first case we can define also the category $\mathcal{L}og\mathcal{S}ch$ as the category of log locally ringed spaces locally isomorphic to spectra of logarithmic rings. In many application, however, th useful notion is that of étale topology, and this is the sense used in what follows.

The inclusion of categories $\mathcal{L}og\mathcal{S}ch^{\mathrm{f}} \hookrightarrow \mathcal{L}og\mathcal{S}ch$ admits a right adjoint defined on log schemes with a global chart $(X, \mathscr{M}) \longrightarrow \operatorname{Spec}k[P]$ by $(X^{\mathrm{int}}, \mathscr{M}^{\mathrm{int}})$ with $X^{\mathrm{int}} := X \times_{\operatorname{Spec}k[P]} \operatorname{Spec}k[P^{\mathrm{int}}]$ and $\mathscr{M}^{\mathrm{int}}$ the log structure defined by taking the canonical projection $X^{\mathrm{int}} \longrightarrow \operatorname{Spec}k[P^{\mathrm{int}}]$ as a chart. The canonical morphism $(X^{\mathrm{int}}, \mathscr{M}^{\mathrm{int}}) \longrightarrow (X, \mathscr{M})$ gives by composition a bijection

$$\operatorname{Hom}_{\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}}\left((Y,\mathscr{N}),(X^{\mathrm{int}},\mathscr{M}^{\mathrm{int}})\right) \underset{\cong}{\longrightarrow} \operatorname{Hom}_{\mathcal{L}og\mathcal{S}ch}\left((Y,\mathscr{N}),(X,\mathscr{M})\right)$$

for any fine log scheme $(Y, \mathscr{N})$. We see directly that the canonical morphism is a closed immersion.

Similarly, the inclusion $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}} \hookrightarrow \mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$ admits a right adjoint defined on log schemes with a global chart $(X, \mathscr{M}) \longrightarrow \operatorname{Spec}k[P]$, with P an integral monoid, by $(X^{\mathrm{sat}}, \mathscr{M}^{\mathrm{sat}})$ with $X^{\mathrm{sat}} := X \times_{\operatorname{Spec}k[P]} \operatorname{Spec}k[P^{\mathrm{sat}}]$ and $\mathscr{M}^{\mathrm{sat}}$ the log structure defined by taking the canonical projection $X^{\mathrm{sat}} \longrightarrow \operatorname{Spec}k[P^s at]$ as a chart. The canonical morphism $(X^{\mathrm{sat}}, \mathscr{M}^{\mathrm{sat}}) \longrightarrow (X, \mathscr{M})$ gives by composition a bijection

$$\operatorname{Hom}_{\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}}\left((Y,\mathscr{N}),(X^{\mathrm{sat}},\mathscr{M}^{\mathrm{sat}})\right) \underset{\cong}{\longrightarrow} \operatorname{Hom}_{\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}}\left((Y,\mathscr{N}),(X,\mathscr{M})\right)$$

for any fs log scheme $(Y, \mathscr{N})$. We will see, by a criterion due to Kato, that the canonical morphism is a log étale map.

Integral Log Structures. A log structure $\mathscr{M}$ on a scheme X is said to be an integral log structure if the sheaf of monoids is integral ($\Gamma(U, \mathscr{M})$ is integral for any U). This implies that for any point $x \in X$ the monoids $\mathscr{M}_{\bar{x}}$ and $(\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$ are integral. If the log schemes are coherent the condition is equivalent to the following: locally there exist charts of $\mathscr{M}$ with integral monoids.

Saturated Log Structures. A log structure $\mathscr{M}$ on a scheme X is said to be a p-saturated (resp. saturated) log structure if the sheaf of monoids is p-saturated (resp. saturated) ($\Gamma(U, \mathscr{M})$ is p-saturated (resp. saturated) for any U). This is equivalent to requiring that for any point $x \in X$ the monoid $\mathscr{M}_{\bar{x}}$ is p-saturated (resp. saturated); also the condition is equivalent to the following: the log structure is integral and for any $x \in X$ the monoid $(\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}} \cong \mathscr{M}_{\bar{x}}/\mathscr{O}_{X,\bar{x}}^{\times}$ is p-saturated (resp. saturated). In the case that the log schemes are coherent the condition is equivalent to the following: locally there exist charts of $\mathscr{M}$ with p-saturated (resp. saturated) monoids.

Valuative Log Structures. A log structure $\mathscr{M}$ on a scheme X is said to be a valuative if for any point $x \in X$ the monoid $\mathscr{M}_{\bar{x}}$ is valuative.

If X is a quasi-coherent log scheme, there exists a valuative log space (not a scheme in general) X^{val} endowed with a canonical morphism $X^{\mathrm{val}} \longrightarrow X$ giving the following universal property $\operatorname{Hom}(Y, X^{\mathrm{val}}) \cong \operatorname{Hom}(Y, X)$ for any valuative log space Y (more generally there exists the right adjoint of the inclusion of valuative log spaces into integral log spaces).

The space X^{val} is locally constructed as $\varprojlim_{I \in \mathscr{J}} X_I$ where the limit is taken in the category of integral log spaces, $\mathscr{J}$ is the set of all non emplty finitely generates ideal of P (P gives a local chart for X) endowed with the order $I' \geq I$ iff $I' = JI$ for same ideal J of P; $X_I = X \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Proj}(\bigoplus_{n \geq 0} \langle I \rangle^n)$ with $\langle I \rangle$ the ideal generated by I in $\mathbb{Z}[P]$.

If we have a morphism of monoids $P \to Q$, we define $X_Q = X \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$. Then for any $a \in I$ we have $P[a^{-1}I] \subseteq P^{\mathrm{gp}}$, and X_I is covered by the affine open subshemes $X_{P[a^{-1}I]}$, and the log structures on the affine parts coincide on the intersection, so defining a log structure on X_I. If a_i generate I, then the $X_{P[a_i^{-1}I]}$ cover X_I.

Remark that the properties of being integral, saturated, valuatif, quasi-coherent, coherent, fine are stable under inverse image.

Morphisms of Log Schemes. We define and give some characterizations of properties of morphisms of log schemes. Our references are essentially [KK.89] and [Ts]. Take a morphism $(f, \varphi) : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$.

Closed Immersions. The morphism is a *closed immersion* if f is a closed immersion of schemes and φ is a surjective map. It is nilpotent if the kernel of the immersion is nilpotent. In the case of idealized log schemes, we require that the Ideal of X is generated by the image under φ of the Ideal of Y.

Exact Morphisms. The morphism is *exact* if for any point $x \in X$ the map $(f^*\mathscr{N})_{\bar{x}} \longrightarrow \mathscr{M}_{\bar{x}}$ is exact as map of monoids.

Since exactness implies injectivity, a closed immersion is exact if and only if it is strict.

Integral Morphisms. The morphism is *integral* if the morphism $(f^*\mathscr{N})_{\bar{x}} \longrightarrow \mathscr{M}_{\bar{x}}$ (or equivalently $f^{-1}(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{x}} \longrightarrow (\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$) is integral as a map of monoids for any point $x \in X$.

For integral log schemes this is equivalent to the fact that the base change of an integral log scheme by the given morphism is again integral.

Saturated Morphisms. The morphism is *p-saturated* (resp. *saturated*) if for any point $x \in X$, say $y = f(x)$, the map $(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{y}} \longrightarrow (\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$ is p-saturated (resp. saturated) as map of monoids.

For p-saturated (resp. saturated) log schemes this is equivalent to the fact that the base change of a p-saturated (resp. saturated) log scheme by the given morphism is again p-saturated (resp. saturated).

Kummer morphisms. The morphism is *of Kummer type* if the morphism $f^{-1}(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{x}} \longrightarrow (\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$ is integral as a map of monoids for any point $x \in X$.

Vertical morphisms. The morphism is *vertical* if the morphism $(f^*\mathscr{N})_{\bar{x}} \longrightarrow \mathscr{M}_{\bar{x}}$ (or equivalently $f^{-1}(\mathscr{N}/\mathscr{O}_Y^{\times})_{\bar{x}} \longrightarrow (\mathscr{M}/\mathscr{O}_X^{\times})_{\bar{x}}$) is vertical as a map of monoids for any point $x \in X$.

Cartier morphisms. Suppose $(Y, \mathscr{N})$ defined over $\mathbb{F}_p$ for a prime p; let $(X', \mathscr{M}')$ be the base change of $(X, \mathscr{M})$ over the Frobenius morphism of $(Y, \mathscr{N})$. The morphism (f, φ) if of Cartier type if it is integral and the relative Frobenius morphism $(X, \mathscr{M}) \longrightarrow (X', \mathscr{M}')$ is exact. The notion is stable by composition and base change.

(K.Kato) A morphism of fine and saturated log schemes is of Cartier type if and only if it is saturated.

Fiber Products of Log Schemes. In the category of Log Schemes the finite projective limits exist and commute with the "underlying scheme" functor; the construction works because the category of log structure on a fixed scheme admits finite inductive limits.

The construction restricts to the category of coherent Log Schemes. In this category we may write locally the fiber product using charts.

If $X, Y, Z \in \mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$ (resp. $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$), the fiber product $X \times_Z Y$ in the category of (coherent) Log Schemes is no longer fine or fs. However, the functors int and sat being right adjoints of the respective forgetful functors, the fiber products in the categories $\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$ (resp. $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$) are given by $(X \times_Z Y)^{\mathrm{int}}$ (resp. $(X \times_Z Y)^{\mathrm{sat}}$).

The Hierarchy of closed subchemes of a Log Scheme. Let X be in $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$; then there exists a chain of closed subsets of X, $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \cdots$, with $X_m = \varnothing$ if $m > \sup_{x\in X} \mathrm{rk}(\mathscr{C}^{\mathrm{gp}}_{X,x})$, and X_i is the subset of X where $\mathrm{rk}(\mathscr{C}^{\mathrm{gp}}_{X,x})$ is $\geq i$.

The open subscheme $X_{\mathrm{triv}} := X \smallsetminus X_1$, defined in general for any log scheme, is the open subscheme of triviality of the log structure, i.e. the biggest set where the inclusion $\mathscr{O}^{\times}_X \hookrightarrow \mathscr{M}_X$ is an isomorphism. Notice that $X_{\mathrm{triv}} \neq \varnothing$ if X is log smooth over a base field.

Log Derivations and Log Differentials.

Let $(X,\mathscr{M}) \longrightarrow (Y,\mathscr{N})$ be a morphism of log schemes; we define the module of relative derivations $\mathrm{Der}\left((X,\mathscr{M})/(Y,\mathscr{N}),\mathscr{F}\right)$ with values in an $\mathscr{O}_X$-Module $\mathscr{F}$ as the set of pairs (D,δ) with $D \in \mathrm{Der}(X/Y,\mathscr{F})$, $\delta \in \mathrm{Hom}(\mathscr{M},\mathscr{F})$ (monoid morphism) such that $\delta(n) = 0$ for $\forall n \in \mathscr{N}$, subject to the compatibility $D(\alpha(m)) = \alpha(m)\delta(m)$ for $\forall m \in \mathscr{M}$. With the obvious notations we can rewrite this as

$$\mathrm{Der}(X/Y,\mathscr{F}) \times_{\mathrm{Der}(\mathbb{Z}[\mathscr{M}]/\mathbb{Z}[\mathscr{N}],\mathscr{F})} \mathrm{Hom}_{\mathscr{N}}(\mathscr{M},\mathscr{F})$$

where we use the canonical morphisms sending D to $D \circ \mathbb{Z}[\alpha]$ and δ to $\mathbb{Z}[\alpha]\cdot\mathbb{Z}[\delta]$ respectively. From this expression it is immediately seen that the functor $\mathscr{F} \longmapsto \mathrm{Der}\left((X,\mathscr{M})/(Y,\mathscr{N}),\mathscr{F}\right)$ is represented by the $\mathscr{O}_X$-Module $\Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})}$ given by

$$\Omega^1_{X/Y} \oplus_{\Omega^1_{\mathbb{Z}[\mathscr{M}]/\mathbb{Z}[\mathscr{N}]}} (\mathscr{O}_X \otimes_{\mathbb{Z}} \mathscr{M}^{\mathrm{gp}}/\mathscr{N}^{\mathrm{gp}})$$

where we use the canonical morphisms sending $d(m)$ to $d(\alpha(m))$ and $\alpha(m) \otimes m$ resp., or, in explicit terms, by the quotient

$$\Omega^1_{X/Y} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} \mathscr{M}^{\mathrm{gp}}/\mathscr{N}^{\mathrm{gp}}) \big/ \sim$$

where $\sim$ is the sub-$\mathscr{O}_X$-Module generated by the differences $(d(\alpha(m)),0) - (0,\alpha(m)\otimes m)$ for all sections m of $\mathscr{M}$.

The canonical morphisms attached to $\Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})}$ are:

$$\begin{array}{rcll} d: & \mathscr{O}_X & \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} & f \mapsto (d(f),0) \\ \mathrm{dlog}: & \mathscr{M}_X & \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} & m \mapsto (0, 1\otimes m) \end{array}$$

As in the case of schemes we have the following results:

(i) Given a commutative square of log schemes:

$$\begin{array}{ccc} (X',\mathscr{M}') & \xrightarrow{g} & (X,\mathscr{M}) \\ \downarrow & & \downarrow \\ (Y',\mathscr{N}') & \longrightarrow & (X,\mathscr{M}) \end{array}$$

we have a canonical morphism

$$g^*\Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} \longrightarrow \Omega^1_{(X',\mathscr{M}')/(Y',\mathscr{N}')}$$

which is an isomorphism if the square is cartesian.

(ii) Given a composition

$$(X,\mathscr{M}) \xrightarrow{f} (Y,\mathscr{N}) \longrightarrow (Z,\mathscr{L})$$

we have the following exact sequence

$$f^*\Omega^1_{(Y,\mathscr{N})/(Z,\mathscr{L})} \longrightarrow \Omega^1_{(X,\mathscr{M})/(Z,\mathscr{L})} \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} \longrightarrow 0$$

(iii) In case f is an exact closed immersion with Ideal $\mathscr{I}$, then $\Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} = 0$ and we have the exact sequence

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega^1_{(Y,\mathscr{N})/(Z,\mathscr{L})} \otimes \mathscr{O}_X \longrightarrow \Omega^1_{(X,\mathscr{M})/(Z,\mathscr{L})} \longrightarrow 0$$

where the first morphism is induced by the classical map $d : \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega^1_{Y/Z}$.

Log de Rham complex. There is a unique extension of the universal differential $d : \mathscr{O}_X \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})}$ to the exterior algebras $\Omega^i_{(X,\mathscr{M})/(Y,\mathscr{N})} = \bigwedge^i \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})}$ making a complex

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} \longrightarrow \Omega^2_{(X,\mathscr{M})/(Y,\mathscr{N})} \longrightarrow \cdots$$

such that $d \circ d = 0$ and $d \circ \mathrm{dlog} = 0$. By definition this is the de Rham relative complex of $(X,\mathscr{M})$ over $(Y,\mathscr{N})$.

Log Smoothness.

As in the classical case we say that a morphism $f : (X,\mathscr{M}) \longrightarrow (Y,\mathscr{N})$ of log schemes is log smooth (resp. log net, log étale) if it is of finite presentation as morphism of schemes and it is formally is log smooth (resp. log net, log étale), i.e. the following condition is satisfied: for any exact closed nilpotent immersion $(T,\mathscr{L}) \hookrightarrow (T',\mathscr{L}')$ over $(Y,\mathscr{N})$ the induced map:

$$\mathscr{H}\!om_{(Y,\mathscr{N})}((T',\mathscr{L}'),(X,\mathscr{M})) \longrightarrow \mathscr{H}\!om_{(Y,\mathscr{N})}((T,\mathscr{L}),(X,\mathscr{M}))$$

is surjective (resp. injective, bijective); we can use in the definition only affine (exact closed) nilpotent immersions, and in that case we can ask for the surjectivity (resp. injectivity, bijectivity) of the following map:

$$\mathrm{Hom}_{(Y,\mathscr{N})}((T',\mathscr{L}'),(X,\mathscr{M})) \longrightarrow \mathrm{Hom}_{(Y,\mathscr{N})}((T,\mathscr{L}),(X,\mathscr{M}))\,.$$

Notice that the notions are stable under composition, base change (also in the category of fine or fs log schemes), and that if a composition $g \circ f$ is log net, then f is; if the composition is log smooth (resp. log étale) and g is log net, then f is log smooth (resp. log étale).

Moreover for a strict morphism the "log notions" (log smooth, log net, log étale) are equivalent to the classical notions (smooth, net, étale). We can define also the analogous "ideally" log notions by using the category of idealized log schemes and strict (also for the ideal component) nilpotent closed immersions.

Differential properties. If a morphism $f : (X,\mathscr{M}) \longrightarrow (Y,\mathscr{N})$ is log smooth (or ideally log smooth) then the $\mathscr{O}_X$-Module of relative differentials is locally free of finite type. If n is the rank, then there exists étale locally a factorization of f by

$$(X,\mathscr{M}) \xrightarrow{h} \mathbb{A}^n_{(Y,\mathscr{N})} \longrightarrow (Y,\mathscr{N})$$

where h is log étale, $\mathbb{A}^n_{(Y,\mathscr{N})} = \mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} (Y,\mathscr{N})$ (notice that $\mathbb{A}^n_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^n])$ with the canonical log structure) and the second morphism is the canonical projection.

The morphism is log net if and only if $\Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} = 0$; iff étale locally there exists a factorization of f as $f' \circ i$ where i is an exact closed immersion, and f' is a log étale morphism; iff étale locally there exist charts $h : P \to Q$ of f such that h^{gp} is injective and the (strict) morphism $X \longrightarrow Y \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ in net.

We report three criteria for smoothness:

Kato's criterion using charts. The morphism f is log smooth (resp. log étale) locally at $x \in X$ if and only if there exists étale locally at x a chart of f modeled on $h : P \longrightarrow Q$ with:

(i) P, Q fine,

(ii) $\ker(h^{\mathrm{gp}})$ killed by an integer invertible at x (or h^{gp} injective),

(iii) $(\mathrm{coker}(h^{\mathrm{gp}}))_{\mathrm{tor}}$ (resp. $\mathrm{coker}(h^{\mathrm{gp}})$) killed by an integer invertible at x,

such that the canonical (strict) morphism $X \longrightarrow Y \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ is smooth (or étale, because the relative differential module is trivial).

Composition criterion. Let $(X,\mathscr{M}) \xrightarrow{f} (Y,\mathscr{N}) \xrightarrow{g} (S,\mathscr{L})$ be morphisms of log schemes; then

(1) if f is log smooth then the sequence

$$0 \longrightarrow f^*\Omega^1_{(Y,\mathscr{N})/(Z,\mathscr{L})} \longrightarrow \Omega^1_{(X,\mathscr{M})/(Z,\mathscr{L})} \longrightarrow \Omega^1_{(X,\mathscr{M})/(Y,\mathscr{N})} \longrightarrow 0$$

is exact and locally split;

(2) if $g \circ f$ is log smooth and the above sequence is exact and locally split then f is log smooth.

Jacobian criterion. Let $(X, \mathscr{M}) \xrightarrow{i} (Y, \mathscr{N}) \xrightarrow{g} (S, \mathscr{L})$ be morphisms of log schemes with i an exact closed immersion with Ideal $\mathscr{I}$, put $f = g \circ i$; then

(1) if f is log smooth then the sequence

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega^1_{(Y,\mathscr{N})/(Z,\mathscr{L})} \otimes_{\mathscr{O}_Y} \mathscr{O}_X \longrightarrow \Omega^1_{(X,\mathscr{M})/(Z,\mathscr{L})} \longrightarrow 0$$

is exact and locally split;

(2) if g is log smooth and the above sequence is exact and locally split then f is log smooth.

We recall some cases in which the notion of ideally smooth is very useful:

(Ogus' lemma.) Let $(X, \mathscr{M})$ be a fine log scheme, $\mathscr{K}$ a coherent Ideal of $\mathscr{M}$, $X_{\mathscr{K}}$ the closed subscheme of X defined by the coherent Ideal $\alpha(\mathscr{K})\mathscr{O}_X$ of $\mathscr{O}_X$; then the canonical morphism

$$\left(X_{\mathscr{K}}, j^*\mathscr{M}, j^*\mathscr{K}\right) \longrightarrow (X, \mathscr{M}, \varnothing)$$

is ideally log étale.

Let P be a fine monoid with $P^\times = \{0\}$ (sharp); then the log point over Spec$k$ with the morphism $P \to k$ sending 0 to 1 and $P^+ := P \setminus P^\times$ to 0, is ideally log smooth over Spec$k$ with the trivial structure, when equipped with the ideal $K := P^+$.

In fact we can see that $Spec(P \to k[P])$ is smooth over k (with the trivial structure) and the closed subscheme defined by the ideal P^+ of P is exactly Spec $\left(k[P]/k[P^+]\right) \cong$ Spec$k$; then the Ogus lemma says that the morphism

$$(\mathrm{Spec}k, P, P^+) \longrightarrow (\mathrm{Spec}k[P], P, \varnothing)$$

is ideally log étale, so that also $(\mathrm{Spec}k, P, P^+)$ is ideally log smooth over k.

Let $(X, \mathscr{M})$ be a fine log scheme, log smooth over a log point (k, P) with P sharp; then there exists an ideal structure on $\mathscr{M}$ such that $(X, \mathscr{M}, \mathscr{K})$ is ideally log smooth over k (with trivial structure). In fact we endow $(X, \mathscr{M})$ with the Ideal inverse image of P^+.

Log Flatness. A morphism $(f, \varphi) : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$ of fine log schemes is log flat if (classical) *fppf* locally there exists charts $h : P \to Q$ such that h^{gp} is injective and the natural morphism of schemes $X \longrightarrow Y \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ is flat (we have the obvious definition of "log flat at $x \in X$"). Log flatness is stable under composition and base change.

Suppose to have a neat chart at $x \in X$ for (f, φ), and f be of finite presentation (in a neighborhood of x); then the morphism is log flat at $x \in X$ if and only if the morphism $X \longrightarrow Y \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ is flat at x.

We have the fibre criterion for log flatness: given a diagram of fine log schemes $(X, \mathscr{M}) \xrightarrow{(f,\varphi)} (Y, \mathscr{N}) \xrightarrow{(q,\tau)} (S, \mathscr{L})$, $x \in X$, $y = f(x)$ and $s = p(x)$, assume that $f, q, p = q \circ f$ are of finite presentation in neighborhoods of x, y. If (p, π) and $(f, \varphi)_s : (X, \mathscr{M})_s \longrightarrow (Y, \mathscr{N})_s$ (the fibers endowed with the induced log structures) are log flat at x, then the morphism (f, φ) is log flat at x.

Log Quasi-finite. A morphism $(f, \varphi) : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$ of fine log schemes is log quais-finite if (classical) *fppf* locally there exists charts $h : P \to Q$ such that $\mathrm{coker}(h^{\mathrm{gp}})$ is finite and the natural morphism of schemes $X \longrightarrow Y \times_{\mathrm{Spec}\mathbb{Z}[P]} \mathrm{Spec}\mathbb{Z}[Q]$ is quasi-finite. Log quasi-finiteness is stable under composition and base change.

A log smooth and log quasi-finite morphism is log étale.

Remark. Let S be a not empty scheme, $h : G \to H$ a morphism of finitely generated abelian groups; then $\mathscr{O}_S[G] \longrightarrow \mathscr{O}_S[G]$ induces a flat, smooth, étale, quasi-finite morphism if and only if we have respectively that $\ker(h)$ is of finite order invertible in S, $\ker(h)$ and $\mathrm{coker}(h)_{\mathrm{tor}}$ are of finite order invertible in S, $\ker(h)$ and $\mathrm{coker}(h)$ are of finite order invertible in S, $\mathrm{coker}(h)$ is finite.

Therefore we have that $(f, \varphi) : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$ morphism of fine log schemes is log flat, log smooth, log étale, log quasi-finite morphism if and only if we have (resp. *fppf*, étale, étale, *fppf*) locally charts $h : P \to Q$ such that the morphisms of schemes $X \longrightarrow Y \otimes_{\mathbb{Z}[P]} \mathbb{Z}[Q]$ and $\mathrm{Spec}(\mathscr{O}_Y[Q^{\mathrm{gp}}]) \longrightarrow \mathrm{Spec}(\mathscr{O}_Y[P^{\mathrm{gp}}])$ are respectively flat, smooth, étale, quasi-finite.

Factorization of morphisms of Log Schemes.

Let $(f, \varphi) : (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism between log schemes locally of finite type (as schemes) with fine (resp. fs) log structures. Then locally for the étale topology there exists a factorization of the morphism:

$$(X, \mathscr{M}) \xrightarrow{(i,\iota)} (Z, \mathscr{L}) \xrightarrow{(g,\psi)} (Y, \mathscr{N})$$

with $(Z, \mathscr{L})$ a fine (resp. fs) log scheme, (i, ι) a closed immersion and (g, ψ) a log smooth morphism.

The proof is that of [HK]. We can assume that $X = \mathrm{Spec}A$ and $Y = \mathrm{Spec}B$ are affine schemes, and also that we have a global chart for the morphism:

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ \uparrow & & \uparrow \\ Q & \xrightarrow[\varphi]{} & P \end{array} \quad \text{corresponding to} \quad \begin{array}{ccc} (Y, \mathscr{N}) & \xleftarrow{\ f\ } & (X, \mathscr{M}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z}[Q] & \xleftarrow[\mathrm{Spec}\mathbb{Z}[\varphi]]{} & \mathrm{Spec}\mathbb{Z}[P] \end{array}$$

and we define $B_1 := B \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$, $L_1 := P$ so that we have the natural morphism $L_1 \equiv P \to \mathbb{Z}[P] \to B \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P] \equiv B_1$. In terms of log schemes we have $X_1 := Y \times_{\mathrm{Spec}\mathbb{Z}[Q]} \mathrm{Spec}\mathbb{Z}[P]$ with the log structure $\mathscr{L}_1$ induced by the written chart; so we have:

$$\begin{array}{ccccc} B & \longrightarrow & B_1 & \longrightarrow & A \\ \uparrow & & \uparrow & & \uparrow \\ Q & \longrightarrow & L_1 & \xrightarrow[=]{} & P \end{array} \quad \text{vs} \quad \begin{array}{ccccc} (Y, \mathscr{N}) & \longleftarrow & (X_1, \mathscr{L}_1) & \longleftarrow & (X, \mathscr{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z}[Q] & \longleftarrow & \mathrm{Spec}\mathbb{Z}[P] & \xleftarrow[=]{} & \mathrm{Spec}\mathbb{Z}[P] \end{array}$$

Now we choose a surjective map $\mathbb{N}^r \overset{v}{\twoheadrightarrow} P$ and we factorize the map $\varphi : Q \to P$ by $Q \to Q \oplus \mathbb{N}^r \to P$ using the maps $q \mapsto (q, 0)$ and $(q, n) \mapsto \varphi(q) + v(n)$; we can then define $B_2 := B \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q \oplus \mathbb{N}^r]$ and $L_2 := Q \oplus \mathbb{N}^r$ with the obvious map, corresponding to the log scheme $X_2 := Y \times_{\mathrm{Spec}\mathbb{Z}[Q]} \mathrm{Spec}\mathbb{Z}[Q \oplus \mathbb{N}^r]$ with log structure $\mathscr{L}_2$ induced by chart given by $Q \oplus \mathbb{N}^r$; now we have a diagram

$$\begin{array}{ccccccc} B & \longrightarrow & B_2 & \twoheadrightarrow & B_1 & \longrightarrow & A \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ Q & \longrightarrow & L_2 & \twoheadrightarrow & L_1 & \xrightarrow[=]{} & P \end{array}$$

corresponding to

$$\begin{array}{ccccccc} (Y, \mathscr{N}) & \longleftarrow & (X_2, \mathscr{L}_2) & \longleftarrow & (X_1, \mathscr{L}_1) & \longleftarrow & (X, \mathscr{M}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\mathbb{Z}[Q] & \longleftarrow & \mathrm{Spec}\mathbb{Z}[Q \oplus \mathbb{N}^r] & \longleftarrow & \mathrm{Spec}\mathbb{Z}[P] & \xleftarrow[=]{} & \mathrm{Spec}\mathbb{Z}[P] \end{array}$$

where the (last) morphism $(Y, \mathscr{N}) \leftarrow (X_2, \mathscr{L}_2)$ is log smooth by Kato's criterion.

Finally we can choose a surjective morphism of algebras $C_1 := B_1[T_1, \ldots, T_n] \twoheadrightarrow A$. Then $C_2 := B_2[T_1, \ldots, T_n]$, so that by the natural morphisms we can deduce the diagram:

$$\begin{array}{ccccccc} B & \longrightarrow & C_2 & \twoheadrightarrow & C_1 & \twoheadrightarrow & A \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ Q & \longrightarrow & L_2 & \twoheadrightarrow & L_1 & \xrightarrow[=]{} & P\ . \end{array}$$

For the log schemes Z_i, obtained by base change of X_i by C_i, with the induced log structures again denoted $\mathscr{L}_i$, this gives the factorization

$$\begin{array}{ccccccc}
(Y,\mathscr{N}) & \longleftarrow & (Z_2,\mathscr{L}_2) & \longleftarrow & (Z_1,\mathscr{L}_1) & \longleftarrow & (X,\mathscr{M}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}\mathbb{Z}[Q] & \longleftarrow & \mathrm{Spec}\mathbb{Z}[Q\oplus\mathbb{N}^r] & \longleftarrow & \mathrm{Spec}\mathbb{Z}[P] & \underset{=}{\longleftarrow} & \mathrm{Spec}\mathbb{Z}[P]
\end{array}$$

which has the required properties: the composite $(X,\mathscr{M})\to(Z_2,\mathscr{L}_2)$ is a closed immersion and $(Z_2,\mathscr{L}_2)\to(Y,\mathscr{N})$ is a log smooth morphism (composition of log smooth morphisms).

Note that starting with saturated monoids, the proof gives at every step charts with saturated monoids, so that we have the factorization in the category of fs log schemes.

Factorization of closed immersions. Let $\varphi : N \to M$ be a morphism of integral monoids and define $N' := (\varphi^{\mathrm{gp}})^{\leftarrow}(M) \subseteq N^{\mathrm{gp}}$ with the canonical maps $N \hookrightarrow N' \to M$ factorizing φ. Then N' is always integral and it is saturated if M is.

In fact N' is a submonoid of a group, so it is integral without any hypothesis; moreover suppose that $\eta \in (N')^{\mathrm{gp}} \equiv N^{\mathrm{gp}}$ with $\eta^e \in N'$ for a positive integer e; then $\varphi^{\mathrm{gp}}(\eta^e) = \varphi^{\mathrm{gp}}(\eta)^e \in M$ and since M is saturated we have $\varphi^{\mathrm{gp}}(\eta) \in M$, hence $\eta \in N'$.

Let $\varphi : N \to M$ be a morphism of integral monoids, and suppose that it is surjective and exact (i.e. $(\varphi^{\mathrm{gp}})^{\leftarrow}(M) = N$). Then the associated morphism of log schemes is an exact closed immersion.

Let $(f,\varphi) : (X,\mathscr{M}) \to (Y,\mathscr{N})$ be a closed immersion of log schemes locally of finite type (as schemes) with fine (resp. fs) log structures. Then locally for the étale topology there exists a factorization of the morphism:

$$(X,\mathscr{M}) \xrightarrow{(i,\iota)} (Z,\mathscr{L}) \xrightarrow{(g,\psi)} (Y,\mathscr{N})$$

with $(Z,\mathscr{L})$ a fine (resp. fs) log scheme, (i,ι) an exact closed immersion and (g,ψ) a log étale morphism.

The proof is that of [KK.89]. We can assume that $X = \mathrm{Spec}A$ and $Y = \mathrm{Spec}B$ are affine schemes, and also that we have a global chart for the morphism:

$$\begin{array}{ccc}
B & \overset{f}{\twoheadrightarrow} & A \\
\uparrow & & \uparrow \\
Q & \underset{\varphi}{\twoheadrightarrow} & P
\end{array}
\quad \text{corresponding to} \quad
\begin{array}{ccc}
(Y,\mathscr{N}) & \xleftarrow{\quad f \quad} & (X,\mathscr{M}) \\
\downarrow & & \downarrow \\
\mathrm{Spec}\mathbb{Z}[Q] & \underset{\mathrm{Spec}\mathbb{Z}[\varphi]}{\longleftarrow} & \mathrm{Spec}\mathbb{Z}[P]\,.
\end{array}$$

We define $Q' := (\varphi^{\mathrm{gp}})^{\leftarrow}(P)$, and the scheme $Z := Y \times_{\mathrm{Spec}\mathbb{Z}[Q]} \mathrm{Spec}\mathbb{Z}[Q']$ with the log structure $\mathscr{L}$ induced from the canonical structure of $\mathrm{Spec}\mathbb{Z}[Q']$, so that we have the factorization

$$\begin{array}{ccccc}
B & \to & B \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[Q'] & \twoheadrightarrow & A \\
\uparrow & & \uparrow & & \uparrow \\
Q & \longrightarrow & Q' & \longrightarrow\!\!\!\!\!\rightarrow & P
\end{array}
\quad \text{vs} \quad
\begin{array}{ccccc}
(Y,\mathscr{N}) & \longleftarrow & (Z,\mathscr{L}) & \longleftarrow & (X,\mathscr{M}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}\mathbb{Z}[Q] & \leftarrow & \mathrm{Spec}\mathbb{Z}[Q'] & \leftarrow & \mathrm{Spec}\mathbb{Z}[P]\,.
\end{array}$$

Then we see that the last morphism is étale by Kato's criterion, while the first is an exact closed immersion, because the morphism $Q' \to P$ is surjective and exact.

In general the fiber product in the category of log schemes is different from that of $\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$ and $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$ (because a fiber product of fine or fs log schemes is not longer integral or saturated, if we don't have some hypothesis on the morphisms). Thus, in order to understand the diagonal embedding of log schemes, we have to study the given factorizations in these categories. The situation is very simple: essentially the local factorization in $\mathcal{L}og\mathcal{S}ch$ works also in $\mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$ and $\mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$. Let $(f,\varphi) : (X,\mathscr{M}) \to (Y,\mathscr{N})$ be a closed immersion of log schemes locally of finite type (as schemes):

Suppose $(X,\mathscr{M}) \in \mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$; then we can factorize (f,φ) by $(X,\mathscr{M}) \to (Y^{\mathrm{int}},\mathscr{N}^{\mathrm{int}}) \to (Y,\mathscr{N})$ where the first morphism is again a closed immersion. The construction of the local factorization gives $Y^{\mathrm{int}\prime} = Y'$, because for a local chart $P \overset{\varphi}{\leftarrow} Q$ we have $P = P^{\mathrm{int}} \overset{\varphi^{\mathrm{int}}}{\leftarrow} Q^{\mathrm{int}}$ and then $Q^{\mathrm{int}\prime} = \left((\varphi^{\mathrm{int}})^{\mathrm{gp}}\right)^{\leftarrow} P = (\varphi^{\mathrm{gp}})^{\leftarrow} P = Q'$. Then we have

$$(X,\mathscr{M}) \xrightarrow{(i,\iota)} (Y',\mathscr{N}') \xrightarrow{(g,\psi)} (Y^{\mathrm{int}},\mathscr{N}^{\mathrm{int}})$$

where the first morphism an exact closed immersion, and the second a log étale morphism (this last point is easily seen also by considering the factorization of the étale map $(Y',\mathscr{N}') \longrightarrow (Y,\mathscr{N})$ as $(Y',\mathscr{N}') \to (Y^{\mathrm{int}},\mathscr{N}^{\mathrm{int}}) \to (Y,\mathscr{N})$, being $(Y',\mathscr{N}')$ a fine log scheme where the second map is a closed immersion).

Suppose now $(X,\mathscr{M}) \in \mathcal{L}og\mathcal{S}ch^{\mathrm{fs}}$ and $(Y,\mathscr{N}) \in \mathcal{L}og\mathcal{S}ch^{\mathrm{f}}$; then we can factorize (f,φ) by $(X,\mathscr{M}) \to (Y^{\mathrm{sat}},\mathscr{N}^{\mathrm{sat}}) \to (Y,\mathscr{N})$ where the first morphism is again a closed immersion. The construction of the local factorization gives $Y^{\mathrm{sat}\prime} = Y'$, so that we have

$$(X,\mathscr{M}) \xrightarrow{(i,\iota)} (Y',\mathscr{N}') \xrightarrow{(g,\psi)} (Y^{\mathrm{sat}},\mathscr{N}^{\mathrm{sat}})$$

where the first morphism an exact closed immersion, and the second a log étale morphism (this last point is easily seen also by considering the factorization of the étale map $(Y',\mathscr{N}') \longrightarrow (Y,\mathscr{N})$ as $(Y',\mathscr{N}') \to (Y^{\mathrm{sat}},\mathscr{N}^{\mathrm{sat}}) \to (Y,\mathscr{N})$, being $(Y',\mathscr{N}')$ a fs log scheme where the second map is a log étale morphism).

Let $(f,\varphi) : (X,\mathscr{M}) \to (Y,\mathscr{N})$ be a morphism between log schemes locally of finite type (as schemes) with fine (resp. fs) log structures. Then locally for the étale topology there exists a factorization of the morphism

$$(X,\mathscr{M}) \xrightarrow{(i,\iota)} (Z,\mathscr{L}) \xrightarrow{(g,\psi)} (Y,\mathscr{N})$$

with $(Z,\mathscr{L})$ a fine (resp. fs) log scheme, (i,ι) an exact closed immersion and (g,ψ) a log smooth morphism.

Kato-Nakayama Spaces.

We present here the basic definition of the Kato-Nakayama spaces associated to a log analytic space. Using this spaces they define a Riemann-Hilbert correspondence for a special class of log connections.

The Log space. We start by the following log structure on the point $\mathrm{Spec}\mathbb{C}$: let $\mathbb{S}^1 \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{C}$ be the morphism $(\zeta, r) \mapsto r\zeta$ where $\mathbb{S}^1$ and $\mathbb{R}_{\geq 0}$ have the multiplicative structure (remark that $\mathbb{R}_{\geq 0}$ is not integral: $\mathbb{R}_{\geq 0}^{\mathrm{gp}} = 0$); clearly this is a log structure.

Call T the log point $\mathrm{Spec}(\mathbb{C}, \mathbb{S}^1 \times \mathbb{R}_{\geq 0})$ and for a coherent log scheme X over $\mathbb{C}$, whose scheme is of finite type, define the set $X^{\log} := \mathrm{Hom}_{\mathbb{C}}(T, X)$. An explicit expression for this set is the following:

$$X^{\log} = \left\{ (x,h) \ : \ x \in X_{\mathrm{an}},\ h : \mathscr{M}^{\mathrm{gp}}_{X,x} \longrightarrow \mathbb{S}^1,\ h(f) = \tfrac{fx}{|fx|}\ \forall f \in \mathscr{O}^{\times}_{X,x} \right\}$$

so that if we have locally a chart modeled on P, we can see that $X^{\log}$ is locally a closed subset of the topological product $X_{\mathrm{an}} \times \mathrm{Hom}(P^{\mathrm{gr}}, \mathbb{S}^1)$.

This local definition for the topology globalizes to give a topology on $X^{\log}$.

We have a canonical morphism $\tau : X^{\log} \longrightarrow X_{\mathrm{an}}$; suppose X fine, then τ is continuous and proper as a map of topological spaces. If moreover $\mathscr{C}^{\mathrm{gp}}_X$ is torsion free, then the fibers of τ are connected and the canonical $\mathrm{id} \to \tau_* \tau^{-1}$ is an isomorphism.

For example for $X = \mathrm{Spec}\mathbb{C}[P]$ with P a fine monoid, then $X^{\log} = \mathrm{Hom}_{\mathcal{M}on}(P, \mathbb{R}_{\geq 0} \times \mathbb{S}^1) = \mathrm{Hom}_{\mathcal{M}on}(P, \mathbb{R}_{\geq 0}) \times \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{S}^1)$; for $x \in X^{\log}$, $\tau x = \pi \circ x$ as morphism $P \longrightarrow \mathbb{C}$. The topology is the natural one, making $X^{\log}$ into a locally compact space.

From the definition we have trivially that $\left(\prod_i X_i\right)^{\log} \simeq \prod_i X_i^{\log}$. Moreover if $f : X \longrightarrow Y$ is a strict morphism of coherent log schemes, then the natural diagram

$$\begin{array}{ccc} X^{\log} & \xrightarrow{f^{\log}} & Y^{\log} \\ \tau \downarrow & & \downarrow \tau \\ X_{\mathrm{an}} & \underset{f_{\mathrm{an}}}{\longrightarrow} & Y_{\mathrm{an}} \end{array}$$

is a cartesian square of topological spaces. So the topologies on the spaces $X^{\log}$ can be characterized as the topologies which are the canonical one on $\mathrm{Spec}\mathbb{C}[P]$ and make cartesian the above written diagrams.

If the log scheme X is fine, then the fiber of τ in a point $x \in X$ is given by a product of a finite group with a product of $r(x)$ copies of $\mathbb{S}^1$; if X is fs, then the fiber is given only by a product of $r(x)$ copies of $\mathbb{S}^1$ where $r(x)$ is the rank of the characteristic sheaf at x.

Moreover for X fine log scheme, the canonical adjonction morphism $X^{\mathrm{sat}} \longrightarrow X$ induce an homeomorphism of topological spaces $\left(X^{\mathrm{sat}}\right)^{\log} \xrightarrow{\sim} X^{\log}$.

Ogus' lemma. For any $y \in X^{\log}$ there exists a basis of neighborhood of y such that the intersection with $(X_{\mathrm{triv}})^{\log} \simeq X_{\mathrm{triv}}$ is contractile.

In the diagram

$$\begin{array}{ccc} (X_{\mathrm{triv}})^{\log} & \xrightarrow{j^{\log}} & X^{\log} \\ \| & & \downarrow \tau \\ (X_{\mathrm{triv}})_{\mathrm{an}} & \underset{j}{\longrightarrow} & X_{\mathrm{an}} \end{array}$$

we have that $\mathbb{Z} \simeq Rj_*^{\log}\mathbb{Z}$, $Rj_*\mathbb{Z} \simeq R\tau_*\mathbb{Z}$. moreover for any locally constant sheaf of $\mathbb{C}$ vector spaces on $X_{\mathrm{triv,an}}$ we have that $Rj_*^{\log}V \simeq j_*^{\log}V$ is locally constant on $X^{\log}$.

The ringed structures. We define the sheaf of logarithmic section of $\mathscr{M}_X$ as $\mathscr{L}_X = \mathscr{C}(X^{\log}, i\mathbb{R}) \times_{\mathscr{C}(X^{\log},\mathbb{S}^1)} \tau^{-1}(\mathscr{M}_X^{\mathrm{gp}})$, i.e. by the cartesian square in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow 2\pi i\mathbb{X}_{X^{\log}} \longrightarrow & \mathscr{C}(X^{\log}, i\mathbb{R}) & \xrightarrow{exp} & \mathscr{C}(X^{\log}, \mathbb{S}^1) & \longrightarrow 0 \\ & \uparrow & & \uparrow c & \\ & \mathscr{L}_X & \longrightarrow & \tau^{-1}(\mathscr{M}_X^{\mathrm{gp}}) & \end{array}$$

where $\mathscr{C}(X^{\log}, -)$ means "continuous function from $X^{\log}$, and the morphism $c : \tau^{-1}(\mathscr{M}_X^{\mathrm{gp}}) \longrightarrow \mathscr{C}(X^{\log}, \mathbb{S}^1)$ corresponds by adjonction to $\mathscr{M}_X^{\mathrm{gp}} \longrightarrow \tau_*\mathscr{C}(X^{\log}, \mathbb{S}^1)$ given by sending a to the function $(x, h) \mapsto h(a_x)$.

From the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow 2\pi i\mathbb{X}_{X^{\log}} \longrightarrow & \mathscr{C}(X^{\log}, i\mathbb{R}) & \xrightarrow{exp} & \mathscr{C}(X^{\log}, \mathbb{S}^1) & \longrightarrow 0 \\ & & & \uparrow c & \\ & a \Big\uparrow & & \tau^{-1}(\mathscr{M}_X^{\mathrm{gp}}) & \\ & & & \uparrow & \\ & \mathscr{L}_X & \underset{exp}{\longrightarrow} & \tau^{-1}(\mathscr{O}_X^\times) & \end{array}$$

where $a : \tau^{-1}(\mathscr{O}_X) \longrightarrow \mathscr{C}(X^{\log}, i\mathbb{R})$ corresponds by adjonction to $\mathscr{O}_X \longrightarrow \tau_*\mathscr{C}(X^{\log}, i\mathbb{R})$ given by sending f to the function $f - \Re(f)$, and $b : \tau^{-1}(\mathscr{O}_X^\times) \longrightarrow \mathscr{C}(X^{\log}, \mathbb{S}^1)$ corresponds by adjonction to $\mathscr{O}_X^\times \longrightarrow \tau_*\mathscr{C}(X^{\log}, \mathbb{S}^1)$ given by sending f to the function $f/|f|$, we have the canonical morphism $h : \tau^{-1}(\mathscr{O}_X) \longrightarrow \mathscr{L}_X$ sneding f to $(a(f), exp(f))$ and the exact sequence

$$0 \longrightarrow \tau^{-1}(\mathscr{O}_X) \xrightarrow{h} \mathscr{L}_X \xrightarrow{exp} \tau^{-1}(\mathscr{C}_X^{\mathrm{gp}}) \longrightarrow 0 .$$

If X is fs, then we have $\tau^{-1}(\mathscr{C}_X^{\mathrm{gp}})_y \simeq \mathscr{C}_{X,x}^{\mathrm{gp}} \simeq \mathbb{Z}^r$ and $\mathscr{L}_y \simeq \tau^{-1}(\mathscr{O}_X)_y \oplus \mathbb{Z}^r \simeq \mathscr{O}_{X,x} \oplus \mathbb{Z}^r$.

Now we deefine the sheaf of ring $\mathscr{O}_X^{\log}$ on $X^{\log}$ by the following universal property: for any $\tau^{-1}(\mathscr{O}_X)$-Algebra $\mathscr{A}$ there is a canonical bijection $\mathrm{Hom}_{\tau^{-1}(\mathscr{O}_X)}(\mathscr{L}, \mathscr{A}) \simeq \mathrm{Hom}_{\tau^{-1}(\mathscr{O}_X)}(\mathscr{O}_X^{\log}, \mathscr{A})$, so that we have $\mathscr{O}_X^{\log} \simeq (\tau^{-1}(\mathscr{O}_X) \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}(\mathscr{L}))/\mathscr{J}$ where $\mathscr{J}$ is the ideal generated by the sections of the form $1 \otimes h(f) - f \otimes 1$ for f section of $\tau^{-1}(\mathscr{O}_X)$.

Remark that the morphism τ is now a morphism of ringed spaces; moreover $X^{\log}$ is not a locally ringed space, in fact we have that, if t_i are section of $\mathscr{L}_y$ such that the $exp(t_i)$ give a $\mathbb{Z}$-bases of $\mathscr{C}_{X,x}^{\mathrm{gp}}$, then $\mathscr{O}_{X,y}^{\log} \simeq \mathscr{O}_{X,x}[T_1, \ldots, T_n]$ by $t_i \mapsto T_i$.

We have a natural filtration on $\mathscr{O}_X^{\log}$ given by

$$\mathrm{fil}_r(\mathscr{O}_X^{\log}) = \mathrm{im}(\tau^{-1}(\mathscr{O}_X) \otimes_{\mathbb{Z}} \mathrm{Sym}_{\mathbb{Z}}^{\leq r}(\mathscr{L}) \longrightarrow \mathscr{O}_X^{\log})$$

so that we have $\mathrm{fil}_0(\mathscr{O}_X^{\log}) = \tau^{-1}(\mathscr{O}_X)$ and

$\tau^{-1}(\mathscr{C}_X^{\mathrm{gp}}) \simeq \mathscr{L}/\tau^{-1}(\mathscr{O}_X) \subseteq \mathrm{fil}_1(\mathscr{O}_X^{\log})/\tau^{-1}(\mathscr{O}_X) = \mathrm{gr}_1(\mathscr{O}_X^{\log})$;

moreover we have $\tau^{-1}(\mathscr{O}_X) \otimes_{\mathbb{Z}} \tau^{-1}\mathrm{Sym}_{\mathbb{Z}}^r(\mathscr{C}_X^{\mathrm{gp}}) \simeq \mathrm{gr}_r(\mathscr{O}_X^{\log})$.

The Riemann-Hilbert correspondence. We define the de Rham complex of $X^{\log}$ by

$$\Omega_X^{\bullet,\log} = \tau^*\Omega_X^\bullet = \mathscr{O}_X^{\log} \otimes_{\tau^{-1}\mathscr{O}_X} \tau^{-1}\Omega_X^\bullet ,$$

with differentials
$d : \mathscr{O}_X^{\log} \longrightarrow \Omega_X^{1,\log}$ defined by $d(\tau^{-1}f \otimes 1) = 1 \otimes \tau^{-1}d\tau^{-1}f$
and
$d : \Omega_X^{q,\log} \longrightarrow \Omega_X^{q+1,\log}$ defined by $d(x \otimes \eta) = dx \wedge \eta + x \otimes d\eta$.

Suppose now that X is a log analytic space over $\mathbb{C}$ satisfying the following property: there exists a covering with open sets U_λ, fs monoids P_λ and Σ_λ ideals of P_λ such that every U_λ is isomorphic to an open set of $\mathrm{Spec}(\mathbb{C}[P_\lambda]/(\Sigma_\lambda))_{\mathrm{an}}$, with the log structure induced by $P_\lambda \to \mathscr{O}_{U_\lambda}$ (in other words, X is idally log smooth over $\mathbb{C}$ endowed with the trivial log structure).

We have the log Poincaré lemma: the canonical morphism $\mathbb{C} \longrightarrow \Omega_X^{\bullet,\log}$ is a quasi-isomorphism.

Define $\mathscr{D}(X)$ the category of vector boundles over X endowed with an integrable log connection, and $\mathscr{L}(X^{\log})$ the category of local systems of finite dimensional $\mathbb{C}$-vector spaces over $X^{\log}$. Let $\mathscr{D}_{\mathrm{nilp}}(X)$ be the subcategory of $\mathscr{D}(X)$ of objects endowed with a finite increasing filtration such that the graded objects have log connections without poles (i.e. the log connection factorizes through the classical differentials); and let $\mathscr{L}_{\mathrm{unip}}(X^{\log})$ be the subcategory of $\mathscr{L}(X^{\log})$ of objects endowed with a finite increasing filtration such that the graded objects are inverse image by τ of local systems of finite dimensional $\mathbb{C}$-vector spaces on X.

Then there exists an equivalence of categories

$$\Phi : \mathscr{D}_{\mathrm{nilp}}(X) \longrightarrow \mathscr{L}_{\mathrm{unip}}(X^{\log})$$

such that

(*i*) $\Phi(V) = \tau^{-1}\ker(\nabla : V \to \Omega_X^1 \otimes_{\mathscr{O}_X} V)$ if $V \in \mathscr{D}_{\mathrm{nilp}}(X)$ without poles;

(*ii*) we have a canonical isomorphism $\Omega_X^1 \otimes_{\mathscr{O}_X} V \simeq R\tau_*\Phi(V)$ in $D(X, \mathbb{C})$ for any $V \in \mathscr{D}_{\mathrm{nilp}}(X)$;
in particular we have $\mathbb{H}^q(X, \Omega_X^1 \otimes_{\mathscr{O}_X} V) \simeq H^q(X^{\log}, \Phi(V))$ and $\mathbb{H}^q(X, \Omega_X^1) \simeq H^q(X^{\log}, \mathbb{C})$.

In general for $V \in \mathscr{D}_{\mathrm{nilp}}(X)$, the functor is defined by $\Phi(V) = \ker(\nabla^{\log})$ where $\nabla^{\log} : V^{\log} \longrightarrow \Omega_X^{1,\log} \otimes_{\mathscr{O}_X^{\log}} V^{\log}$ is induced by $\nabla : V \longrightarrow \Omega_X^1 \otimes_{\mathscr{O}_X} V$ (and $V^{\log} = \mathscr{O}_X^{\log} \otimes_{\tau^{-1}\mathscr{O}_X} \tau^{-1}V$).

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