Geometria Superiore Reference Cards

Monoids

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The category of Monoids.

A monoid $(M,\cdot,1)$ (commutative with unit) is a set M with a composition law $\cdot : M\times M\to M$ and an element $1\in M$ such that: the composition is associative: $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ (for all $a,b,c\in M$), commutative: $a\cdot b = b\cdot a$ (for all $a,b\in M$), and 1 is a neutral element for the composition: $1\cdot a = a$ (for all $a\in M$).

A morphism $(M,\cdot,1_M)\xrightarrow{\varphi}(N,\cdot,1_N)$ of monoids is a map $M\xrightarrow{\varphi}N$ of sets commuting with $\cdot$ and 1: $\varphi(ab)=\varphi(a)\varphi(b)$ for all $a,b\in M$ and $\varphi(1_M)=1_N$. Thus we have defined the category $\mathcal{M}on$ of monoids.

$\{1\}$ is a monoid, initial and final object of the category $\mathcal{M}on$.

$(\mathbb{N},+,0)$ and $(\mathbb{Z},\cdot,1)$ are monoids.

Free monoids. For any set S there is a free monoid with generators the elements of S: the *free monoid on* S: $\mathbb{N}^S$. The natural morphism $S\to\mathbb{N}^S$ given by $s\mapsto e_s$ (element of N^S sending s to 1 and the other elements to zero) realizes a (bifunctorial) bijection $\mathrm{Hom}_{\mathcal{S}et}(S,M)\cong\mathrm{Hom}_{\mathcal{M}on}(\mathbb{N}^S,M)$ for any monoid M. This construction gives a left adjoint to the forgetful functor from $\mathcal{M}on$ to $\mathcal{S}et$.

Associated Group. There exists a left adjoint to the natural forgetful functor from $\mathcal{A}b$ to $\mathcal{M}on$, given by $M\mapsto M^{\mathrm{gp}}$ where: $M^{\mathrm{gp}} := M\times M\big/\sim$ and $\sim$ is the equivalence relation on $M\times M$ defined by $(x,y)\sim(m,n)$ if and only if there exists $a\in M$ such that $a\cdot x\cdot n = a\cdot y\cdot m$. We have the natural morphism $M\longrightarrow M^{\mathrm{gp}}$, $x\mapsto(x,1)$.

Projective limits. In the category $\mathcal{M}on$ all projective limits exist, and the projective limit functor commutes with the forget functor from $\mathcal{M}on$ to $\mathcal{S}et$. In fact this functor is a right adjoint, and so commutes with all projective limits, and we have a canonical monoid structure (component by component) on the (set theoretical) projective limits of a system of monoids.

In particular we have (arbitrary) products and fiber products of monoids.

Warning: Kernel. Given a morphism $M\xrightarrow{f}N$ of monoids, we have also the kernel $\ker f$ which is a particular projective limit; being the kernel of the double map f and the constant 1 map, it is realized as for abelian groups by $\ker f = f^{-1}(1)$. But the notion of monomorphism (i.e. left eraseable map) in $\mathcal{M}on$ coindices with the notion of monomorphism in $\mathcal{S}et$ (i.e. injectivity): so a monomorphism has a trivial kernel, while a map with a trivial kernel is not necessarily a monomorphism (consider for example $\mathbb{N}:=(\mathbb{N},+,0)$ and the map $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ defined by the sum $(x,y)\mapsto x+y$).

Inductive limits. In the category $\mathcal{M}on$ all inductive limits exist, but in general it is difficult to calculate them.

Direct sums. Given a family $\{M_i\}_{i\in\mathscr{I}}$ the direct sum of the family is the submonoid of the product $\bigoplus_{i\in\mathscr{I}}M_i\subseteq\prod_{i\in\mathscr{I}}M_i$ of $\mathscr{I}$-ple (s_i) where $s_i=1_{M_i}$ for almost all components.

Cokernel of double arrows. We consider a double arrow $u,v: P\rightrightarrows Q$. The cokernel of u and v is the quotient $\mathrm{coker}(u,v) := Q/\mathcal{R}$ were $\mathcal{R}$ is the minimal equivalent relation stable by product and containing $u(P)\times_P v(P) := \Big\{(u(p),v(p)) \;:\; p\in P\Big\}$. The canonical morphism $Q\to Q/\mathcal{R}$ has the attended universal property.

Push out (Sommes amalgamées). Given a diagram $i: P\to Q_i$ $(i=1,2)$ we can define

$$Q_1\oplus_P Q_2 := \mathrm{coker}\left(P \overset{\binom{1}{u_2}}{\underset{\binom{u_1}{1}}{\rightrightarrows}} Q_1\oplus Q_2\right)$$

and it is a standard fact that the natural arrows $j_i: Q_i\to Q_1\oplus_P Q_2$ define a cocartesian square:

$$\begin{array}{ccc} P & \xrightarrow{u_1} & Q_1 \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ Q_2 & \xrightarrow[j_2]{} & Q_1\oplus_P Q_2\,. \end{array}$$

More explicitly we have $Q_1\oplus_P Q_2 = (Q_1\oplus Q_2)/\mathcal{R}$ with $\mathcal{R}$ the smallest equivalence relation stable under product (in $Q_1\oplus_P Q_2$) and making $(u_1(p),1)\sim_{\mathcal{R}}(1,u_2(p))$ for all $p\in P$.

In general it is not easy to understand the relation involved in the description of $Q_1\oplus_P Q_2$, but in the case in which one of the Q_i is a group we have a nice description: $Q_1\oplus_P Q_2 = (Q_1\oplus Q_2)/\sim$ where $(x_1,x_2)\sim(y_1,y_2)$ if and only if:

$$\text{there exist } c,d\in P \text{ such that } \begin{cases} x_1u_1(d)=y_1u_1(c) \\ x_2u_2(c)=y_2u_2(d) \end{cases}$$

i.e. if and only if there exist $c,d\in P$ such that $(x_1,x_2)(u_1(d),u_2(c)) = (y_1,y_2)(u_1(c),u_2(d))$ in $Q_1\oplus Q_2$. We see in fact that $\sim$ is an equivalent relation (without hypothesis) and, by the given hypothesis, it is just the relation described in the previous number.

Cokernel. Given a morphism $P\xrightarrow{u}Q$ of monoids, we can consider the diagram $P\xrightarrow{u}Q\xleftarrow{1}\{1\}$ and put $\mathrm{coker}(u)=\mathrm{coker}(u,1) := \mathrm{coker}(\binom{u}{1},\binom{1}{1})$ that is: $Q\oplus\{1\}/\sim\, = Q/\sim$ where

$x\sim y$ if and only if there exist $c,d\in P$ such that $xu(c)=yu(d)$

(note that $\{1\}$ is an abelian group!).

It is useful to observe that if we have a cocartesian diagram

$$\begin{array}{ccc} P & \xrightarrow{u_1} & Q_1 \\ {\scriptstyle u_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ Q_2 & \xrightarrow[j_2]{} & Q \end{array}$$

the induced morphism between the cokernel: $\mathrm{coker}(u_1)\to\mathrm{coker}(j_2)$ is always an isomorphism.

Warning: Cokernel. The notion of epimorphism (i.e. right erasable map) in $\mathcal{M}on$ is weaker than the notion of epimorphism in $\mathcal{S}et$ (i.e. surjectivity): moreover an epimorphism has a trivial cokernel, while a map with a trivial cokernel is not necessarily an epimorphism.

Consider for example $\mathbb{N}:=(\mathbb{N},+,0)$ and the map $\mathbb{N}\times\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ defined by $(x,y)\mapsto(x,x+y)$; the map is not surjective, but its cokernel is trivial. Composing with $\mathbb{N}\times\mathbb{N}\xrightarrow{s}(\mathbb{N},\cdot,1)$ defined by $s(e_1)=1$ and $s(e_2)=0$, i.e. sending (x,y) to 1 if $y=0$ and 0 if $y\neq0$, we find the same map as composing with the map r defined by $r(e_1)=0$ and $r(e_2)=0$, i.e. sending all $(x,y)\neq(0,0)$ to 0.

We can see however that in the category of *integral* monoids the notion of epimorphism coincides with that of "map with trivial cokernel" while the notion of surjectivity is stronger.

Example: the canonical morphism $P\to P^{\mathrm{gp}}$ is epi, because $P^{\mathrm{gp}}/P=0$ (remark however that the morphism is epi also in the category $\mathcal{M}on$).

Quotients. In particular, if we have a submonoid Q of a monoid P, we can define the quotient $Q/P := \mathrm{coker}(Q\hookrightarrow P) = Q/\sim$ where $x\sim y$ if and only if there exist $c,d\in P$ such that $xc=yd$.

Presentation of monoids by generators and relations. We note that there is a bijection

$$\mathrm{Hom}_{\mathcal{M}on}\left(\mathbb{N}^{(I)}, M\right) \xrightarrow{\cong} M^{(I)} \qquad \varphi \longmapsto \left(\varphi(e_i)\right)_{i\in I}$$

where $(e_i)_{i\in I}$ is the canonical base of $\mathbb{N}^{(I)}$: $e_i(j) = \delta_{i,j}$.

A family $(x_i)_{i\in I}$ of elements of a monoid M generates M if the morphism $\mathbb{N}^{(I)} \to M$ sending $e_i \mapsto x_i$ is surjective.

This means that every element x of M can be written as a product $x = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_\ell}^{n_\ell}$ with $i_j \in I$ and $n_j \in \mathbb{N}$.

Given two sets I and J, and elements $r_j \in \mathbb{N}^{(I)} \times \mathbb{N}^{(I)}$ for every $j \in J$, we can define the monoid M with generators $(x_i)_{i\in I}$ and relations $(r_j)_{j\in J}$ as the cokernel of the double arrow:

$$\begin{array}{ccccc} \mathbb{N}^{(J)} & \xrightarrow{r} & \mathbb{N}^{(I)} \times \mathbb{N}^{(I)} & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & \mathbb{N}^{(I)} \\ e_j & \longmapsto & r_j & & \end{array}$$

and we will write the "relations" r_j also as "equalities" $p_1(r_j) = p_2(r_j)$.

Algebra of monoids over a ring. Let R be a commutative ring with unit. The forgetful functor from the category of R-algebra to $\mathcal{M}on$ given by $(A, +, \cdot) \mapsto (A, \cdot)$ admits a left adjoint $M \mapsto R[M]$ where $R[M]$ is the free R-module with bases e_x for $x \in M$ and multiplication defined by $e_x e_y = e_{x\cdot y}$ and R-linearity.

In particular $R[\cdot]$ commutes with inductive limits.

$R[\mathbb{N}] \cong R[T]$ by $1 \leftrightarrow T$.

$R[\mathbb{N}^\ell] \cong R[T_1, \ldots, T_\ell]$ by $e_i \leftrightarrow T_i$.

If $P = \mathbb{N}^\ell/\mathbb{N}^r$ for a presentation $\mathbb{N}^r \rightrightarrows \mathbb{N}^\ell$ sending e_j to $\sum_i n_{j,i} e_i$ and $\sum_i m_{j,i} e_i$ for $j = 1, \ldots, r$, then

$$R[P] \cong R[T_1, \ldots, T_\ell] \Big/ \left(\prod_i T_i^{n_{j,i}} - \prod_i T_i^{m_{j,i}}\right)_{j=1,\ldots,r}$$

Subsets of monoids.

Ideals and Prime Ideals. We say that a subset I of a Monoid P is an ideal if one of the following, equivalent, conditions is satisfied:

(i_0) $I \cdot P \subseteq I$

(i_1) for all $x \in I$ and $a \in P$ one has $ax \in I$

(i) for all $x, y \in P$: $x \in I$ or $y \in I$ implies $xy \in I$

We say that $J \subseteq P$ is a prime subset if $1 \notin J$ and the converse of condition (i) holds, i.e. if $xy \in J$ implies that x or y belongs to J.

So a prime ideal (i.e. an ideal and a prime subset) is a subset K of P such that $1 \notin K$ (i.e. $K \neq P$) and $xy \in K$ if and only if $x \in K$ or $y \in K$.

Height and valuations. Define the dimension $\dim(P)$ of a monoid P as the maximal r such that there exists a chain $J_0 \subsetneq J_1 \subsetneq \cdots \subsetneq J_r$ of prime ideals of P. Let J a prime ideal of P; the height is defined by $\mathrm{ht}(J) = \dim(P_J)$, where $P_J = (P \smallsetminus J)^{-1}P$, i.e. $\mathrm{ht}(J)$ is the maximal lenght r of a chain of prime ideals starting with J.

Notice that a finitely generated integral monoid P is the intersection in P^{gp} of all P_J where J varies in the ideals of P with $\mathrm{ht}(J) = 1$. Moreover we have $\dim(P) = \mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}}/P^\times)$ and $\dim(P) = \mathrm{ht}(J) + \dim(P \smallsetminus J)$.

Let P be a finitely generated saturated monoid, J a prime ideal of heght 1; then P_J and $P_J/P_J^\times$ are saturated, $P_J/P_J^\times \cong \mathbb{N}$, $P^{\mathrm{gp}}/P_J^\times \cong \mathbb{Z}$. We define the valuation at J as the projection $v_J : P^{\mathrm{gp}} \longrightarrow P^{\mathrm{gp}}/P_J^\times \cong \mathbb{Z}$.

Then we have $P_J = \{x \in P^{\mathrm{gp}} | v_J(x) \geq 0\}$.

If $f : P \to Q$ is a morphism of finitely generated saturated monoids, K a prime of Q with $\mathrm{ht}(K) = 1$ and $J = f^{\leftarrow}(K)$ with $\mathrm{ht}(J) = 1$, then the ramification index of f at K is the integer $n \geq 0$ such that $v_K \circ f^{\mathrm{gp}} = n v_J$.

Submonoids and Faces. We say that a subset R of a Monoid P is a submonoid if $1 \in R$ and one of the following, equivalent, conditions is satisfied:

(s_0) $R \cdot R \subseteq R$

(s) for all $x \in R$ and $y \in R$ one has $xy \in R$

We say that $Q \subseteq P$ is a saturated subset if the converse of condition (s) holds, i.e. if $xy \in Q$ implies that x and y belong to Q.

A face of P is a submonoid and a saturated subset, i.e. is a subset F of P such that $1 \in F$ and $xy \in F$ if and only if $x \in F$ and $y \in F$.

We see immediately that taking the complement gives a bijection between the set of ideals (resp. submonoids) and the set of saturated subsets (resp. prime subsets) of a monoid. The same is true for the set of prime ideals and the set of faces: a face is the complement of a prime ideal.

Let P be a monoid:

(i) If an ideal contains 1 it is the monoid P.

(ii) The invertible elements $P^\times$ form a submonoid and a face. It is the smallest face of the monoid.

(iii) The subset $P \setminus P^\times$ is the maximal prime ideal.

Localization. Let S be a subset of M; then there exists a commutative monoid $S^{-1}M$ endowed with a canonical morphism $M \longrightarrow S^{-1}M$ such that the image of any element of S is invertible, and universal for this property in the category of commutative monoids. We can construct $S^{-1}M$ as the quotient $(S \times M)/\sim$ where $(s, m) \sim (s', m')$ if there exists $t \in S$ such that $tsm' = ts'm$.

Remark that $S^{-1}M \cong F^{-1}M$ if F is the face of M generated by S. If J is a prime ideal of M, then we use the notation $M_J = (M \smallsetminus J)^{-1}M$.

The category of M-Set.

Let M be a monoid. An M-Set is a set X with an action of M, i.e. a morphism of monoids $M \to \mathrm{End}(X)$. The category M-Set admits arbitrary projective limits, and they commute with the forgetful functor from M-Set to Set.

The category M-Set admits arbitrary inductive limits, but the construction does not commute in general with the forgetful functor. The direct sum of M-Sets is given by the disjoint union (as for Sets) and we can make the quotient of an M-Set by a congruent relation (i.e. an equivalent relation stable under the action of M).

Tensor product. The universal problem $\mathrm{Bil}_M(X \times Y, Z) \cong \mathrm{Hom}_M(X \otimes_M Y, Z)$ for any $Z \in M$-Set define an M-set $X \otimes_M Y \cong X \times Y/\sim$ where $\sim$ is the congruence relation generated by $(mx, y) \sim (x, my)$ for $x \in X$, $y \in Y$ and $m \in M$. The bifunctor $\otimes_M$ is commutative and associative in the obvious sense; $M \otimes_M X \cong X$ for any X.

If $M \to N$ is a morphism of monoids, and X in an M-set, then $N \otimes_M X$ (resp. $\mathrm{Hom}_M(N, X)$) has a natural structure of N-Set. The functor $N \otimes_M$ - (resp. $\mathrm{Hom}_M(N, \text{-})$) is the right (resp. left) adjoint of the "restriction of monoid" from N to M.

Tensor and Symmetric monoids. The tensor monoid over the M-set X is the (non commutative) monoid $T_M(X) := \coprod_{i=0}^{\infty} T_M^i(X)$ where $T_M^0(X) := M$, $T_M^1(X) := X$, and $T_M^{i+1}(X) :=$

$T_M^i(X)\otimes_M X$. It is endowed in a natural way of a (non commutative) monoid structure, with a canonical monoid morphism $M \to T_M(X)$.

The symmetric monoid $S_M(X)$ over the M-set X is the quotient of $T_M(X)$ by the congruent relation generated by the symmetric group of order i acting on $T_M^i(X)$ for any i.

Modules and Algebras. Let $M \to A$ be a log ring, i.e. a morphism between a monoid M and the moltiplicative monoid of a ring A. Then the forgetful functor from A-Mod to M-Set admits the right adjoint sending X to $A^{(X)}$, and the forgetful functor from A-Alg to M-Mon admits the right adjoint sending N (M-monoid by $M \to N$) to $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[L]$.

We have the canonical morphisms $A\otimes_{\mathbb{Z}[M]}\mathbb{Z}[T_M(X)] \longrightarrow T_A(A^{(X)})$ and $A \otimes_{\mathbb{Z}[M]} \mathbb{Z}[S_M(X)] \longrightarrow S_A(A^{(X)})$.

Properties of monoids.

Invertible elements. An element $x \in M$ is invertible if there exists $y \in M$ with $xy = 1 = yx$. The set of invertible elements of a monoid M is a submonoid (in fact a subgroup and a face, see below) and will be indicated by $M^\times$. The correspondence $M \to M^\times$ gives a functor $\mathcal{M}on \to \mathcal{A}b$.

A monoid is **sharp** if $M^\times$ is trivial, i.e. the only invertible element of M is the identity. For any monoid M, we put $\overline{M} = M/M^\times$, which is sharp.

We have the obvious relations: $\{1\} \subseteq M^\times \subseteq M$, $(P/P^\times)^\times = \{1\}$, $(P^\times)^{\mathrm{gp}} = P^\times$.

But in general the composite morphism $M^\times \hookrightarrow M \longrightarrow M^{\mathrm{gp}}$ is not injective, as the case $(\mathbb{Z},\cdot)^\times = \{1,-1\} \hookrightarrow (\mathbb{Z},\cdot) \to (\mathbb{Z},\cdot)^{\mathrm{gp}} = \{1\}$ shows.

Torsion elements. An element $x \in M$ is a torsion element if there exists $n \in \mathbb{N}\setminus\{0\}$ such that $x^n = 1$. Any torsion element is invertible. The set of torsion elements of a monoid M is a submonoid and will be indicated as $M_{\mathrm{tor}} \subseteq M^\times$. The correspondence $M \to M_{\mathrm{tor}}$ gives a functor $\mathcal{M}on \to \mathcal{A}b$.

M is says to be without torsion if $M_{\mathrm{tor}} = \{1\}$. Obviously M is without torsion if and only if $M^\times$ is ($M_{\mathrm{tor}} = (M^\times)_{\mathrm{tor}}$).

We have that M/M_{tor} is torsion free, and the functorial correspondence $M \mapsto M/M_{\mathrm{tor}}$ gives a left adjoint of the forgetful functor from the (sub)category of torsion-free monoids to $\mathcal{M}on$, using the canonical morphism $M \to M/M_{\mathrm{tor}}$.

Integrality. A monoid M is **integral** (resp. **quasi-integral**) if the canonical map $M \longrightarrow M^{\mathrm{gp}}$ is injective (resp. has a trivial kernel); i.e. if and only if for any $a,b,c \in M$ we have $ab = ac \Longrightarrow b = c$ (resp. $ab = a \Longrightarrow b = 1$).

A monoid M is **pre-integral** if the composite of canonical morphisms $M^\times \longrightarrow M^{\mathrm{gp}}$ has trivial kernel (i.e. is injective); the condition is equivalent to the following: for any $u \in M^\times$, if $ua = a$ for same $a \in M$, then $u = 1$. This means that the action of $M^\times$ on M is free.

The definition $M^{\mathrm{int}} := \mathrm{im}(M \to M^{\mathrm{gp}})$ defines a functor from $\mathcal{M}on$ to the subcategory of integral monoids which is the left adjoint of the forgetful functor.

The canonical morphism $M \to M^{\mathrm{int}}$, factorization of $M \to M^{\mathrm{gp}}$, induces an isomorphism $M^{\mathrm{gp}} \to (M^{\mathrm{int}})^{\mathrm{gp}}$.

Saturation. A monoid M is p-saturated, p being a prime, if it is integral and for any $x \in M^{\mathrm{gp}}$, if $x^p \in M$ then $x \in M$. In other words if the diagram

$$\begin{array}{ccc} M & \xrightarrow{p} & M \\ {\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\ M^{\mathrm{gp}} & \xrightarrow[p]{} & M^{\mathrm{gp}} \end{array}$$

is cartesian.

A monoid M is saturated if it is integral and for any $x \in M^{\mathrm{gp}}$, if $x^n \in M$ for $n \geq 1$ then $x \in M$, i.e. if the previous diagram is cartesian for any p, or for any prime p, so that M is saturated if and only if it is p-saturated for all primes p.

For an integral monoid M define the saturation of M as $M^{\mathrm{sat}} := \{x \in M^{\mathrm{gp}} \ : \ \exists n \geq 1 \ s.t.\ x^n \in M\}$; for a generic monoid M we put $M^{\mathrm{sat}} := \left(M^{\mathrm{int}}\right)^{\mathrm{sat}}$.

The correspondence $M \mapsto M^{\mathrm{sat}}$ defines a functor which is a left adjoint of the (inclusion) forgetful functor from the subcategory of saturated monoids to $\mathcal{M}on$.

Valuatif monoids. An integral monoid M is valuatif if for any $a \in M^{\mathrm{gp}}$ we have that either $a \in M$ or $a^{-1} \in M$. A valuatif monoid is saturated.

Consider the class of all submonoid $N \subseteq M^{\mathrm{gp}}$ endowed with the order relation defined by $N \leq N'$ iff $N \subseteq N'$ and $N^\times = N \cap (N')^\times$. Then M is valuatif if and only if it is maximal in that class.

If M is an integral monoid, then there exists a valuative monoid $V \subseteq M^{\mathrm{gp}}$ such that $M \subseteq V$ and $M = M \cap V^\times$; moreover M^sat is characterized as the intersection of all such valuative submonoids of M^{gp}.

Remark the analogy between monoids and local rings, where integral monoids corresponds to local domains, saturated monoids to integrally closed local domains, valuatif monoids to valuation (local) rings.

Finiteness. A monoid M is of finite type if it admits a finite family of generators, i.e. if and only if there exists a surjective morphism: $\mathbb{N}^n \longrightarrow M$.

A monoid M is of finite presentation if it admits a finite family of generators, subject to a finite family of relations, i.e. if and only if there exists an isomorphism: $M \cong \operatorname{coker}(\mathbb{N}^r \rightrightarrows \mathbb{N}^n)$.

(Gabber) If M is of finite type, then it is of finite presentation. So the two notions are equivalent and a monoid of finite type can be presented with a finite number of generators and relations.

Remark that if P is finitely generated, also P^{sat} is.

Warning. A submonoid of a monoid of finite type is not necessarily of finite type. For example a "line with irrational slope" define twos submonoids of $\mathbb{N}^2$ which are not of finite type.

A monoid M is finitely generated iff $M^\times$ and $\overline{M}$ are finitely generated (as monoids).

A monoid M is fine if it is of finite type and integral; fs (fine and saturated) if it fine and saturated; toric if it is fs and torsion-free. If M is toric, then $M^{\mathrm{gp}} \cong \mathbb{Z}^d$.

If P is a fine and sharp monoid, then it has a unique minimal system of generators, given by the *irreducible* elements of P, i.e. the non invertible elements p such that if $p = q + r$, then q or r is invertible (so the other is p).

Properties of maps of monoids.

Sharp morphisms. A morphism $f : M \to N$ is sharp if $f^\times : M^\times \longrightarrow N^\times$ is an isomorphism.

Strict morphisms. A morphism $f : M \to N$ is strict if $\overline{f} : M/M^\times \longrightarrow N/N^\times$ is an isomorphism. Remark that $\overline{f}$ is injective iff $f^{\leftarrow}(N^\times) \subseteq M^inv$, and it is surjective iff $N = N^\times \cdot f(M)$.

If f is a sharp and strict morphism, then it is surjective; moreover, if N is pre-integral, then f is a bijection.

Local morphisms. A morphism $f : P \longrightarrow Q$ of monoids is local if $f^{\leftarrow}(Q^{\times}) = P^{\times}$ (the trivial inclusion $P^{\times} \subseteq f^{\leftarrow}(Q^{\times})$ is always true).

Exact morphisms. Let M and N be integral monoids; we say that a morphism $f : M \longrightarrow N$ is exact if the following condition holds: $(f^{\mathrm{gp}})^{\leftarrow}(N) = M$, i.e. if the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^{\mathrm{gp}} & \underset{f^{\mathrm{gp}}}{\longrightarrow} & N^{\mathrm{gp}} \end{array}$$

is cartesian. A morphism f is exact iff the following condition holds: for any $a, b \in M$, if $f(a)|f(b)$ then $a|b$ (the converse is true for any morphism of monoids). Note that composition of exact morphisms is exact, and also that for an exact morphism we have $f^{\leftarrow}(N^{\times}) = M^{\times}$, i.e. it is local. Finally, if a composition $g \circ f$ and g are exact, then f is too. A sharp and exact morphism is injective.

Let f be an exact morphism of monoids; then the canonical morphism from $\ker(f)$ to $\ker(f^{\mathrm{gp}})$ is a bijection, in particular f is a monomorphism (i.e. injective) if and only if $\ker(f)$ is trivial.

Let $\mathcal{E}x$ be the category of exact morphism of integral monoids, a (full) subcategory of $\mathscr{F}\ell(\mathcal{M}on)$ (morphism of the category of integral monoids). Then the inclusion functor $\mathcal{E}x \hookrightarrow \mathscr{F}\ell(\mathcal{M}on)$ admits a left adjoint $M \xrightarrow{f} N \longmapsto M^{\mathrm{ex}} \xrightarrow{f^{\mathrm{ex}}} N$ defined by $M^{\mathrm{ex}} := (f^{\mathrm{gp}})^{\leftarrow}(N)$ and $f^{\mathrm{ex}} := f^{\mathrm{gp}}_{|M^{\mathrm{ex}}}$.

Integral morphisms. A morphism $f : M \longrightarrow N$ of integral monoids is integral if the following equivalent conditions hold:

(i) for any other morphism $M \longrightarrow M'$ of integral monoids, the push-out in the category of monoids is integral;

(ii) for any other morphism $M \longrightarrow M'$ of integral monoids, the push-out $M' \oplus_M N = M' \oplus N/\sim$ in the category of monoids is described by the equivalence relation defined as in the case of "push-out with a group";

(iii) for any $a_1, a_2 \in M$, $b_1, b_2 \in N$ with $f(a_1)b_1 = f(a_2)b_2$ there exist $a_3, a_4 \in M$ and $b \in N$ such that

$$\begin{cases} b_1 = f(a_3)b \\ b_2 = f(a_4)b \\ a_1 a_3 = a_2 a_4 \ . \end{cases}$$

The following properties are equivalent:

(i) $f : M \longrightarrow N$ is injective and integral;

(ii) $\mathbb{Z}[f] : \mathbb{Z}[M] \longrightarrow \mathbb{Z}[N]$ is flat;

(iii) for any field k, $k[f] : k[M] \longrightarrow k[N]$ is flat.

The composition of integral morphisms is integral, and the notion is stable for cocartesian diagrams, localization and quotient. Moreover if a composition $g \circ f$ is integral and g is exact, then f is integral; if f is surjective, then g is integral.

We have that f is integral iff $\overline{f}$ is integral. If M is valuatif, or M or N is a group, then f is integral.

Saturated morphisms. Let $f : M \longrightarrow N$ be a morphism of integral monoids. It is p-quasi-saturated (resp. quasi-saturated) i.e. if the following condition holds: $h : N \oplus_M (M, p) \longrightarrow N$, defined by the cocartesian diagram

$$\begin{tikzcd} M \arrow[r, "p"] \arrow[d, "f"'] & M \arrow[d] \arrow[dr, "f"] & \\ N \arrow[r] & N \oplus_M (M,p) \arrow[r, "h"'] & N \end{tikzcd}$$

(where the lower horizontal composition is the multiplication by p) is an exact morphism (resp. is p-quasi-saturated for any p). It is p-saturated for a prime p (resp. saturated) if it is integral and p-quasi-saturated (resp. quasi-saturated).

For M and N p-saturated (resp. saturated) monoids and $f : M \longrightarrow N$ an integral morphism, the following conditions are equivalent:

(i) f is p-saturated (resp. saturated);

(ii) for any $M \to M'$ with M' p-saturated (resp. saturated), the push-out in the category of monoids $M' \oplus_M N$ is p-saturated (resp. saturated);

(iii) $N \oplus_M (M, p) := \mathrm{cocart}\left(M \xleftarrow{p} M \xrightarrow{f} N\right)$ is p-saturated.

Let M and N be p-saturated monoids and $f : M \longrightarrow N$ an integral morphism; then f is p-saturated if and only if the following condition holds: for any $a \in M$ and $b \in N$ such that $f(a)|b^p$ there exists $c \in M$ with $a|c^p$ and $f(c)|b$.

Composition of p-quasi-saturated morphisms is p-quasi-saturated, All the notions of p-quasi-saturated, quasi-saturated, p-saturated and saturated morphisms are stable for composition, cocartesian diagrams, localization and quotient. Moreover if a composition $g \circ f$ is p-saturated (resp. saturated) and g is exact, then f is p-saturated (resp. saturated).

Valuative criterion. Let F be a morphism of finitely generated saturated monoids; then f is saturated iff for any prime ideal K of height 1 sach that $J = f^{\leftarrow}(K)$ has height 1, the ramification index of f at K is 1. In particular the only saturated morphism of $\mathbb{N}$ into itself is the identity.

Kummer morphisms. A morphism $f : M \to N$ of integral monoids is "of Kummer type" if it is injective and for any $b \in N$ there exists an integer $n \geq 1$ such that $b^n \in f(M)$.

If f is Kummer, then $\mathrm{coker}(f)$ is a torsion group; in case f is exact, then it is Kummer iff $\mathrm{coker}(f)$ is a torsion group.

Vertical morphisms. A morphism $f : M \to N$ is vertical if for any $b \in N$ there exists $a \in M$ such that $b|f(a)$ (i.e. $f(a) \in bN$); if and only if for any $b \in N$ we have $bN \cap f(M) \neq \varnothing$.

If f is vertical, then $\mathrm{coker}(f)$ is a group; in case f is exact, then it is vertical iff $\mathrm{coker}(f)$ is a group.

If M, N are finitely generated integral monoids, then a vertical and p-saturated morphism is saturated.

Duality of monoids.

For a monoid P we define its dual $P^* := \mathrm{Hom}_{\mathcal{M}on}(P, \mathbb{N})$. Using the fact that $\mathbb{N}$ is integral, sharp and saturated we can see immediately that $(P^{\mathrm{int}})^* \cong P^*$, $(P/P^{\times})^* \cong P^*$ and $(P^{\mathrm{sat}})^* \cong P^*$. Moreover P^* is automatically integral, sharp and saturated; it is also finitely generated if P is.

Obviously we have $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Z}) \cong \mathrm{Hom}(P, \mathbb{Z})$ (universal property of P^{gp}) which is a group, so that we have a canonical morphism: $(P^*)^{\mathrm{gp}} \longrightarrow \mathrm{Hom}(P, \mathbb{Z})$ (which factorizes the injection $P^* = \mathrm{Hom}(P, \mathbb{N}) \hookrightarrow \mathrm{Hom}(P, \mathbb{Z})$); it is an isomorphism if and only if P is sharp. In particular, we have a canonical isomorphism $(M^*)^{\mathrm{gr}} \longrightarrow \mathrm{Hom}(\overline{M}, \mathbb{Z})$.

The canonical morphism $P \longrightarrow (P^*)^*$ sending p to the map $p^* \mapsto p^*(p)$ is injective if and only if P is integral and sharp, and surjective if and only if P is saturated. In particular if M is saturated we have a canonical isomorphism $\overline{M} \to M^{**}$; if M is arbitrary, then we have the isomorphism $\overline{M^{\mathrm{sat}}} \to M^{**}$.

We can recover the invertible elements of P as $P^{\times} = \ker(P \to (P^*)^*)$ and for an integral P we can describe its saturation as $P^{\mathrm{sat}} = \{p \in P^{\mathrm{gp}} \ : \ p^*(p) \in \mathbb{N} \ \forall p^* \in P^*\}$.

Suppose M be a fine and sharp monoid; if M^{gp} is torsion free, then M is isomorphic to a submonoid of $\mathbb{N}^r$ for some r; if M

is saturated then it is isomorphic to an exact submonoid of $\mathbb{N}^r$ for some r.

Geometry of Monoids.

Let $\mathrm{Spec}(M)$ be the set of prime ideals of a monoid M; then $\varnothing, M^+ = M \smallsetminus M^\times \in \mathrm{Spec}(M)$ (the minimal and maximal prime ideals of M).

If $J \in \mathrm{Spec}(M)$ we have the following identifications:
$\mathrm{Spec}(M_J) = \{K \in \mathrm{Spec}(M) | K \subsetneq J\} \subseteq \mathrm{Spec}(M)$ and
$\mathrm{Spec}(M \smallsetminus J) = \{K \in \mathrm{Spec}(M) | K \supsetneq J\} \subseteq \mathrm{Spec}(M)$.

A morphism $f : M \to N$ of monoids induces a map $\mathrm{Spec}(f) = f^{\leftarrow} : \mathrm{Spec}(N) \longrightarrow \mathrm{Spec}(M)$.

If M is finitely generated, then $\mathrm{Spec}(M)$ is finite; moreover we have that the morphism $M \to \overline{M}$ induces a canonical bijection $\mathrm{Spec}(\overline{M}) \longrightarrow \mathrm{Spec}(M)$

We define on $\mathrm{Spec}(M)$ the Zariski topology: the closed sets are the subset of the form $V(I) = \{K \in \mathrm{Spec}(M) | I \subsetneq K\}$ for I a subset (or an ideal) of M. Remark that $V(\varnothing) = \mathrm{Spec}(M)$ (i.e. $\varnothing$ is the generic point of $\mathrm{Spec}(M)$) and $V(M^+) = \{M^+\}$ (M^+ is the closed point of $\mathrm{Spec}(M)$). Bare for the open sets of the topology are the subsets of the form $D(f) = \{K \in \mathrm{Spec}(M) | f \notin K\}$ arying $f \in M$.

Now we define on the topological space $X = \mathrm{Spec}(M)$ a sheaf $\mathscr{M}_X$ of monoids by setting $\mathscr{M}_X(D(f)) = S_f^{-1}P$ where $S_f = \{f^n | n \geq 0\}$.

The contravariant functor Spec gives an equivalence of categories between the category of monoids and the essential image in the category of monoidal spaces (by definition the "affine monoidal spaces"), with inverse given by the global sections of the monoidal sheaf.

The affine monoidal space $\mathrm{Spec}(M)$ represents the functor sending a monoidal space T to the set $\mathrm{Hom}(M, \Gamma(T, \mathscr{M}_T))$; so we have that $\mathrm{Hom}(T, \mathrm{Spec}(M)) \cong \mathrm{Hom}(M, \Gamma(T, \mathscr{M}_T))$ and in particular $\mathrm{Hom}(\mathrm{Spec}(N), \mathrm{Spec}(M)) \cong \mathrm{Hom}(M, N)$.