# Poincaré Duality for Algebraic De Rham Cohomology. 

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#### Abstract

We discuss in some detail the algebraic notion of De Rham cohomology with compact supports for singular schemes over a field of characteristic zero. We prove Poincaré duality with respect to De Rham homology as defined by Hartshorne [H.75], so providing a generalization of some results of that paper to the non proper case. In order to do this, we work in the setting of the categories introduced by Herrera and Lieberman [HL], and we interpret our cohomology groups as hyperext groups. We exhibit canonical morphisms of "cospecialization" from complex-analytic De Rham (resp. rigid) cohomology groups with compact supports to the algebraic ones. These morphisms, together with the "specialization" morphisms [H.75, IV.1.2] (resp. [BB, 1]) going in the opposite direction, are shown to be compatible with our algebraic Poincaré pairing and the analogous complex-analytic (resp. rigid) one (resp. [B.97, 3.2]).


## Introduction.

In his paper on "De Rham cohomology of algebraic varieties"[H.75], Hartshorne defined De Rham cohomology and homology groups for singular schemes over a field $K$ of characteristic zero. He proved a global Poincaré duality theorem in the proper case. Naturally, one expects to have a notion of (algebraic) "De Rham cohomology with compact supports" Poincaré dual to De Rham homology, extending Hartshorne's De Rham cohomology and Poincaré duality to not necessarily proper (nor regular) schemes. To be useful, such a theory should admit an independent (i.e. not based on duality with homology) and easily computable definition.

The method for doing this in general has been "well-known" since Deligne's lectures on crystals in characteristic zero at IHES in March 1970. He sketched there an algebraic theory of the functor $f_{!}$for crystals, providing the natural setting for a general answer to the above problem. An account of those lectures, however, has never been made available: we are not aware of any satisfactory reference for algebraic De Rham cohomology with compact supports, not even in the standard case of constant coefficients on a smooth open variety.

In view of the generality of Deligne's lectures, the idea of publishing an independent account of the case of constant coefficients may seem obsolete. On the other hand, even the choice of a suitable level of generality for coefficients is embarassing, even if only because of the simultaneous need of pro-coherent and ind-coherent $\mathscr{O}$-modules. We have preferred to introduce the minimal amount of technique necessary to give a good and flexible definition of algebraic cohomology with compact supports for singular $K$-varieties. Hartshorne's choice in [H.75] was to ignore the open case altogether, concentrating on the proper singular case, despite his experience with Serre duality for open varieties over a field of characteristic zero [H.72].

The main point of this article is the following. The case of a proper singular $K$-scheme $X$, presents the big technical advantage that one can develop all considerations in the category of abelian sheaves on the Zariski space $X$. In the open case, $X$ will have to be compactified to $\bar{X}$, and $\bar{X}$ will be embedded as a closed subscheme of a scheme $P$, smooth in a neighbourhood of $X$. The category $\mathscr{A} b(\bar{X})$ is suitable for the definition of $H_{\mathrm{DR}, c}^{\bullet}(X)=\mathbf{H}^{\bullet}\left(\bar{X},\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X} \backslash X}\right)_{\text {tot }}\right)$ but not so for proving that this definition is independent of the embeddings $X \hookrightarrow \bar{X} \hookrightarrow P$.

We are aware of at least three methods for proving that the given definition of $H_{\dot{\mathrm{DR}}, \mathrm{c}}(X)$ is good. The most general one thrives in the previously mentioned context of crystals in characteristic zero. A second method is based on Grothendieck's linearization of differential operators [AB, Appendix D] (dual to Saito's linearization [S]). A third approach is via the filtered refinement of Herrera and Lieberman's $\mathscr{C}^{\bullet}$-Modules [HL] proposed by Du Bois [DB.90]. We plan to investigate the full formalism of Grothendieck operations for De Rham coefficients following Du Bois, in the near future. For the limited scope of the present article however,

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we chose to simply revisit the original article of Herrera and Lieberman [HL], which has the virtue of making the relation between Serre and Poincaré duality very explicit. This theory is of course not sufficiently flexible to permit the introduction of Grothendieck's operations in general. But constructions are very explicit and the comparison with analytic theories becomes natural.

We hesitated on whether we should work with pro-objects (of the category of coherent sheaves on a scheme or analytic space), as Deligne does, or take their derived inverse limits on a formal scheme or formal analytic space, as in Hartshorne [H.72], [H.75]. The point of a Mittag-Leffler type of result as [H.75, I.4.5] (see also lemma 1.3 .1 below) is that for suitable pro-objects, derived direct images commute with inverse limits, so that applying the functor lim to those pro-objects is harmless. As a consequence, in our situation we take indifferently one attitude or the other. It is worthwhile mentioning that when the base scheme is $\operatorname{Spec} K$, the procategory of finite dimensional $K$-vector spaces is equivalent via lim to the category of linearly compact topological $K$-vector spaces. So, Hartshorne's Serre duality of [H.72], may be regarded as a perfect topological pairing between a discrete topological $K$-vector space and a linearly compact one (both in general infinite dimensional). In the case of De Rham cohomology [H.75], this specializes to a duality of finite dimensional $K$-vector spaces. We use this topological pairing when describing a duality of spectral sequences converging to Poincaré duality. One should compare this with the analytic problem posed by Herrera and Lieberman [HL, §5].

We want to mention that very recently, Chiarellotto and Le Stum have included in their wide-ranging article [CLS] a short account of Deligne's method for defining De Rham cohomology with compact supports for open smooth $K$-varieties.

In the case of a smooth $K$-scheme $X$ and of a connection $(\mathscr{E}, \nabla)$, with $\mathscr{E}$ a coherent, hence locally free $\mathscr{O}_{X}$-module, the De Rham cohomology groups with compact supports $H_{D R, c}^{q}(X / K,(\mathscr{E}, \nabla))$ were defined in [AB, Def. D.2.16] and Poincaré duality was proved [AB, D.2.17], in fact in a more general relative situation. The definition of $f_{!} \mathscr{M}$ proposed by Katz-Laumon [KL, section 7], Du Bois [DB.90, 6.9] and Mebkhout [M, I.5.3], for $f: X \longrightarrow Y$ a morphism of smooth $K$-varieties and $\mathscr{M}$ an object of $D_{h o l}^{b}\left(\mathscr{D}_{X}\right)$, is for us less satisfactory, since duality is built into its definition. The experience with Dwork's dual theory [AB, appendix D ], shows that an independent definition of $f_{!}$is of great help in calculations and for arithmetic applications, and is therefore very desirable.

We should mention here the crucial help we received from P. Berthelot who provided us with his notes of Deligne's course and whose treatement of rigid cohomology was also very useful, somewhat paradoxically, to organize our discussion in the algebraic case.

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## 0. Notation and Preliminaries.

0.1. Schemes. Let $K$ be a field of characteristic 0 . By scheme we will mean a separated $K$-scheme of finite type. Morphisms and products of schemes will be taken over Spec $K$, unless otherwise specified. For a scheme $X,|X|$ will denote the underlying topological space.
0.2. Closed Immersions. If $X$ is a closed subscheme of the scheme $Y$, we indicate by $\mathscr{I}_{X \subseteq Y}$, or simply by $\mathscr{I}_{X}$, the coherent Ideal of $\mathscr{O}_{Y}$ associated to $X$.
0.3. Infinitesimal neighborhoods. If $i: X \hookrightarrow Y$ is a closed immersion of schemes, we denote by $X_{Y}^{(M)}$ the $M$-th infinitesimal neighborhood of $X$ in $Y$, that is the scheme having $|X|$ as topological space
and $\mathscr{O}_{Y} / \mathscr{I}_{X}^{M+1}$ (restricted to $\left.|X|\right)$ as structural sheaf.
Let $Y_{/ X}$ be the formal completion of $Y$ along $X$, i.e. the formal scheme inductive limit of all infinitesimal neighborhoods of $X$ in $Y$; its underlying topological space is $|X|$ and its structural sheaf is $\mathscr{O}_{Y / X}:=\left(\mathscr{O}_{Y}\right)_{/ X}=$ $\lim _{M} \mathscr{O}_{X_{Y}^{(M)}}$. Following [EGA I, 10.8], if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{Y}$-Module, $\mathscr{F} / X$ will be the sheaf $\left(\right.$ an $\left(\mathscr{O}_{Y}\right)_{X^{-}}$ Module) of formal sections of $\mathscr{F}$ along $X$ (i.e. the projective limit $\varliminf_{M} i^{(M) * \mathscr{F}}$ where $i^{(M)}: X_{Y}^{(M)} \rightarrow Y$ is the canonical immersion; if $\mathscr{F}$ is coherent, $\mathscr{F} / X=i_{/ X}^{*} \mathscr{F}$ where $i_{/ X}: Y_{/ X}^{M} \rightarrow Y$ is the canonical morphism of ringed spaces, since our schemes are locally noetherian).
0.4. Differentials on infinitesimal neighborhoods. The closed immersion of $X_{Y}^{(M)}$ in $Y$, of Ideal $\mathscr{I}_{X_{Y}^{(M)}}=\mathscr{I}_{X_{Y}^{(M)} \subseteq Y}=\mathscr{I}_{X}^{M+1}$, gives rise to the exact sequence

$$
\mathscr{I}_{X_{Y}^{(M)}} / \mathscr{I}_{X_{Y}^{(M)}}^{2} \xrightarrow{d^{(M)}} \Omega_{Y}^{1} \otimes \mathscr{O}_{Y} \mathscr{O}_{X_{Y}^{(M)}} \longrightarrow \Omega_{X_{Y}^{(M)}}^{1} \longrightarrow 0
$$

The exact sequence

$$
0 \longrightarrow \mathscr{I}_{X_{Y}^{(M)}} / \mathscr{I}_{X_{Y}^{(M)}}^{2} \longrightarrow \mathscr{O}_{X_{Y}^{(2 M+1)}} \longrightarrow \mathscr{O}_{X_{Y}^{(M)}} \longrightarrow 0
$$

shows that "lim" ${ }_{M} \mathscr{I}_{X_{Y}^{(M)}} / \mathscr{I}_{X_{Y}^{(M)}}^{2} \cong 0$. Therefore, the canonical morphism

$$
" \lim _{M} " \Omega_{Y}^{1} \otimes \mathscr{O}_{Y} \mathscr{O}_{X_{Y}^{(M)}} \longrightarrow{ }_{M}^{\lim } " \Omega_{X_{Y}^{(M)}}^{1}
$$

is an isomorphism. This isomorphism still holds in higher degrees, i.e.

$$
" \lim _{M} " \Omega_{Y}^{i} \otimes_{\mathscr{O}_{Y}} \mathscr{O}_{X_{Y}^{(M)}} \xrightarrow{\cong} "_{\underset{M}{\lim }} " \Omega_{X_{Y}^{(M)}}^{i},
$$

for all $i$. On the other hand, $\Omega_{Y_{/ X}}^{i}={\underset{\longleftarrow}{幺}}_{\lim _{M}} \Omega_{X_{Y}^{(M)}}^{i}$, so that

$$
\Omega_{Y / X}^{i}=\varliminf_{M}^{\lim } \Omega_{X_{Y}^{(M)}}^{i} \cong \Omega_{Y}^{i} \otimes_{\mathscr{O}_{Y}} \mathscr{O}_{Y_{/ X}} \cong\left(\Omega_{Y}^{i}\right)_{/ X}
$$

0.5. Categories of Differential Operators. Let $X$ be a scheme. We recall ([HL] or [B.74, II.5]) that the category $\mathscr{C}(X)$ of complexes of differential operators of order (less or equal to) one is defined as follows: the objects of $\mathscr{C}(X)$ are complexes whose terms are $\mathscr{O}_{X}$-Modules and whose differentials are differential operators of order less or equal to one. Morphisms between such complexes are $\mathscr{O}_{X}$-linear maps of degree zero of graded $\mathscr{O}_{X}$-modules, compatible with the differentials.

We recall that for any complex $\mathscr{F} \bullet \cdots \longrightarrow \mathscr{F}^{\stackrel{d_{\mathscr{F}}^{i}}{d^{i}}} \mathscr{F}^{i+1} \longrightarrow \cdots$ of abelian sheaves on $X$ and $k \in \mathbb{Z}$, $\mathscr{F} \bullet[k]$ is defined by $(\mathscr{F} \bullet[k])^{i}=\mathscr{F}^{i+k}$ and $d_{\mathscr{F}}^{i} \bullet[k]=(-1)^{k} d_{\mathscr{F} \bullet}^{i+k}$, for all $i \in \mathbb{Z}$. If $f=f \bullet: \mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ is such a morphism and $k$ is an integer,

$$
f \bullet[k]: \mathscr{F} \bullet[k] \longrightarrow \mathscr{G} \bullet[k]
$$

is usually defined by $\left(f^{\bullet}[k]\right)^{j}=f^{j+k}$, for all $j$, and may therefore be identified with $f^{\bullet}$. These conventions are used in particular for $\mathscr{F} \bullet$ and $f^{\bullet}$ an object and a morphism in $\mathscr{C}(X)$.

A homotopy between two morphisms in $\mathscr{C}(X)$ is a homotopy in the sense of the category of complexes of abelian sheaves, except that the homotopy operator (of degree -1 ) is taken to be $\mathscr{O}_{X}$-linear. We denote by $\mathscr{C}_{c}(X)$ (resp. $\left.\mathscr{C}_{q c}(X)\right)$ the full subcategory of $\mathscr{C}(X)$ consisting of complexes with coherent (resp. quasicoherent) terms.

We slightly generalize the previous definitions as follows. Let "lim" $\mathscr{F}_{\alpha}$ and "lim" $\mathscr{F}_{\beta}$ be two procoherent $\mathscr{O}_{X}$-Modules (i.e. two objects of the pro-category $\operatorname{ProCoh}(X)$ ). A morphism $f: "$ lim" $\mathscr{F}_{\alpha} \longrightarrow " \lim _{\beta} \mathscr{G}_{\beta}$ of $\operatorname{Pro} \mathscr{A} b(X)$ is a differential operator of order one if $f$ factors via the commutative diagram

$$
\begin{align*}
& " \varliminf_{i m}{ }_{\alpha} \mathscr{P}_{X}^{1}\left(\mathscr{F}_{\alpha}\right) \\
& " \lim _{\ddagger}^{\rightleftarrows} d_{X, \mathscr{F}_{\alpha}}^{1} \nearrow \quad \searrow \bar{f}  \tag{0.5.1}\\
& " \mathrm{lim}{ }_{\alpha} \mathscr{F}_{\alpha} \longrightarrow \lim _{\beta} \mathscr{G}_{\beta}
\end{align*}
$$

where $d_{X, \mathscr{F}_{\alpha}}^{1}$ is the universal differential operator of order one of source $\mathscr{F}_{\alpha}$, and $\bar{f}$ is a morphism of $\operatorname{ProCoh}(X)$. Now, $\mathscr{C}_{p c}(X)$ will denote the category of complexes of pro-coherent $\mathscr{O}_{X}$-Modules whose differentials are differential operators of order one. Morphisms in $\mathscr{C}_{p c}(X)$ will be maps of degree zero of
graded objects of $\operatorname{ProCoh}(X)$, compatible with the differentials. There is a natural exact and faithful functor $\operatorname{Pro} \mathscr{C}_{c}(X) \longrightarrow \mathscr{C}_{p c}(X)$ compatible with homotopical equivalence. Similarly, there is a natural exact and faithful functor, compatible with homotopical equivalence, from the ind-category of $\mathscr{C}_{c}(X), \operatorname{Ind} \mathscr{C}_{c}(X)$, to $\mathscr{C}_{q c}(X)$, since $\operatorname{Ind} \operatorname{Coh}(X)$ is equivalent to the category of quasi-coherent $\mathscr{O}_{X}$-Modules [H.RD, Appendix]. We point out that any object of $\mathscr{C}_{q c}(X)$ (resp. $\left.\mathscr{C}_{p c}(X)\right)$ which is a bounded complex, is in fact in the essential image of $\operatorname{Ind} \mathscr{C}_{c}(X)$ in $\mathscr{C}_{q c}(X)$ (resp. of $\operatorname{Pro} \mathscr{C}_{c}(X)$ in $\mathscr{C}_{p c}(X)$ ).
0.5.2. (This subsection wants to motivate the definition of $\mathscr{H} o m^{k}(-,-)$ in [HL, §2]). We recall that for a graded left $\Omega_{\dot{X}}^{\bullet}$-Module $\mathscr{F} \bullet$ and an integer $k$, one sets $\mathscr{F} \bullet[k]$ to be the graded left $\Omega_{X}^{\bullet}$-Module $\mathscr{E} \bullet$ such that $\mathscr{E}^{j}=\mathscr{F}^{j+k}$ and such that for $\alpha$ a section of $\mathscr{E}^{j}=\mathscr{F}^{j+k}$ and $\varphi$ a section of $\Omega_{X}^{i}$, the scalar product $\varphi_{\mathscr{E}} \cdot \alpha$ of $\varphi$ and $\alpha$ in $\mathscr{E} \bullet$ is $(-1)^{i k}$ times the scalar product $\varphi_{\mathscr{P}} \cdot \alpha$ of $\varphi$ and $\alpha$ in $\mathscr{F} \bullet$. Morphisms of graded left $\Omega_{\dot{X}}^{\bullet}$-Modules are meant to be of degree zero. They form a $\Gamma\left(X, \mathscr{O}_{X}\right)$-module $\operatorname{Hom}_{\Omega_{\dot{X}}}(-,-)$; we denote by $\mathscr{H}$ om $\Omega_{\dot{X}}(-,-)$ the sheafified version. We point out that if $\omega$ is a section of $\Omega_{X}^{k}$, and $\mathscr{F} \bullet$ is a graded left $\Omega_{X^{\bullet}}^{\bullet}$-Module, left multiplication by $\omega$ is a morphism $\mathscr{F} \bullet \longrightarrow \mathscr{F} \cdot[k]$. Once again, if $f^{\bullet}$ is a morphism of graded left $\Omega_{X}^{\bullet}$-Modules and $k$ is an integer, $f^{\bullet}[k]$ identifies with $f^{\bullet}$. For $\mathscr{F} \bullet$ and $\mathscr{G} \bullet$ as before, and $k$ an integer, one sets

$$
\operatorname{Hom}_{\Omega_{\dot{X}}}^{k}(\mathscr{F} \bullet, \mathscr{G} \bullet)=\operatorname{Hom}_{\Omega_{\dot{X}}}(\mathscr{F} \bullet, \mathscr{G} \bullet[k])=\operatorname{Hom}_{\Omega_{\dot{X}}}\left(\mathscr{F} \bullet[-k], \mathscr{G}^{\bullet}\right)
$$

and similarly for $\mathscr{H} m_{\Omega_{\dot{X}}}^{k}\left(\mathscr{F} \bullet, \mathscr{G}^{\bullet}\right)=\mathscr{H}_{\mathrm{H}_{\Omega_{X}}}(\mathscr{F} \bullet, \mathscr{G} \bullet[k])$.
The skew-commutativity of $\Omega_{X}^{\bullet}$ (i.e. $\alpha \beta=(-1)^{i j} \beta \alpha$, for $\alpha \in \Omega^{i}$ and $\beta \in \Omega^{j}$ ) permits to interpret any graded left or right $\Omega_{X}^{\bullet}$-Module $\mathscr{F} \bullet$ as two-sided. In fact, if $\mathscr{F} \bullet$ is a graded left $\Omega_{X}^{\circ}$-Module, we can define a structure of graded right $\Omega_{X}^{\bullet}$-Module on it by setting

$$
\alpha \cdot \varphi=(-1)^{i j} \varphi \cdot \alpha
$$

for $\varphi$ a section of $\Omega_{X}^{i}$ and $\alpha$ a section of $\mathscr{F}^{j}$. It is then clear that a morphism of graded left $\Omega_{X}^{\bullet}$-Modules is also right $\Omega_{\dot{X}}^{\bullet}$-linear, and the other way around. This is why the notation $\operatorname{Hom}_{\Omega_{\dot{X}}}^{k}(-,-)$ does not carry any indication on whether left or right linearity is assumed. We observe that an element $\Phi$ of $\operatorname{Hom}_{\Omega_{\mathbf{X}}}^{k}(\mathscr{F} \bullet, \mathscr{G} \bullet)$ turns out to be a collection $\Phi=\left(\varphi_{j}\right)_{j}$ of maps of abelian sheaves $\varphi_{j}: \mathscr{F}^{j} \longrightarrow \mathscr{G}^{j+k}$ satisfying

$$
\varphi_{i+j}\left(\omega_{\mathscr{F} \bullet}^{\bullet} \alpha\right)=(-1)^{i k} \omega_{\mathscr{G} \bullet} \varphi_{j}(\alpha)
$$

for sections $\omega$ of $\Omega_{X}^{i}$ and $\alpha$ of $\mathscr{F}^{j}$ (cf. [HL, loc. cit.]).
The possibility of interchanging the left and right $\Omega_{X}^{*}$-Module structure gives a meaning and a structure of $\Omega_{X^{\bullet}}$-Module to $\mathscr{M}^{\bullet} \otimes_{\Omega_{\dot{X}}^{\bullet}} \mathscr{N}^{\bullet}$, for two $\Omega_{X^{-}}^{\bullet}$ modules $\mathscr{M}^{\bullet}$ and $\mathscr{N}^{\bullet}$. Here one uses the right (resp. left) $\Omega_{X^{-}}$ Module structure on $\mathscr{M}^{\bullet}$ (resp. $\mathscr{N}^{\bullet}$ ) to take the tensor product, while the left (resp. right) $\Omega_{X^{\bullet}}$-Module structure on the tensor product is given by left (resp. right) $\Omega_{X}^{\bullet}$-Module structure of $\mathscr{M} \bullet$ (resp. $\mathscr{N} \bullet$ ). Similarly, $\mathscr{H}$ om $\dot{\Omega}_{\dot{X}}^{\bullet}\left(\mathscr{M}^{\bullet}, \mathscr{N}^{\bullet}\right)$ has a structure of $\Omega_{X_{X}}^{\bullet}$-module.
0.5.3. The category $\mathscr{C}(X)$ is equivalent to the category of graded left $\Omega_{X^{\bullet}}$-Modules $\mathscr{F} \cdot$, endowed with a morphism of graded abelian sheaves $D=D_{\mathscr{F} \bullet}: \mathscr{F} \bullet \longrightarrow \mathscr{F} \bullet[1]$ satisfying $D_{\mathscr{F}} \bullet[1] \circ D_{\mathscr{F} \bullet}=0$ and

$$
\begin{equation*}
D_{\mathscr{F}} \bullet\left(\varphi_{\mathscr{F} \bullet} \dot{\bullet}\right)=\varphi_{\mathscr{P} \cdot[1]} D_{\mathscr{F}} \bullet(\alpha)+\left(d_{X} \varphi\right)_{\mathscr{\mathscr { P }}} \dot{\bullet} \tag{0.5.4}
\end{equation*}
$$

for sections $\varphi$ of $\Omega_{X}^{\bullet}$ and $\alpha$ of $\mathscr{F} \bullet$. If $\mathscr{F} \bullet$ and $\mathscr{G} \bullet$ are two objects of $\mathscr{C}(X)$, a morphism $f: \mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ is then a morphism of graded $\Omega_{X}^{\bullet}$-Modules, such that

$$
f[1] \circ D_{\mathscr{F} \bullet}=D_{\mathscr{G}} \bullet \circ f .
$$

For any integer $k, \mathscr{F} \bullet[k]$ as an object of $\mathscr{C}(X)$, is the graded left $\Omega_{X}^{\bullet}$-Module $\mathscr{F} \bullet[k]$, endowed with $D_{\mathscr{F} \bullet[k]}=$ $(-1)^{k} D_{\mathscr{F}} \cdot$.

The category $\mathscr{C}(X)$ is endowed with a natural tensor product $-\otimes_{\Omega_{\dot{x}}}$ - and with an internal hom $\mathscr{H} 0 m_{\Omega_{X}^{\bullet}}^{\bullet}(-,-)$. One sets $D_{\mathscr{F} \bullet} \otimes_{\Omega_{X}} \mathscr{G} \bullet(\alpha \otimes \beta)=D_{\mathscr{F}} \cdot(\alpha) \otimes \beta+(-1)^{i} \alpha \otimes D_{\mathscr{G}} \bullet(\beta)$, for sections $\alpha$ of $\mathscr{F}^{i}$ and $\beta$ of $\mathscr{G} \bullet$. One also defines

$$
D=D_{\mathscr{H} o m_{\Omega_{\dot{X}}}^{\bullet}}(\mathscr{F} \bullet, \mathscr{G} \bullet): \mathscr{H} a m_{\Omega_{\dot{X}}^{\bullet}}^{\bullet}\left(\mathscr{F} \cdot \mathscr{G}^{\bullet}\right) \longrightarrow \mathscr{H}_{\Omega_{\dot{X}}}^{\bullet}\left(\mathscr{F} \cdot \mathscr{G}^{\bullet}\right)[1]
$$

by

$$
D \Phi=D_{\mathscr{G}} \bullet \circ \Phi-\Phi[1] \circ D_{\mathscr{F} \bullet[-k]}
$$

if $\Phi$ is a section of $\mathscr{H} 0 m_{\Omega_{\dot{X}}}^{k}(\mathscr{F} \bullet, \mathscr{G} \bullet)=\mathscr{H} 0 m_{\Omega_{X}}(\mathscr{F} \bullet[-k], \mathscr{G} \bullet)$.
0.5.5. Obviously, two morphisms $f$ and $g: \mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ in $\mathscr{C}(X)$ are homotopic via the homotopy operator $\vartheta: \mathscr{F} \bullet[1] \longrightarrow \mathscr{G} \bullet$ (a morphism of graded $\mathscr{O}_{X}$-Modules) if and only if

$$
g-f=D_{\mathscr{G}} \bullet \circ-\vartheta[1] \circ D_{\mathscr{F} \bullet[1]} .
$$

The previous formula means in fact that for any $i$

$$
g^{i}-f^{i}=D_{\mathscr{G} \bullet}^{i-1} \circ \vartheta^{i}+\vartheta^{i+1} \circ D_{\mathscr{F} \bullet}^{i} \cdot
$$

It is easy to check by induction on the degree of differential forms, that $\vartheta$ is automatically $\Omega_{\dot{X}}^{\bullet}$-linear, and that the previous formula simply means that $g-f \in D\left(\operatorname{Hom}_{\Omega_{X}}^{-1}(\mathscr{F} \bullet, \mathscr{G} \bullet)\right)$. Similarly, for $f \in \operatorname{Hom}_{\Omega_{\dot{X}}}^{0}(\mathscr{F} \bullet, \mathscr{G} \bullet)$, $D f=0$ is equivalent to $f$ being a morphism $\mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ in the category $\mathscr{C}(X)$. So, (cf. [HL, remark p.104]) $H^{0}\left(\operatorname{Hom}_{\Omega_{X}}^{\bullet}(\mathscr{F} \bullet, \mathscr{G} \bullet)\right)$ is the group of $\mathscr{C}(X)$-morphisms $\mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ up to homotopy.
0.5.6. It is also possible to interpret $\mathscr{C}(X)$ as the category of graded left $\mathscr{C}_{\dot{X}}$-Modules where $\mathscr{C}_{\dot{X}} \cong$ $\Omega_{X}^{\bullet-1} D \oplus \Omega_{X}^{\bullet}$ is the "mapping cylinder" of the identity map of $\Omega_{X}^{\bullet}$. It is a graded $\mathscr{O}_{X}$-Algebra, whose product is defined using the wedge product of $\Omega_{X}^{\bullet}$ and $D^{2}=0$, while the structure of complex is defined by $D \alpha=\left(d \alpha_{1}+(-1)^{i} \alpha_{2}\right) D+d \alpha_{2}$ if $\alpha=\alpha_{1} D+\alpha_{2}$ with $\alpha_{1} \in \Omega_{X}^{i-1}$ and $\alpha_{2} \in \Omega_{X}^{i}$. Therefore, the category $\mathscr{C}(X)$ has enough injectives.
0.5.7. For a morphism $\pi: X \longrightarrow Y$ of schemes, we have canonical morphisms of graded differential Rings

$$
T_{\pi}: \Omega_{Y}^{\bullet} \longrightarrow \pi_{*} \Omega_{X}^{\bullet} \quad, \quad S_{\pi}: \pi^{-1} \Omega_{Y}^{\bullet} \longrightarrow \Omega_{X}^{\bullet}
$$

We deduce from this a pair of adjoint functors

$$
\pi_{*}: \mathscr{C}(X) \longrightarrow \mathscr{C}(Y) \quad, \quad \pi^{*}: \mathscr{C}(Y) \longrightarrow \mathscr{C}(X)
$$

For an object $\mathscr{F} \bullet$ of $\mathscr{C}(X)$, the complex of abelian sheaves $\pi_{*}(\mathscr{F} \bullet)$ coincides with the usual direct image of the complex of abelian sheaves $\mathscr{F} \bullet$. For $\mathscr{G} \bullet$ in $\mathscr{C}(Y)$ instead, $\pi^{*}(\mathscr{G} \bullet)=\Omega_{X}^{\bullet} \otimes_{\pi^{-1} \Omega_{Y}^{\bullet}} \pi^{-1}(\mathscr{G} \bullet)$ and

$$
D_{\pi^{*}(\mathscr{G} \bullet)}\left(\varphi \otimes \pi^{-1}(\alpha)\right)=d_{X} \varphi \otimes \pi^{-1}(\alpha)+(-1)^{i} \varphi \otimes \pi^{-1}\left(D_{\mathscr{G}} \bullet \alpha\right),
$$

for $\varphi$ a section of $\Omega_{X}^{i}$ and $\alpha$ a section of $\mathscr{G} \bullet$.
0.5.8. Lemma. Let $\pi: X \longrightarrow Y$ be a morphism of schemes, and assume that two morphisms $f$ and $g: \mathscr{F} \bullet \longrightarrow \mathscr{G} \bullet$ in $\mathscr{C}(Y)$ are homotopic via the homotopy operator $\vartheta: \mathscr{F} \bullet[1] \longrightarrow \mathscr{G} \bullet$. Then $\pi^{*}(f)$ and $\pi^{*}(g): \pi^{*}(\mathscr{F} \bullet) \longrightarrow \pi^{*}(\mathscr{G} \bullet)$ in $\mathscr{C}(X)$ are homotopic via the homotopy operator $\pi^{*}(\vartheta): \pi^{*}(\mathscr{F} \bullet)[1] \longrightarrow \pi^{*}(\mathscr{G} \bullet)$.

Proof. The main point is that, for any section $\varphi$ of $\Omega_{X}^{i}$ and any section $\alpha$ of $\mathscr{F} \bullet$,

$$
\begin{equation*}
\pi^{*}(\vartheta)\left(d_{X} \varphi \otimes \pi^{-1}(\alpha)\right)+(-1)^{i} d_{X} \varphi \otimes \pi^{-1}(\vartheta(\alpha))=0 \tag{0.5.9}
\end{equation*}
$$

(since more generally $\pi^{*}(\vartheta)\left(\varphi \otimes \pi^{-1}(\alpha)\right)=(-1)^{i} \varphi \otimes \pi^{-1}(\vartheta(\alpha))$, $\vartheta$ being a morphism of degree -1 ), so that

$$
\begin{aligned}
& \left(D_{\pi^{*}(\mathscr{G} \bullet)} \circ \pi^{*}(\vartheta)+\pi^{*}(\vartheta) \circ D_{\pi^{*}(\mathscr{F} \bullet)}\right)\left(\varphi \otimes \pi^{-1}(\alpha)\right)= \\
= & D_{\pi^{*}(\mathscr{G} \bullet)}\left((-1)^{i} \varphi \otimes \pi^{-1}(\vartheta \alpha)\right)+\pi^{*}(\vartheta)\left(d_{X} \varphi \otimes \pi^{-1}(\alpha)+(-1)^{i} \varphi \otimes \pi^{-1}\left(D_{\mathscr{F}} \bullet \alpha\right)\right) \\
= & (-1)^{i} d_{X} \varphi \otimes \pi^{-1}(\vartheta \alpha)+\varphi \otimes \pi^{-1}\left(D_{\mathscr{G}} \bullet \vartheta \alpha\right)+(-1)^{i+1} d_{X} \varphi \otimes \pi^{-1}(\vartheta \alpha)+\varphi \otimes \pi^{-1}\left(\vartheta D_{\mathscr{F} \bullet \alpha)}\right) \\
= & \varphi \otimes \pi^{-1}\left(D_{\mathscr{G}} \bullet \vartheta+\vartheta D_{\mathscr{F} \bullet}\right) \alpha \\
= & \varphi \otimes \pi^{-1}(g-f) \alpha \\
= & \left(\pi^{*}(g)-\pi^{*}(f)\right)\left(\varphi \otimes \pi^{-1}(\alpha)\right) .
\end{aligned}
$$

We point out that if $\pi$ is a locally closed immersion, then $\pi^{*}(\mathscr{F} \bullet)$ coincides, as a graded $\mathscr{O}_{X}$-Module, with the usual inverse image in the sense of graded $\mathscr{O}_{X}$-Modules.
0.5.10. Let $\mathscr{F}$ and $\mathscr{G}$ be two objects of $\mathscr{C}(X)$, and let $\mathscr{G} \rightarrow \mathscr{J} \cdot$ be an injective resolution of $\mathscr{G}$ in $\mathscr{C}(X)$. The local (resp. global) hyperext functors of Herrera-Lieberman are defined in [B.74, II.5.4.3] as the abelian sheaves (resp. groups), for any $p \in \mathbb{Z}$,

$$
\underline{\mathscr{E} x t}^{p} \Omega_{\dot{X}}(\mathscr{F}, \mathscr{G})=H^{p}\left(\mathscr{H} o m_{\Omega_{\dot{X}}^{\bullet}}(\mathscr{F}, \mathscr{I} \cdot)_{\mathrm{tot}}\right)
$$

(resp.

$$
\left.\underline{\underline{\operatorname{Ext}}}^{p} \Omega_{X}^{\bullet}(\mathscr{F}, \mathscr{G})=H^{p}\left(\operatorname{Hom}_{\Omega_{\dot{X}}^{\bullet}}^{\bullet}(\mathscr{F}, \mathscr{I} \bullet)_{\text {tot }}\right)\right),
$$

where an object of $\mathscr{C}(X)$ is naturally regarded as a complex of abelian sheaves on $X$, and $H^{p}$ is taken in that sense. This definition is not exactly the original one of Herrera and Lieberman [HL, §3], but leads to isomorphic objects; this subtlety on the definition of hyperext functors will be explained in section 2 below.

The local (resp. global) hyperext functors are limits of a spectral sequence

$$
\mathscr{E}_{1}^{p, q}=R^{q} \mathscr{H} o m_{\Omega_{\dot{X}}}^{p}(\mathscr{F}, \mathscr{G})=: \mathscr{E} x t_{\Omega_{\dot{X}}}^{p, q}(\mathscr{F}, \mathscr{G}) \Longrightarrow \mathscr{E}^{p+q}=\underline{\underline{\mathscr{E}} x t}_{\Omega_{\dot{X}}^{\dot{\dot{C}}}}^{p+q}(\mathscr{F}, \mathscr{G})
$$

(resp.

$$
E_{1}^{p, q}=R^{q} \operatorname{Hom}_{\Omega_{\dot{X}}^{\mathscr{~}}}^{p}(\mathscr{F}, \mathscr{G})=: \operatorname{Ext}_{\Omega_{\dot{X}}}^{p, q}(\mathscr{F}, \mathscr{G}) \Longrightarrow E^{p+q}=\underline{\underline{\operatorname{Ext}}}_{\Omega_{\dot{X}}^{p+q}}^{p+\mathscr{F}, \mathscr{G}) \quad), ~, ~}
$$

obtained as the first spectral sequence of the bicomplex under consideration.
As a particular case we obtain, for any object $\mathscr{F}^{\bullet}$, the functors

$$
H^{p}(\mathscr{F} \bullet)=\underline{\underline{\mathscr{E}} x t}_{\Omega_{X}^{p}}^{p}\left(\Omega_{X}^{\bullet}, \mathscr{F} \bullet\right)
$$

and

$$
\mathbf{H}^{p}(X, \mathscr{F} \bullet)={\underline{\operatorname{Ext}_{x_{X}}^{p}}}_{\dot{X}}^{p}\left(\Omega_{X}^{\bullet}, \mathscr{F} \bullet\right)
$$

In the last case, the spectral sequence above is the usual first spectral sequence of hypercohomology

$$
H^{q}\left(X, \mathscr{F}^{p}\right) \Longrightarrow \mathbf{H}^{p+q}(X, \mathscr{F} \bullet)
$$

The hyperext functors naturally extend to functors

$$
\underline{\mathscr{E} x t}^{p} \Omega_{X}^{\bullet}(-,-): \operatorname{Pro} \mathscr{C}(X)^{\circ} \times \operatorname{Ind} \mathscr{C}(X) \longrightarrow \operatorname{Ind} \mathcal{A} b(X)
$$

and

$$
\underline{\underline{\mathscr{E} x t}}_{\Omega_{X}^{\bullet}}^{p}(-,-): \operatorname{Ind} \mathscr{C}(X)^{\circ} \times \operatorname{Pro} \mathscr{C}(X) \longrightarrow \operatorname{Pro} \mathcal{A} b(X)
$$

(resp.

$$
\underline{\underline{\operatorname{Ext}}}_{\Omega_{X}^{\bullet}}^{p}(-,-): \operatorname{Pro} \mathscr{C}(X)^{\circ} \times \operatorname{Ind} \mathscr{C}(X) \longrightarrow \operatorname{Ind} \mathcal{A} b
$$

and

$$
\left.\underline{\underline{\operatorname{Ext}}}_{\Omega_{X}}^{p}(-,-): \operatorname{Ind} \mathscr{C}(X)^{\circ} \times \operatorname{Pro} \mathscr{C}(X) \longrightarrow \operatorname{Pro} \mathcal{A} b\right)
$$

0.6. Direct image with compact supports (Deligne). We recall that in the appendix to [H.RD], Deligne defines for an open immersion $j: U \hookrightarrow X$ of locally noetherian schemes the functor $j$ ! ("prolongement par zéro") in the following way. Let $\mathscr{F}$ be a coherent $\mathscr{O}_{U}$-Module, and take $\overline{\mathscr{F}}$ a coherent extension on $X$; let $\mathscr{I}$ the coherent Ideal of $\mathscr{O}_{X}$ defining $X \backslash U$. Then $j!\mathscr{F}$ is the pro-coherent sheaf on $X$ given by "lim" ${ }_{N} \mathscr{I}^{N} \overline{\mathscr{F}}$. Deligne proves that the definition is independent of the choice of the coherent extension $\overline{\mathscr{F}}$.

The functor $j!: \operatorname{Coh}\left(\mathscr{O}_{U}\right) \rightarrow \operatorname{ProCoh}\left(\mathscr{O}_{X}\right)$ naturally extends to the category $\operatorname{ProCoh}\left(\mathscr{O}_{U}\right)$ by the condition of commuting to all projective "limits".

The functor $j$ ! extends to the subcategory of $\mathscr{C}_{p c}(U)$ whose objects (regarded as complexes) are bounded below, with values in $\mathscr{C}_{p c}(X)$ and preserves homotopical equivalence of morphisms. If $f, g: \mathscr{E} \rightarrow \mathscr{F}$ are homotopically equivalent morphisms in $\mathscr{C}_{p c}(X)$, then we have

$$
R \varliminf_{\rightleftarrows} H^{i}(f)=R \lim _{\leftrightarrows} H^{i}(g): R \lim H^{i}(\mathscr{E}) \longrightarrow R \npreceq H^{i}(\mathscr{F})
$$

(maps of abelian sheaves) and

$$
R \varliminf_{\leftrightarrows} \mathbf{H}^{i}(X, f)=R \npreceq \varliminf^{i}(X, g): R \varliminf_{\leftrightarrows} \mathbf{H}^{i}(X, \mathscr{E}) \longrightarrow R \varliminf_{\mathbf{H}^{i}}(X, \mathscr{E})
$$

(maps of abelian groups).
0.7. Cousin complex. We recall from [H.RD, chap. IV], that for any abelian sheaf $\mathscr{F}$ one functorially defines a complex $E^{\bullet}(\mathscr{F})$ (the Cousin complex of $\mathscr{F}$ ) uniquely defined by suitable conditions of support w.r.t. the stratification of $X$ by the codimension of its points (see [H.RD, IV.2.3] or [H.75, II.2]). Moreover there is a functorial augmentation morphism $\mathscr{F} \rightarrow E^{\bullet}(\mathscr{F})$, where $\mathscr{F}$ is regarded as a complex concentrated in degree zero. The functor restricts to a functor from the category of $\mathscr{O}_{X}$-Modules into complexes of such (i.e. with $\mathscr{O}_{X}$-linear differentials).

Under suitable conditions on the sheaf $\mathscr{F}$ (see [H.RD,IV.2.6]), the Cousin complex is a flabby resolution of $\mathscr{F}$.

Moreover, if $X$ is smooth $E^{\bullet}\left(\mathscr{O}_{X}\right)$ admits an explicit description (see [H.RD, example p. 239]) proving that it is an injective resolution of $\mathscr{O}_{X}$. In particular if $\mathscr{F}$ is a locally free $\mathscr{O}_{X}$-Module of finite type, we have a canonical isomorphism of complexes $E^{\bullet}(\mathscr{F}) \cong E^{\bullet}\left(\mathscr{O}_{X}\right) \otimes_{\mathscr{O}_{X}} \mathscr{F}$. In fact these complexes both satisfy the conditions to be the Cousin complex of $\mathscr{F}$, so that by [H.RD, IV.3.3] the canonical morphism is an isomorphism.

As in [H.75] we extend the definition of Cousin complex by associating to each complex $\mathscr{F} \bullet$ of $\mathscr{O}_{X^{-}}$ Modules (with $\mathbb{Z}$-linear differentials) the total complex $E(\mathscr{F} \bullet)$ associated to the double complex $E \bullet(\mathscr{F} \bullet)$ defined by the Cousin complexes of its components.
0.7.1. Notice that $E^{r}\left(\Omega_{X}^{\bullet}\right)$, for any $r \in \mathbb{N}$, and $E\left(\Omega_{X}^{\bullet}\right)$ are naturally objects of $\mathscr{C}_{q c}(X)$. More generally, if $D: \mathscr{F} \rightarrow \mathscr{G}$ is a differential operator then, for any $r, E^{r}(D): E^{r}(\mathscr{F}) \rightarrow E^{r}(\mathscr{G})$ is a differential operator of the same order (this is easly seen using [EGA IV,16.8.8]).

## 1. De Rham Cohomology with compact supports.

1.1. Setting. Let $j: X \longrightarrow \bar{X}$ be an open immersion of the scheme $X$ into a proper scheme $\bar{X}$, and assume $i: \bar{X} \longrightarrow P$ is a closed immersion in a scheme $P$ smooth in a neighborhood of $X$. We have then the following embeddings

$$
X \stackrel{j}{\longleftrightarrow} \bar{X} \stackrel{i}{\longrightarrow} P
$$

Let $C$ be the complement of $X$ in $\bar{X}$ endowed with some structure of closed subscheme of $\bar{X}$, and let $h: C \rightarrow \bar{X}$ be the closed immersion. We do not suppose that $X$ be dense in $\bar{X}$ (as the symbol may suggest), even if we can always reduce to that case, replacing $\bar{X}$ by the closure of $X$ in $\bar{X}$.
1.1.1. Let $W$ be an open smooth subscheme of $P$ containing $X$ as a closed subset. The closed immersion $X \hookrightarrow W$ can be used to calculate the De Rham cohomology and homology of $X$, as defined by Hartshorne [H.75]. More generally, any open subscheme $W^{\prime}$ of $P$ containing $X$ as a closed subscheme (e.g. the not necessarily smooth open subscheme $P \backslash C$ ) would work for Hartshorne's computation of De Rham homology and cohomology. In fact, $W^{\prime}$ would contain an open smooth subscheme $W$ containing $X$ as a closed subscheme. The open immersion $u: W \hookrightarrow W^{\prime}$ then induces isomorphisms of infinitesimal neighbourhoods $X_{W}^{(N)} \xrightarrow{\cong} X_{W^{\prime}}^{(N)}$ of $X$ in $W$ and $W^{\prime}$, respectively. On the other hand, the trace map $\operatorname{Tr}_{u}$ induces an isomorphism of Cousin complexes $\operatorname{Tr}_{u}: \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right) \stackrel{\cong}{\Longrightarrow} \Gamma_{X} E\left(\Omega_{W^{\prime}}^{\bullet}\right)$.
1.1.2. Consider now the infinitesimal neighborhoods of $X$ and $\bar{X}$ in $W$ and $P$, respectively. In the following diagram

the two squares are cartesian. If we put $\mathscr{I}_{X} \stackrel{j^{(M)}}{=} \mathscr{I}_{X \subseteq W}, \mathscr{I}_{\bar{X}}^{h(M)}=\mathscr{I}_{\bar{X} \subseteq P}$ and $\mathscr{I}_{C}=\mathscr{I}_{C \subseteq P}$, the ideal of the closed immersion $h^{(M)}$ is $\mathscr{I}_{M}:=\mathscr{I}_{C \subseteq \bar{X}_{P}^{(M)}}=\left(\mathscr{I}_{C}+\mathscr{I}_{\bar{X}}^{M+1}\right) / \mathscr{I}_{\bar{X}}^{M+1}$.
1.2. Definition. In the previous notation, we define the De Rham cohomology of $X$ with compact supports $H_{\mathrm{DR}, c}^{\bullet}(X)$ as the hypercohomology of the simple complex of abelian sheaves on $\bar{X}$ associated to the bicomplex

$$
\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \longrightarrow h_{*}\left(\Omega_{P}^{\bullet}\right)_{/ C},
$$

where $\left(\mathscr{O}_{P}\right)_{/ \bar{X}}$ sits in bidegree $(0,0)$. Namely, $H_{\mathrm{DR}, c}^{\bullet}(X)=\mathbf{H}^{\bullet}\left(\bar{X},\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow h_{*}\left(\Omega_{P}^{\bullet}\right)_{/ C}\right)_{\text {tot }}\right)$.
1.3. Proposition. We may rewrite the previous definition as

$$
\begin{aligned}
& \cong \mathbf{H}^{\bullet}\left(\bar{X}, \underset{M}{\operatorname{Rim}} j_{!}^{(M)} \Omega_{X_{W}^{(M)}}^{\bullet}\right) \cong{\underset{M}{M}}_{\lim ^{\bullet}} \mathbf{H}^{\bullet}\left(\bar{X}, j_{!}^{(M)} \Omega_{X_{W}^{(M)}}\right)
\end{aligned}
$$

where $\mathscr{I}_{C} \supseteq \mathscr{I}_{\bar{X}}$ are the Ideals of $\mathscr{O}_{P}$ corresponding to the closed subschemes $C \subseteq \bar{X}$, respectively.
We are using the compact notation $\mathscr{I}_{C}^{N-} \bullet \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-}$ • for the complex

$$
0 \longrightarrow \mathscr{I}_{C}^{N} / \mathscr{I}_{\bar{X}}^{M} \longrightarrow \mathscr{I}_{C}^{N-1} \Omega_{P}^{1} / \mathscr{I}_{\bar{X}}^{M-1} \Omega_{P}^{1} \longrightarrow \mathscr{I}_{C}^{N-2} \Omega_{P}^{2} / \mathscr{I}_{\bar{X}}^{M-2} \Omega_{P}^{2} \longrightarrow \ldots
$$

and similarly for other complexes of this type.
Proof. For each $M \geqslant N$ the short exact sequence of complexes

$$
0 \longrightarrow \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \longrightarrow \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \longrightarrow \Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet} \longrightarrow 0
$$

gives the exact sequence of projective systems of complexes

$$
0 \longrightarrow\left\{\mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right\}_{M \geqslant N} \longrightarrow\left\{\Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right\}_{M} \longrightarrow\left\{\Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet}\right\}_{N} \longrightarrow 0
$$

and the isomorphism in $\operatorname{Pro} \mathscr{C}_{c}(P)$

$$
" \breve{M \geqslant N}^{\lim _{M}} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \xrightarrow{\cong} \operatorname{limm}_{M \geqslant N} "\left(\Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \rightarrow \Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet}\right)_{\text {tot }}
$$

We apply the functor $\mathbf{R} \underset{\rightleftarrows}{\leftrightarrows}$ to get

$$
\underset{M \geqslant N}{\mathrm{R} \varliminf_{\overparen{m}}} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \cong\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow\left(\Omega_{P}^{\bullet}\right) / C\right)_{\text {tot }},
$$

where $\left(\mathscr{O}_{P}\right)_{/ \bar{X}}$ sits in bidegree $(0,0)$. This proves the first isomorphism of the proposition.
If we take hypercohomology first and then projective limits we obtain instead isomorphisms

We now have the following generalization of [H.75, I.4.5] to suitable complexes of abelian sheaves, already used in the proof of [loc. cit., III.5.2]. For further use in the rigid-analytic context, we express the result for $G$-topological spaces. We recall that a base $\mathfrak{B}$ for a $G$-topology on $X$ is a class of admissible open subsets such that for any admissible open subset there is an admissible covering with elements in $\mathfrak{B}$.
1.3.1. Lemma. Let $\left(F_{n}^{\bullet}\right)_{n \in \mathbb{N}}$ be an inverse system of complexes in degrees $\geqslant 0$ of abelian sheaves on the $G$-topological space $X$. Let $T$ be a functor on the category of complexes of abelian sheaves on $X$, taking its values in an abelian category $\mathscr{A}$, where arbitrary direct products exist. We assume that the functor $T$ commutes with arbitrary direct products and that there is a base $\mathfrak{B}$ for the $G$-topology of $X$ such that:
(a) For each $U \in \mathfrak{B}$, the inverse system $\left(F_{n}^{\bullet}(U)\right)_{n}$ is surjective,
(b) For each $U \in \mathfrak{B}, H^{i}\left(U, F_{n}^{j}\right)=0$ for all $i>0$ and all $j, n$.

Then, for each $i$, there is an exact sequence

$$
0 \longrightarrow \varliminf_{\rightleftarrows}{ }^{(1)} \mathbf{R}^{i-1} T\left(F_{n}^{\bullet}\right) \longrightarrow \mathbf{R}^{i} T\left(\lim _{\leftrightarrows} F_{n}^{\bullet}\right) \xrightarrow{\alpha_{i}^{\bullet}} \varliminf_{\rightleftarrows}^{\lim } \mathbf{R}^{i} T\left(F_{n}^{\bullet}\right) \longrightarrow 0
$$

In particular, if for some $i, \mathbf{R}^{i-1} T\left(F_{n}^{\bullet}\right)$ satisfies the Mittag-Leffler condition, then $\alpha_{i}^{\bullet}$ is an isomorphism.
Proof (LEMMA). One may reason precisely as in the case of an ordinary topological space, and simply follow the proof of [H.75, I.4.5], taking into account the structure of injectives in the category of complexes in degrees $\geqslant 0$ over any abelian category [ T, II.2.4].

We apply the previous lemma to the topological space $P$ (or $\bar{X}$, if one prefers), the functor $\Gamma(P,-)$ and the simple complex
(More precisely, one takes $F_{N}^{\bullet}=\left(\Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{2 N-\bullet} \rightarrow \Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet}\right)_{\text {tot }}$ in the lemma.)
We observe that, since the coherent sheaves appearing in the complex have support in the proper subscheme $\bar{X}$, the $K$-vector spaces $\mathbf{H}^{i}\left(P,\left(\Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{2 N-\bullet} \rightarrow \Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet}\right)_{\text {tot }}\right)$ are finite dimensional, so that they satisfy, for variable $N$, the Mittag-Leffler condition. Therefore
which is $H_{\mathrm{DR}, c}^{\bullet}(X)$ by definition. This proves the isomorphisms in the first line of the statement. To check the isomorphism on the second line, we write in the notation of 1.1

$$
"{\underset{M}{\lim } "}_{{ }_{M}} j^{(M)} \Omega_{X_{W}^{(M)}}^{\bullet( }=\underset{M \geqslant N}{" \lim _{\overleftarrow{\prime}}} \mathscr{I}_{M}^{N-\bullet} \Omega_{\bar{X}_{P}^{(M)}} \cong " \underset{M \geqslant N}{\lim "} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \text {. }
$$

In order to prove that the given definition of De Rham cohomology with compact supports is good, we need some enhancements to proposition II.1.1 of [H.75].
1.4. Lemma. Let $Z \hookrightarrow X \hookrightarrow Y$ be a sequence of closed immersions of schemes. For any $M \geqslant 0$, we have a cartesian diagram of closed immersions


Proof. The equality $Z_{Y}^{(M)}=Z_{X_{Y}^{(M)}}^{(M)}$ follows from

$$
\left(\mathscr{O}_{Y} / \mathscr{I}_{X \subset Y}^{M+1}\right) /\left(\mathscr{I}_{Z \subset Y} / \mathscr{I}_{X \subset Y}^{M+1}\right)^{M+1} \cong \mathscr{O}_{Y} / \mathscr{I}_{Z \subset Y}^{M+1}
$$

The square is cartesian because so is

1.5. In the situation of the previous lemma, for any section $s: X_{Y}^{(M)} \rightarrow X$ of $X \hookrightarrow X_{Y}^{(M)}$, there is a unique section $s^{(M)}: Z_{Y}^{(M)} \rightarrow Z_{X}^{(M)}$ of $i^{(M)}: Z_{X}^{(M)} \hookrightarrow Z_{Y}^{(M)}$ fitting in a commutative diagram, necessarily cartesian,


To the morphism $i$ (resp. s) we associate the $\mathscr{C}_{c}\left(X_{Y}^{(M)}\right)$-morphism

$$
T_{i}: \Omega_{X_{Y}^{(M)}}^{\bullet} \longrightarrow i_{*} \Omega_{X}^{\bullet}
$$

(resp. the $\mathscr{C}_{c}(X)$-morphism

$$
\left.T_{s}: \Omega_{X}^{\bullet} \longrightarrow s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet}\right)
$$

Similarly, to the morphism $i^{(M)}$ (resp. $s^{(M)}$ ) we associate the $\mathscr{C}_{c}\left(Z_{Y}^{(M)}\right)$-morphism

$$
T_{i(M)}: \Omega_{Z_{Y}^{(M)}}^{\bullet} \longrightarrow i_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet}
$$

(resp. the $\mathscr{C}_{c}\left(Z_{X}^{(M)}\right)$-morphism

$$
\left.T_{s^{(M)}}: \Omega_{Z_{X}^{(M)}}^{\bullet} \longrightarrow s_{*}^{(M)} \Omega_{Z_{Y}^{(M)}}^{\bullet} \quad\right)
$$

Since $s \circ i=\operatorname{id}_{X}\left(\right.$ resp. $\left.s^{(M)} \circ i^{(M)}=\operatorname{id}_{Z_{X}^{(M)}}\right)$, we have

$$
s_{*}\left(T_{i}\right) \circ T_{s}=\operatorname{id}_{\Omega_{X}^{\bullet}}
$$

(resp.

$$
\left.s_{*}^{(M)}\left(T_{i(M)}\right) \circ T_{s^{(M)}}=\operatorname{id}_{\Omega_{X}^{\bullet}}\right)
$$

We will be interested in the composite $\mathscr{C}_{c}(X)$-morphism (resp. $\mathscr{C}_{c}\left(Z_{X}^{(M)}\right)$-morphism)

$$
\begin{equation*}
s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet} \xrightarrow{s_{*}\left(T_{i}\right)} \Omega_{X}^{\bullet} \xrightarrow{T_{s}} s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet} \tag{1.5.2}
\end{equation*}
$$

(resp.

$$
\begin{equation*}
\left.s_{*}^{(M)} \Omega_{Z_{Y}^{(M)}} \xrightarrow{s_{*}^{(M)}\left(T_{i}(M)\right)} \Omega_{Z_{X}^{(M)}}^{\bullet} \xrightarrow{T_{s}(M)} s_{*}^{(M)} \Omega_{Z_{Y}^{(M)}} \quad\right) \tag{1.5.3}
\end{equation*}
$$

1.5.4. Local situation. In the notation of the previous lemma, assume furthermore that $X$ and $Y$ are affine and smooth. Let $I$ be the ideal of $\mathscr{O}(Y)$ corresponding to $X \hookrightarrow Y$ and assume moreover that the $\mathscr{O}(X)$ module $I / I^{2}$ is free on the generators $x_{1}, \ldots, x_{n}$. Let us denote by $D_{n}^{(M)}:=\operatorname{Spec} K\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1}, \ldots, x_{n}\right)^{M+1}$, the $M$-th infinitesimal neighborhood of the origin in the affine $K$-space of dimension $n$. Then a section $s$ as in (1.5.1) certainly exists and that cartesian diagram can be identified with the standard diagram

$$
\begin{array}{rlll}
X_{Y}^{(M)} & \cong & D_{n}^{(M)} \times X & \mathrm{pr}_{2}  \tag{1.5.5}\\
\beta^{(M)} \uparrow & X \\
\mathrm{id}_{D_{d}^{(M)}} \times \alpha^{(M)} \mid & & \\
Z_{Y}^{(M)} & & & \\
\cong & \alpha_{n}^{(M)} \times Z_{X}^{(M)} \xrightarrow[\mathrm{pr}_{2}]{\longrightarrow} & Z_{X}^{(M)}
\end{array}
$$

1.5.6. LEMMA. In the local situation above, the composite morphism $T_{s^{(M)}} \circ s_{*}^{(M)}\left(T_{i(M)}\right)$ in formula (1.5.2) is homotopic to the identity of $s_{*}^{(M)} \Omega_{Z_{Y}^{(M)}}^{\bullet}$ in the category $\mathscr{C}_{C}\left(Z_{X}^{(M)}\right)$.

Proof. The canonical morphisms

$$
D_{n}^{(M)} \xrightarrow{\sigma} \operatorname{Spec} K \xrightarrow{\iota} D_{n}^{(M)},
$$

where $\iota$ corresponds to the canonical augmentation $K\left[x_{1}, \ldots, x_{n}\right] /\left(x_{1}, \ldots, x_{n}\right)^{M+1} \longrightarrow K$, fit in the diagram with cartesian squares

where $\pi: X \longrightarrow$ Spec $K$ is the structural morphism.
We have as usual a $\mathscr{C}_{c}\left(D_{n}^{(M)}\right)$-morphism

$$
T_{\iota}: \Omega_{D_{n}^{(M)}}^{\bullet} \longrightarrow \iota_{*} K=K_{D_{n}^{(M)}}=K
$$

and a morphism of $K$-vector spaces

$$
T_{\sigma}: K \longrightarrow \sigma_{*} \Omega_{D_{n}^{(M)}}^{\bullet}
$$

We observe that $\Omega_{D_{n}^{(M)}}^{\bullet}$ is freely generated over $K=K_{D_{n}^{(M)}}$ by its global sections

$$
\left\{\underline{x}^{\underline{\alpha}} d x_{\lambda_{1}} \wedge \cdots \wedge d x_{\lambda_{p}}|p+|\underline{\alpha}| \leqslant M\}\right.
$$

We have

$$
T_{\iota}\left(\underline{x}^{\underline{\alpha}} d x_{\lambda_{1}} \wedge \cdots \wedge d x_{\lambda_{p}}\right)= \begin{cases}1 & \text { if } \underline{\alpha}=0 \text { and } p=0 \\ 0 & \text { otherwise }\end{cases}
$$

while

$$
T_{\sigma}(1)=1
$$

So, $\sigma_{*}\left(T_{\iota}\right) \circ T_{\sigma}=\operatorname{id}_{K}$, while

$$
\begin{equation*}
\operatorname{id}_{\sigma_{*} \Omega_{D_{n}^{(M)}}}-T_{\sigma} \circ \sigma_{*}\left(T_{\iota}\right)=d \circ \vartheta+\vartheta \circ d \tag{1.5.8}
\end{equation*}
$$

where $d:=d_{D_{n}^{(M)}}$ and

$$
\vartheta: \sigma_{*} \Omega_{D_{n}^{(M)}}^{\bullet} \longrightarrow \sigma_{*} \Omega_{D_{n}^{(M)}}^{\bullet}[-1]
$$

is the morphism of $K$-vector spaces such that

$$
\vartheta\left(\underline{x}^{\underline{\alpha}} d x_{\lambda_{1}} \wedge \cdots \wedge d x_{\lambda_{p}}\right)= \begin{cases}0 & \text { if } \underline{\alpha}=0 \text { and } p=0 \\ \frac{1}{p+|\underline{\alpha}|} \sum_{j=1}^{p}(-1)^{j+1} \underline{x}^{\underline{\alpha}} d x_{\lambda_{1}} \wedge \cdots \wedge \widehat{d x_{\lambda_{j}}} \wedge \cdots \wedge d x_{\lambda_{p}} & \text { otherwise } .\end{cases}
$$

We now observe that $\pi^{*} \sigma_{*} \Omega_{D_{n}^{(M)}}^{\bullet}=s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet}$, that $\pi^{*}\left(T_{\sigma}\right)=T_{s}$ and that $\pi^{*} \sigma_{*}\left(T_{\sigma}\right)=s_{*}\left(T_{i}\right)$ and take

$$
\vartheta_{X}:=\pi^{*}(\vartheta): s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet} \longrightarrow s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet}[-1],
$$

a morphism of graded $\Omega_{X}^{\bullet}$-Modules. Applying the functor $\pi^{*}$ to (1.5.8) we conclude that

$$
\operatorname{id}_{s_{*} \Omega_{X_{Y}^{(M)}}^{\bullet}}-T_{s} \circ s_{*}\left(T_{i}\right)=d_{X_{Y}^{(M)} / X} \circ \vartheta_{X}+\vartheta_{X} \circ d_{X_{Y}^{(M)} / X}=d_{X_{Y}^{(M)}} \circ \vartheta_{X}+\vartheta_{X} \circ d_{X_{Y}^{(M)}}
$$

by (0.5.9). This proves the claim.
Now we can handle the case of smooth morphisms, which will be used in the proof of the theorem.
1.6. Proposition. Let $f: X \rightarrow Y$ be a smooth morphism of smooth schemes. Let $Z$ be a closed subscheme of $X$ such that the composition with $f$ gives a closed immersion of $Z$ in $Y$. Then the canonical map

$$
T_{f^{(M)}}: \Omega_{Z_{Y}^{(M)}}^{\bullet} \longrightarrow f_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet}
$$


Proof. Since the morphism $f: X \rightarrow Y$ is smooth, and $Z$ is a closed subset of both $X$ and $Y$, we may factorize the diagram

$$
\begin{aligned}
Z \xrightarrow{i_{1}} & X \\
i_{2} \searrow & \downarrow f \\
& Y
\end{aligned}
$$

locally on $X$ and $Y$ at the points of $Z$, as

where $i_{1}^{\prime}$ and $i_{2}^{\prime}$ are closed immersions such that $i_{1}=i_{1}^{\prime} \circ i_{2}^{\prime}$, and $f^{\prime}$ is an étale morphism (see [H.75, II.1.3]). We may and will insist that $X$ be affine and that the ideal $\mathscr{I}$ of $Y^{\prime}$ in $X$ satisfy the condition that $\mathscr{I} / \mathscr{I}^{2}$ be a free $\mathscr{O}_{Y^{\prime}}$-Module. Taking the infinitesimal neighborhoods of $Z$ we have

$$
\begin{aligned}
& Z_{Y^{\prime}}^{(M)} \xrightarrow{i_{1}^{\prime(M)}} Z_{X}^{(M)} \\
& Z_{Y}^{(M)}
\end{aligned} \quad \swarrow f^{(M)}
$$

where $f^{\prime(M)}$ is an isomorphism. The morphism $i_{1}^{\prime(M)}$ admits therefore the retraction $s^{(M)}=f^{\prime(M)-1} f^{(M)}$, for which $f^{\prime(M)} s^{(M)}=f^{(M)}$. Notice that, $f^{\prime}$ being étale, the retraction $s^{(M)}$ comes from the unique section $s: Y_{X}^{\prime(M)} \rightarrow Y^{\prime}$ of the canonical inclusion $Y^{\prime} \hookrightarrow Y_{X}^{\prime(M)}$ fitting in the first commutative square here below. The second square below is obtained by taking $M$-th infinitesimal neighborhoods of $Z$ in the objects of the first.

(the lower maps of the above diagrams are given by the composition $Y_{X}^{\prime(M)} \hookrightarrow X \xrightarrow{f} Y$, the left vertical arrows are nilpotent closed immersions, while the right vertical maps are étale morphisms).

Now, lemma 1.5.6 applied to the closed immersion $i_{1}^{(M)}$ (and its section $s^{(M)}$ ) shows that the composite morphism

$$
s_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet} \xrightarrow{s_{*}^{(M)}\left(T_{i_{1}^{\prime}(M)}\right)} s_{*}^{(M)} i_{1 *}^{\prime(M)} \Omega_{Z_{Y^{\prime}}^{(M)}} \cong \Omega_{Z_{Y^{\prime}}^{(M)}} \xrightarrow{T_{s}(M)} s_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet}
$$

is homotopic to the identity of $s_{*}^{(M)} \Omega_{Z_{X}^{(M)}}$ in the category $\mathscr{C}_{c}\left(Z_{Y^{\prime}}^{(M)}\right)$, while

$$
s_{*}^{(M)}\left(T_{i_{1}^{\prime(M)}}\right) \circ T_{s^{(M)}}=\operatorname{id}_{\substack{Z_{Y^{\prime}}}}
$$

Taking direct images via $f^{\prime(M)}$, and identifying $f_{*}^{\prime(M)} \Omega_{Y_{Y^{\prime}}^{(M)}}$ with $\Omega_{Z_{Y}^{(M)}}^{\bullet}$, so that $f_{*}^{\prime(M)}\left(T_{s}(M)\right)$ is identified with $T_{f(M)}$, we conclude that

$$
f_{*}^{(M)}\left(T_{i_{1}^{\prime(M)}}\right) \circ f_{*}^{\prime(M)}\left(T_{s^{(M)}}\right)=\operatorname{id}_{\Omega_{Y}^{\bullet}}
$$

while

$$
f_{*}^{\prime(M)}\left(T_{s^{(M)}}\right) \circ f_{*}^{(M)}\left(T_{i_{1}^{\prime(M)}}\right): f_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet} \longrightarrow f_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet}
$$

is homotopically equivalent the identity of $f_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet}$. In other words

$$
f_{*}^{(M)}\left(T_{i_{1}^{\prime(M)}}\right): f_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet} \cong f_{*}^{\prime(M)} s_{*}^{(M)} \Omega_{Z_{X}^{(M)}}^{\bullet} \xrightarrow{f_{*}^{\prime(M)} s_{*}^{(M)}\left(T_{i_{1}^{\prime(M)}}^{(M)}\right.} f_{*}^{\prime(M)} \Omega_{Z_{Y^{\prime}}^{(M)}}^{\bullet} \cong \Omega_{Z_{Y}^{(M)}}^{\bullet}
$$

is a homotopic inverse of the canonical morphism $T_{f^{(M)}}$.
1.7. Lemma. Let $X$ be a scheme and $\mathscr{U}=\left\{U_{\alpha}\right\}$ be an open covering. For an object $\mathscr{F} \bullet$ of $\mathscr{C}_{p c}(X)$ we define the Čech (co-) complex $\mathscr{C} \cdot(\mathscr{U}, \mathscr{F})$ of $\mathscr{F} \bullet$ on $\mathscr{U}$ as follows:

$$
\mathscr{C}_{p}(\mathscr{U}, \mathscr{F} \bullet)=\bigoplus_{|\underline{\alpha}|=p+1} j_{\underline{\alpha}!} \mathscr{F}_{\mid U_{\underline{\alpha}}}
$$

where $\underline{\alpha}=\left(\alpha_{0}, \ldots, \alpha_{p}\right)$ is a multi-index, $U_{\underline{\alpha}}=\bigcap_{i} U_{\alpha_{i}}$ and $j_{\underline{\alpha}}$ is the inclusion of $U_{\underline{\alpha}}$ in $X$. The differentials $\mathscr{C}_{p}(\mathscr{U}, \mathscr{F} \bullet) \rightarrow \mathscr{C}_{p-1}(\mathscr{U}, \mathscr{F} \bullet)$ are defined as usual by the simplicial structure. Then we have a canonical augmentation morphism $\mathscr{C}_{0}(\mathscr{U}, \mathscr{F} \bullet) \rightarrow \mathscr{F} \bullet$ making $\mathscr{C} \cdot(\mathscr{U}, \mathscr{F} \bullet)$ into a left resolution of $\mathscr{F} \bullet$ in the category $\mathscr{C}_{p c}(X)$. In other words, the sequence

$$
\begin{equation*}
\cdots \longrightarrow \mathscr{C}_{2}\left(\mathscr{U}, \mathscr{F}^{\bullet}\right) \longrightarrow \mathscr{C}_{1}\left(\mathscr{U}, \mathscr{F}^{\bullet}\right) \longrightarrow \mathscr{C}_{0}\left(\mathscr{U}, \mathscr{F}^{\bullet}\right) \longrightarrow \mathscr{F}^{\bullet} \longrightarrow 0 \tag{1.7.1}
\end{equation*}
$$

is exact.
Proof. In order to prove the exactness of (1.7.1), we have to prove that for any $i$ the sequence in $\operatorname{ProCoh}(X)$ given by $\mathscr{C}_{\bullet}\left(\mathscr{U}, \mathscr{F}^{i}\right)$ is a left resolution of $\mathscr{F}^{i}$. Since all the constructions involved commute with the functor "lim" we may assume that each $\mathscr{F}^{i}$ is a coherent $\mathscr{O}_{X}$-Module. So, let $\mathscr{F}$ be in $\operatorname{Coh}(X)$. The complex

$$
\mathscr{C}_{\bullet}(\mathscr{U}, \mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow 0
$$

is exact if and only if for any injective quasi-coherent $\mathscr{O}_{X}$-Module $\mathscr{I}$ the sequence

$$
\mathscr{H} o m_{\mathscr{O}_{X}}(\mathscr{C} \cdot(\mathscr{U}, \mathscr{F}), \mathscr{I}) \longleftarrow \mathscr{H}_{\mathscr{O}_{X}}(\mathscr{F}, \mathscr{I}) \longleftarrow 0
$$

is exact. The exact functors $\mathscr{H}^{\prime} m_{\mathscr{O}_{X}}(-, \mathscr{I}): \operatorname{Coh}(X) \longrightarrow \mathrm{QCoh}(X)$, for $\mathscr{I}$ an injective of $\mathrm{QCoh}(X)$, form in fact a conservative family. Using the adjunction of $j_{\underline{\alpha}!}$ and $j_{\underline{\alpha}}^{-1}$ (see the Deligne appendix in [H.RD]) the last sequence is just the usual Čech resolution of the sheaf $\mathscr{H}_{a_{0}} \mathscr{\mathscr { O }}_{X}(\mathscr{F}, \mathscr{I})$
1.8. Theorem. The definition of $H_{\mathrm{DR}, c}^{\bullet}(X)$ is independent of the choice of the compactification $\bar{X}$ and of the closed immersion of $\bar{X}$ in $P$ smooth around $X$.

Proof. Given $X \rightarrow \bar{X}_{1} \rightarrow P_{1}$ and $X \rightarrow \bar{X}_{2} \rightarrow P_{2}$, and $C_{i}=\bar{X}_{i} \backslash X_{i}$, as in the definition, we consider $\bar{X}$ the closure of (the diagonal immersion of) $X$ in $\bar{X}_{1} \times \bar{X}_{2}$, and the product $P_{1} \times P_{2}$. So we have a diagram

where all the horizontal maps are closed immersions, except $X \rightarrow \bar{X}$ which is an open one. Moreover, since the natural morphisms from $X$ to $X \times \bar{X}_{2}$ and $\bar{X}_{1} \times X$ are closed immersions, we have that $C=\bar{X} \backslash X$ is contained in $C_{1} \times C_{2}$. We then obtain diagrams

for $i=1,2$. Therefore we are reduced to the case of a proper morphism $g: \bar{X}_{1} \rightarrow \bar{X}_{2}$ with $g^{-1}\left(C_{2}\right)=C_{1}$ (in particular $g\left(C_{1}\right) \subseteq C_{2}$ ) and a morphism $\bar{f}: P_{1} \rightarrow P_{2}$ restricting to a smooth morphism $f: W_{1} \rightarrow W_{2}$, if $W_{1}$ is taken to be sufficiently small. We have then the commutative diagram


As usual, $h_{i}: C_{i} \hookrightarrow \bar{X}_{i}$ will denote the closed immersions for $i=1,2$.
We have to prove that $\mathbf{H}^{\bullet}\left(\bar{X}_{1},\left(\left(\Omega_{P_{1}}^{\bullet}\right)_{/ \bar{X}_{1}} \rightarrow h_{1 *}\left(\Omega_{P_{1}}^{\bullet}\right)_{/ C_{1}}\right)_{\text {tot }}\right) \cong \mathbf{H}^{\bullet}\left(\bar{X}_{2},\left(\left(\Omega_{P_{2}}^{\bullet}\right)_{/ \bar{X}_{2}} \rightarrow h_{2 *}\left(\Omega_{P_{2}}^{\bullet}\right)_{/ C_{2}}\right)_{\text {tot }}\right)$. More precisely we have to prove that the canonical morphism

$$
\begin{equation*}
\left(\left(\Omega_{P_{2}}^{\bullet}\right)_{/ \bar{X}_{2}} \rightarrow h_{2 *}\left(\Omega_{P_{2}}^{\bullet}\right)_{/ C_{2}}\right)_{\text {tot }} \longrightarrow \mathbf{R} g_{*}\left(\left(\Omega_{P_{1}}^{\bullet}\right)_{/ \bar{X}_{1}} \rightarrow h_{1 *}\left(\Omega_{P_{1}}^{\bullet}\right)_{/ C_{1}}\right)_{\text {tot }} \tag{1.8.1}
\end{equation*}
$$

induces isomorphisms on the hypercohomology groups. We will show that the previous morphism is a quasiisomorphism of abelian sheaves. Taking infinitesimal neighborhoods of $X$ and $\bar{X}_{i}$ in $W_{i}$ and $P_{i}$ respectively, we have diagrams

where $j_{1}^{(M)}$ and $j_{2}^{(M)}$ are open immersions, $f^{(M)}$ is smooth and $\bar{f}^{(M)}$ is proper. By proposition 1.3 and remark 0.3 we may study the morphism

$$
\begin{equation*}
" \varliminf_{M, N}^{\lim } " \mathscr{I}_{C_{2}}^{N-\bullet} \Omega_{\bar{X}_{2 P_{2}}^{(M)}}^{\bullet} \longrightarrow \varliminf_{M, N} " \mathbf{R} \bar{f}_{*}^{(M)}\left(\mathscr{I}_{C_{1}}^{N-} \cdot \Omega_{\bar{X}_{1 P_{1}}^{(M)}}^{\bullet}\right) \tag{1.8.2}
\end{equation*}
$$

(corresponding to the morphism (1.8.1)). We explicitly recall, without proof, the following enhanced version of proposition 5 of Deligne's appendix to [H.RD] (whose proof depends on [EGA III, Prop. 3.3.1]).
1.8.3. Proposition. Let

be a cartesian diagram of noetherian schemes, where the horizontal maps are open immersions, $f$ and $\bar{f}$ are proper morphisms and $f$ is acyclic. Let $\mathscr{I}$ be an Ideal of $\mathscr{O}_{Y}$ defining the closed subset $Y \backslash V$, and let $\mathscr{J}$ denote the extension of the $\mathscr{O}_{Y}$-Ideal $\mathscr{I}$ to an Ideal of $\mathscr{O}_{X}$. Then, for any coherent $\mathscr{O}_{X}$-Modules $\mathscr{F}$
(i) if $k>0$, "lim" ${ }_{n} R^{k} \bar{f}_{*} \mathscr{J}^{n} \mathscr{F}=0$,
(ii) if $k=0$, for sufficiently big $n, \bar{f}_{*} \mathscr{J}^{n+1} \mathscr{F}=\mathscr{I} \bar{f}_{*} \mathscr{J}^{n} \mathscr{F}$.

From this proposition, we have that for any $M, i$ and any sufficiently big $\bar{N}$, there is a canonical isomorphism "lim" ${ }_{N} \mathbf{R} \bar{f}_{*}^{(M)} \mathscr{I}_{C_{1}}^{N+\bar{N}} \Omega_{\bar{X}_{1 P_{1}}^{(M)}}^{i} \cong " \varliminf_{\lim ^{\prime}}^{\leftrightarrows}{ }_{N} \mathscr{I}_{C_{2}}^{N} \bar{f}_{*}^{(M)}\left(\mathscr{I}_{C_{1}}^{\bar{N}} \Omega_{\bar{X}_{1}}^{i}{ }_{P_{1}}^{(M)}\right)$. We are then reduced to proving that, for any $M$, the canonical morphism
is a quasi-isomorphism. By 1.3 and flat base change, we may rewrite (1.8.4) as

$$
\begin{equation*}
j_{2!}^{(M)} \Omega_{X_{W_{2}}^{\bullet(M)}}^{\longrightarrow} \longrightarrow j_{2!}^{(M)} f_{*}^{(M)} \Omega_{X_{W_{1}}^{(M)}}^{\bullet} \tag{1.8.5}
\end{equation*}
$$

and we know by proposition 1.6 that $\Omega_{X_{W_{2}}^{(M)}}^{\bullet} \longrightarrow f_{*}^{(M)} \Omega_{X_{W_{1}}^{(M)}}$ is locally a homotopic isomorphism. So there exists an open covering $\mathscr{U}^{(M)}$ of $X_{W_{2}}^{(M)}$ such that for any $i$ the canonical morphism

$$
\mathscr{C}_{i}\left(\mathscr{U}^{(M)}, \Omega_{X_{W_{2}}^{(M)}}^{\bullet}\right) \longrightarrow \mathscr{C}_{i}\left(\mathscr{U}^{(M)}, f_{*}^{(M)} \Omega_{X_{W_{1}}^{(M)}}^{\bullet}\right)
$$

is a homotopic isomorphism. Therefore, applying the functor $j_{2!}$, which is exact in the category $\operatorname{ProCoh}(X)$, to the diagram

we obtain that (1.8.5) is a quasi-isomorphism in the category $\operatorname{Pro} \mathcal{A} b(\bar{X})$.
1.9. Remark. In the setting 1.1 we may suppose that $P$ is smooth. In fact, if

$$
X \longrightarrow \bar{X} \longrightarrow P
$$

are immersions as in 1.1, we can take a resolution of singularities à la Hironaka $\pi: P^{\prime} \rightarrow P$ (we are in characteristic zero), which restricts to an isomorphism on any open smooth subscheme $W$ of $P$, and therefore on $X$. Taking as $\bar{X}^{\prime}$ the closure of $X$ in $P^{\prime}$, we have a commutative diagram

where $\bar{X}^{\prime}$ (a closed subscheme of the inverse image by $\pi$ of $\bar{X}$ ) is proper and $j^{\prime}$ is an open immersion. We may then calculate the De Rham cohomology with compact supports of $X$ using the first line of the diagram, i.e. we can assume $P$ smooth. Notice that this can also be proven independently of the previous theorem, in a simpler way. In fact, let $W$ be any smooth open subscheme of $P$ containing $X$ as a closed subscheme. Then in the previous diagram $W$ is also an open subscheme of $P^{\prime}$ with the same property. Now,
for any $M$, the proper map $\pi$ induces a proper map $\pi^{(M)}: \bar{X}_{P^{\prime}}^{(M)} \longrightarrow \bar{X}_{P}^{(M)}$, and an isomorphism of functors $\mathbf{R} \pi_{*}^{(M)} \circ j_{!}^{\prime(M)} \xrightarrow{\cong} j_{!}^{(M)}$. We then have a morphism of spectral sequences

$$
\begin{array}{ccc}
H^{q}\left(\bar{X}_{P}^{(M)}, j_{!}^{(M)} \Omega_{X_{W}^{(M)}}^{p}\right) & \Longrightarrow \mathbf{H}^{p+q}\left(\bar{X}_{P}^{(M)}, j_{!}^{(M)} \Omega_{X_{W}^{(M)}}^{\bullet}\right) \\
\downarrow & \downarrow \\
H^{q}\left(\bar{X}_{P^{\prime}}^{\prime(M)}, j_{!}^{\prime(M)} \Omega_{X_{W}^{(M)}}^{p}\right) & \Longrightarrow \mathbf{H}^{p+q}\left(\bar{X}_{P^{\prime}}^{\prime(M)}, j_{!}^{(M)} \Omega_{X_{W}^{(M)}}^{\bullet}\right)
\end{array}
$$

The left hand arrow is an isomorphism, since both source and target identify with the cohomology groups with compact supports for coherent sheaves $H_{c}^{q}\left(X_{W}^{(M)}, \Omega_{X_{W}^{(M)}}^{p}\right)$ as defined in [H.72, §2]. In view of regularity, the limits of the two spectral sequences are also isomorphic.
1.10. Proposition. The De Rham cohomology with compact supports is a contravariant functor w.r.t. proper morphisms and a covariant functor w.r.t. open immersions.

Proof. For an open immersion $j: X_{1} \rightarrow X_{2}$ we may calculate the De Rham cohomologies with compact supports of $X_{1}$ and $X_{2}$ using an open immersion of $X_{2}$ into a proper scheme:

$$
\begin{aligned}
& X_{1} \\
& j \downarrow \\
& X_{2} \xrightarrow[j_{2}]{\searrow} \bar{X}_{2} \xrightarrow[i_{2}]{ } P_{2}
\end{aligned}
$$

since the diagonal arrow is again an open immersion. Now remark that $C_{2}=\bar{X}_{2} \backslash X_{2}$ is contained in $C_{1}=\bar{X}_{2} \backslash X_{1}$, so that $\mathscr{I}_{C_{1}} \subseteq \mathscr{I}_{C_{2}}$. The canonical commutative diagram

where $h_{i}$ is the closed immersion $C_{i} \rightarrow \bar{X}_{2}$, induces a natural morphism $H_{\mathrm{DR}, c}\left(X_{1}\right) \longrightarrow H_{\mathrm{DR}, c}\left(X_{2}\right)$.
Let now $h: X_{1} \rightarrow X_{2}$ be a proper morphism. As in the first step of the proof of the theorem, we may complete a diagram as

$$
\begin{gathered}
X_{1} \longrightarrow \bar{X}_{1} \longrightarrow P_{1} \\
h \downarrow \\
X_{2} \longrightarrow \bar{X}_{2} \longrightarrow P_{2}
\end{gathered}
$$

using $\bar{X}=$ the closure of the image of the canonical morphism $X_{1} \longrightarrow \bar{X}_{1} \times \bar{X}_{2}$, and the product $P=P_{1} \times P_{2}$. So we are reduced to a diagram of the form

where the first square is cartesian. In fact the canonical morphism $X_{1} \rightarrow g^{-1}\left(X_{2}\right)$ is clearly an open immersion and it is a proper morphism since its composition with $g_{\mid g^{-1}\left(X_{2}\right)}: g^{-1}\left(X_{2}\right) \rightarrow X_{2}$ is proper; therefore it is the identity. As a consequence we have that $g\left(C_{1}\right) \subseteq C_{2}$. We then have a commutative diagram

$$
\begin{array}{cc}
g_{*}\left(\Omega_{P_{1}}^{\bullet}\right) / \bar{X}_{1} & g_{*} h_{1 *}\left(\left(\Omega_{P_{1}}^{\bullet}\right) / C_{1}\right) \\
\uparrow & \uparrow \\
\left(\Omega_{P_{2}}^{\bullet}\right) / \bar{X}_{2} & \longrightarrow \\
h_{2 *}\left(\left(\Omega_{P_{2}}^{\bullet}\right) / C_{2}\right)
\end{array}
$$

so that we deduce a natural map $H_{\mathrm{DR}, c}\left(X_{2}\right) \longrightarrow H_{\mathrm{DR}, c}\left(X_{1}\right)$.
1.10.2. Remark. Notice that if $X$ is a proper scheme we have $H_{\dot{\mathrm{DR}, c}}^{\dot{ }}(X) \cong H_{\dot{\mathrm{DR}}}^{\dot{ }}(X)$, since we may choose $X=\bar{X}$ in the setting 1.1 of our definition. In general, let $j: X \rightarrow \bar{X}$ be the open immersion in 1.1. By the proposition we have a canonical map $H_{\mathrm{DR}, c}^{\bullet}(X) \rightarrow H_{\mathrm{DR}, c}^{\bullet}(\bar{X})=H_{\mathrm{DR}}^{\bullet}(\bar{X})$ (the last equality by properness of $\bar{X}$ ). Moreover, by the (contravariant) functoriality of De Rham cohomology (without supports) we have a
canonical morphism $H_{\mathrm{DR}}^{\bullet}(\bar{X}) \rightarrow H_{\mathrm{DR}}^{\dot{\bullet}}(X)$. Therefore, by composition, we have for any scheme $X$ a canonical morphism

$$
H_{\stackrel{\mathrm{DR}, c}{\bullet}}^{\bullet}(X) \longrightarrow H_{\mathrm{DR}, c}^{\bullet}(\bar{X})=H_{\dot{\mathrm{DR}}}^{\bullet}(\bar{X}) \longrightarrow H_{\mathrm{DR}}^{\bullet}(X) .
$$

This morphism is induced by the canonical morphism of complexes $\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \longrightarrow j_{*}\left(\left(\Omega_{W}^{\bullet}\right) / X\right)$ which defines a morphism

$$
\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow h_{*}\left(\Omega_{P}^{\bullet}\right)_{/ C}\right)_{\text {tot }} \longrightarrow j_{*}\left(\Omega_{W}^{\bullet}\right)_{/ X} ;
$$

taking hypercohomology gives the canonical morphism between De Rham cohomologies.
1.11. Proposition. Let $j: U \rightarrow X$ be an open immersion of schemes and $i: Z=X \backslash U \rightarrow X$ be the closed immersion of the complement, endowed with some closed subscheme structure. There exists a long exact sequence

$$
\cdots \longrightarrow H_{\mathrm{DR}, c}^{i-1}(Z) \longrightarrow H_{\mathrm{DR}, c}^{i}(U) \longrightarrow H_{\mathrm{DR}, c}^{i}(X) \longrightarrow H_{\mathrm{DR}, c}^{i}(Z) \longrightarrow H_{\mathrm{DR}, c}^{i+1}(U) \longrightarrow \cdots .
$$

Proof. We can choose a compactification $\bar{X}$ of $X$ and a closed immersion of $\bar{X}$ in a scheme $P$ smooth around $X$. We then construct the diagram

where we remark that $\bar{X} \backslash U$ is closed in $\bar{X}$ and contains $Z$ as an open subset; moreover $(\bar{X} \backslash U) \backslash Z=\bar{X} \backslash X$. Therefore, we are in the situation to calculate the three De Rham cohomologies with compact supports:

$$
\begin{aligned}
& \left.H_{\mathrm{DR}, c}(U)=\mathbf{H}^{\bullet}\left(\bar{X},\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X} \backslash U}\right)_{\mathrm{tot}}\right)={\left.\underset{M, N}{ } \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{I}_{\bar{X} \backslash U}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right)\right)}^{\overleftarrow{N}^{M}}\right) \\
& H_{\mathrm{DR}, c}(X)=\mathbf{H}^{\bullet}\left(\bar{X},\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X}} \rightarrow\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X} \backslash X}\right)_{\mathrm{tot}}\right)=\lim _{M, N} \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{I}_{\bar{X} \backslash X}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right) \\
& H_{\mathrm{DR}, c}(Z)=\mathbf{H}^{\bullet}\left(\bar{X} \backslash U,\left(\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X} \backslash U} \rightarrow\left(\Omega_{P}^{\bullet}\right)_{/ \bar{X} \backslash X}\right)_{\mathrm{tot}}\right)={\underset{M, N}{ } \lim _{M, N} \mathbf{H}^{\bullet}\left(\bar{X} \backslash U, \mathscr{I}_{\bar{X} \backslash X}^{N-\cdot} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X} \backslash U}^{M-\bullet}\right) .}^{M}
\end{aligned}
$$

From the exact sequences

$$
0 \longrightarrow \mathscr{I}_{\bar{X} \backslash U}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \longrightarrow \mathscr{I}_{\bar{X} \backslash X}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \longrightarrow \mathscr{I}_{\bar{X} \backslash X}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X} \backslash U}^{M-\bullet} \longrightarrow 0
$$

we have immediatly the conclusion taking the long exact sequence of hypercohomology.
1.11.1. Remark. It will follow from our Poincaré duality theorem below that the long exact sequence of the proposition is dual of the exact sequence of a closed subset for homology, see [H.75, II.3.3].
1.11.2. Remark. If $X$ is a proper scheme, then also $Z$ is proper and the exact sequence of the proposition can be written as

$$
\cdots \longrightarrow H_{\mathrm{DR}}^{i-1}(Z) \longrightarrow H_{\mathrm{DR}, c}^{i}(U) \longrightarrow H_{\mathrm{DR}}^{i}(X) \longrightarrow H_{\mathrm{DR}}^{i}(Z) \longrightarrow H_{\mathrm{DR}, c}^{i+1}(U) \longrightarrow \cdots
$$

because of the remark 1.10.2. This will be one ingredient for the proof of the Poincaré duality theorem, see below.
1.12. Proposition. Let $X$ be the union of two closed subschemes $X_{1}$ and $X_{2}$; then there exists a Mayer-Vietoris exact sequence for the De Rham cohomology with compact supports
$\cdots \longrightarrow H_{\mathrm{DR}, c}^{i-1}\left(X_{1} \cap X_{2}\right) \longrightarrow H_{\mathrm{DR}, c}^{i}(X) \longrightarrow H_{\mathrm{DR}, c}^{i}\left(X_{1}\right) \oplus H_{\mathrm{DR}, c}^{i}\left(X_{2}\right) \longrightarrow H_{\mathrm{DR}, c}^{i}\left(X_{1} \cap X_{2}\right) \longrightarrow H_{\mathrm{DR}, c}^{i+1}(X) \longrightarrow \cdots$
Proof. We can take a compactification $\bar{X}$ of $X$ which is the union of closed subschemes $\bar{X}_{1}$ and $\bar{X}_{2}$, compactifications of $X_{1}$ and $X_{2}$, respectively. (It suffices to replace $\bar{X}$ by the union of the closure of $X_{1}$ and of $X_{2}$ in $\bar{X}$ ). Since the $X_{i}$ are closed in $X$, we have that $C=\bar{X} \backslash X$ is the union of the $C_{i}=\bar{X}_{i} \backslash X_{i}$. The
proof of the Mayer-Vietoris sequence in De Rham cohomology (without supports), [H.75, II.4.1], gives the following diagram with exact rows

$$
\begin{gathered}
0 \longrightarrow\left(\Omega_{P}^{\bullet}\right) / \bar{X} \longrightarrow\left(\Omega_{P}^{\bullet}\right)^{\bullet} / \bar{X}_{1} \oplus\left(\Omega_{P}^{\bullet}\right) / \bar{X}_{2} \longrightarrow\left(\Omega_{P}^{\bullet}\right) / \bar{X}_{1} \cap \bar{X}_{2} \longrightarrow 0 \\
\downarrow \\
\downarrow \\
0 \longrightarrow\left(\Omega_{P}^{\bullet}\right)_{C} \longrightarrow\left(\Omega_{P}^{\bullet}\right)_{/ C_{1}} \oplus\left(\Omega_{P}^{\bullet}\right)_{/ C_{2}} \longrightarrow\left(\Omega_{P}^{\bullet}\right)_{/ C_{1} \cap C_{2}} \longrightarrow 0 .
\end{gathered}
$$

Then we can deduce the exact sequence of pro-complexes

$$
0 \longrightarrow \varliminf_{M, N} \frac{\lim _{C}}{\mathscr{I}_{\bar{X}}^{M-\bullet}} \longrightarrow \Omega_{M, N}^{\bullet} \varliminf_{\grave{X}} " \frac{\mathscr{I}_{C_{1}}^{N-\bullet} \Omega_{P}^{\bullet}}{\mathscr{I}_{\bar{X}_{1}}^{M-\bullet}} \oplus " \varliminf_{M, N} \frac{\mathscr{I}_{C_{2}}^{N-\bullet} \Omega_{P}^{\bullet}}{\mathscr{I}_{\bar{X}_{2}}^{M-\bullet}} \longrightarrow " \varliminf_{M, N} \frac{\mathscr{I}_{C_{1} \cap C_{2}}^{N-\bullet} \Omega_{P}^{\bullet}}{\mathscr{I}_{\bar{X}_{1} \cap \dot{\bar{X}}_{2}}^{M-\bullet}} \longrightarrow 0
$$

from which the exact sequence of Mayer-Vietoris of De Rham cohomology with compact supports follows.
1.13. Example: De Rham cohomology with compact supports of affine spaces. If $X=\mathbb{A}_{K}^{n}$, we can take $P=\bar{X}=\mathbb{P}_{K}^{n}$, and the exact sequence of the closed subset $C=\bar{X} \backslash X \cong \mathbb{P}_{K}^{n-1}$ is

$$
\cdots \longrightarrow H_{\mathrm{DR}}^{i-1}\left(\mathbb{P}_{K}^{n-1}\right) \longrightarrow H_{\mathrm{DR}, c}^{i}\left(\mathbb{A}_{K}^{n}\right) \longrightarrow H_{\mathrm{DR}}^{i}\left(\mathbb{P}_{K}^{n}\right) \xrightarrow{\delta} H_{\mathrm{DR}}^{i}\left(\mathbb{P}_{K}^{n-1}\right) \longrightarrow H_{\mathrm{DR}, c}^{i+1}\left(\mathbb{A}_{K}^{n}\right) \longrightarrow \cdots
$$

Now, for projective spaces we have $H_{\mathrm{DR}}^{i}\left(\mathbb{P}_{K}^{n}\right) \cong K$ for $0 \leqslant i \leqslant 2 n$ even, and 0 otherwise [H.75, II.7.1]; moreover, the coboundary operator $\delta$ is an isomorphism. Then we deduce

$$
H_{\mathrm{DR}, c}^{i}\left(\mathbb{A}_{K}^{n}\right) \cong \begin{cases}0 & \text { if } i \neq 2 n \\ K & \text { if } i=2 n\end{cases}
$$

1.14. Proposition. For any scheme $X$ of dimension $n$ we have that $H_{\mathrm{DR}, c}^{i}(X)=0$ for $i>2 n$.

Proof. In fact we can take the exact sequence of $X$ as an open subset of $\bar{X}$ with closed complement $C$, and apply the analogous results for ordinary De Rham cohomology of [H.75, II.7.2].

## 2. Hyperext functors and De Rham Homology.

In this section we recall the notion of De Rham homology, which will be essential in the proof of our duality theorem in the next section. For simplicity, in this and the next section we will assume, in the setting 1.1, to have chosen a smooth $P$ (see remark 1.9). In that case, we can use the classical Cousin complex of $\Omega_{P}^{\bullet}$, rather than the more complicated dualizing complex of Du Bois [DB.90], and we can make explicit its relevance in the calculation of certain hyperext groups of Herrera and Lieberman.
2.1. In the notation of our setting 1.1, since $W$ is a smooth scheme containing $X$ as a closed subscheme, the Hartshorne definition of De Rham homology [H.75, II.3] may be expressed as

$$
H_{\bullet}^{\mathrm{DR}}(X)=\mathbf{H}_{X}^{2 n-\bullet}\left(W, \Omega_{W}^{\bullet}\right)
$$

where $n$ is the dimension of $W$, and the right hand side indicates hypercohomology with support in $X$ of the complex $\Omega_{W}^{\bullet}$.
2.1.1. Using the resolution of $\Omega_{W}^{\bullet}$ given by the Cousin complex $E\left(\Omega_{W}^{\bullet}\right)$, Hartshorne interpretes his definition as

$$
H_{\bullet}^{\mathrm{DR}}(X) \cong \mathbf{H}^{2 n-\bullet}\left(W, \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)\right) \cong H^{2 n-\bullet}\left(\Gamma\left(W, \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)\right)\right)
$$

since $\Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)$ is a complex of injective $\mathscr{O}_{W}$-Modules ([H.75, II.3]).
2.2. In the following we will characterize the notion of De Rham homology in terms of the functors $\mathscr{H}$ om $\boldsymbol{\Omega}_{P}^{\bullet}(-,-)$ (which are important in order to understand the differentials of the complex) and $\mathscr{H}$ om $\dot{\mathscr{O}}_{P}(-,-)$ (which are related to the notion of support).
2.2.1. The ring change formula

$$
\mathscr{H} o m_{\mathscr{O}_{P}}\left(\mathscr{F}, \mathscr{M}^{\bullet}\right) \cong \mathscr{H}^{\circ} m_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{F} \otimes_{\mathscr{O}_{P}} \Omega_{P}^{\bullet}, \mathscr{M}^{\bullet}\right)
$$

where $\mathscr{F}$ is an $\mathscr{O}_{P}$-Module and $\mathscr{M}^{\bullet}$ is a graded $\Omega_{P}^{\bullet}$-Module, is an isomorphism of graded $\Omega_{P^{\bullet}}$-Modules. If moreover $\mathscr{M} \bullet$ is a $\mathscr{C}_{\dot{P}}$-Module and $\mathscr{F} \otimes_{\mathscr{O}_{P}} \Omega_{P}^{\bullet}$ admits a structure of $\mathscr{C}_{\dot{P}}$-Module (for example if $P$ is smooth and $\mathscr{F}$ is a left $\mathscr{D}_{P}$-Module), then the ring change formula gives a structure of $\mathscr{C}_{\dot{P}}^{\bullet}$-Module to $\mathscr{N} \bullet=\mathscr{H} o m_{\mathscr{O}_{P}}^{\bullet}\left(\mathscr{F}, \mathscr{M}^{\bullet}\right)$. Notice that in general $d_{\mathscr{N}_{\bullet}}^{i}$ is not induced by $\mathscr{H} o m_{K}^{\bullet}\left(i d{ }_{\mathscr{F}}, d_{\mathscr{M}}^{i}\right)$, since this last
operator does not preserve $\mathscr{O}_{P}$-linearity. As a particular case, we see that if $\mathscr{M} \bullet$ is a $\mathscr{C}_{P}$-Module and $Z$ is closed in $P$, then

$$
\Gamma_{Z} \mathscr{M}^{\bullet}:=\underset{N}{\lim } \mathscr{H}_{\mathrm{C}} \boldsymbol{\bullet}_{\mathscr{O}_{P}}\left(\mathscr{O}_{P} / \mathscr{I}_{Z}^{N}, \mathscr{M}^{\bullet}\right) \cong \underset{N}{\lim } \mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\Omega_{P}^{\bullet} / \mathscr{I}_{Z}^{N-\bullet}, \mathscr{M}^{\bullet}\right)
$$

is a $\mathscr{C}_{P}$-Module.
2.2.2. The same sort of phenomena are explored in detail in [HL], and [B.74, II.5.2] for the other ring change formula. Namely, for $\mathscr{F}$ an $\mathscr{O}_{P}$-Module and $\mathscr{M} \bullet$ an $\Omega_{P}^{\bullet}$-Module, we have a canonical isomorphism of $\Omega_{P}^{\bullet}$-Modules

$$
\mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{M}^{\bullet}, \mathscr{H o m}_{\mathfrak{O}_{P}}\left(\Omega_{P}^{\bullet}, \mathscr{F}\right)\right) \cong \mathscr{H}^{\bullet} m_{\mathscr{O}_{P}}\left(\mathscr{M}^{\bullet}, \mathscr{F}\right)
$$

If moreover $\mathscr{M} \bullet$ is a $\mathscr{C}_{P}^{\bullet}$-Module and $\mathscr{H}_{0} m_{\mathscr{O}_{P}}^{\bullet}\left(\Omega_{P}^{\bullet}, \mathscr{F}\right)$ admits a structure of $\mathscr{C}_{P}^{\bullet}$-Module (for example if $P$ is smooth and $\mathscr{F}$ is a right $\mathscr{D}_{P}$-Module), then the second ring change formula gives a structure of $\mathscr{C}_{\dot{P}}$-Module to $\mathscr{H} m_{\mathscr{O}_{P}}^{\bullet}\left(\mathscr{M}^{\bullet}, \mathscr{F}\right)$. A special case of this isomorphism is implicitely used by [H.75] in the proof of the duality theorem. Assuming $P$ smooth, if $\mathscr{F}=\Omega_{P}^{n}[-n]$ where $n=\operatorname{dim} P, \mathscr{H} o m_{\mathscr{O}_{P}}\left(\Omega_{P}^{\bullet}, \Omega_{P}^{n}[-n]\right) \cong \Omega_{P}^{\bullet}$, the isomorphism becomes
and the terms have a structure of $\mathscr{C}_{\dot{P}}^{\bullet}$-Module if $\mathscr{M} \cdot$ does (see [HL, 2.9]).
2.3. On the definition of HyperExt functors. This section is meant to justify the useful sign convention of $[\mathrm{HL}, \S 3]$, and to modify some incorrect statements in that section ${ }^{(1)}$. We also prove that the definition of hyperext groups of [B.74, II.5.4.3], which we adopted here, is equivalent to the original definition of [HL, §3].

We start with an easy lemma on complexes of $\mathscr{C}_{P}{ }^{\bullet}$-Modules.
2.3.1. Lemma. Let

$$
\mathscr{J} \bullet \bullet: \cdots \longrightarrow \mathscr{J} \bullet q \xrightarrow{d_{\mathscr{\prime}}^{\prime \prime \bullet}, q} \mathscr{J} \bullet, q+1 \longrightarrow \cdots
$$

be a complex of $\mathscr{C}_{\dot{P}}$-Modules, and let $d_{\mathscr{J}}^{\prime \bullet, q}$ indicate the differentials of each term $\mathscr{J}^{\bullet, q}$ (notice that $d_{\mathscr{J}}^{\prime}$ is a differential operator, while $d_{\mathscr{J}}^{\prime \prime}$ is an $\mathscr{O}_{P}$-linear map). We define a new complex $\widetilde{\mathscr{J}} \bullet \bullet$ of $\mathscr{C}_{P}$-Modules in the following way: for any $p$ and $q$ let $\widetilde{\mathcal{J}^{p}, q}=\mathscr{J}^{p, q}, d_{\widetilde{J}}^{\prime p, q}=(-1)^{q} d_{\mathscr{J}}^{\prime p, q}$ and $d_{\widetilde{\mathcal{J}}}^{\prime \prime p, q}=(-1)^{p} d_{\mathscr{J}}^{\prime \prime p, q}$. Then the canonical map $\sigma_{\mathscr{J}}^{\bullet \bullet}: \mathscr{J} \bullet \bullet \longrightarrow \widetilde{\mathscr{J}} \bullet \bullet$ defined by $\sigma_{\mathscr{J}}^{p, q}=(-1)^{p q_{\mathrm{id}_{\mathscr{F}}, q}}$ is an isomorphism of complexes of $\mathscr{C}_{P_{P}}$-Modules. Moreover, the functor sending $\mathscr{J}^{\bullet \bullet}$ to $\mathscr{J}^{\bullet \bullet}$ is an involution of the category of complexes of $\mathscr{C}_{P}^{\bullet}$-Modules.

Proof. Clearly, for any $q, \widetilde{\mathscr{J}} \cdot q$ with the differentials $d_{\widetilde{\mathscr{J}}}^{\prime \bullet, q}$ is a $\mathscr{C}_{\dot{P}}$-Module. Moreover we have the commutativity $d_{\widetilde{\mathcal{J}}}^{\prime} d_{\widetilde{\mathscr{J}}}^{\prime \prime}=d_{\widetilde{\mathscr{J}}}^{\prime \prime} d_{\widetilde{\mathscr{J}}}^{\prime}$, so that $\widetilde{\mathscr{J}} \bullet \bullet$ is a complex of $\mathscr{C}_{\mathbf{P}^{-}}$-Modules.

Therefore we have only to prove that $\sigma_{\mathscr{G}}^{\bullet \bullet \bullet}$ is a morphism of complexes of $\mathscr{C}_{\dot{P}}^{\bullet}$-Modules, that is it commutes with the differentials, which is an easy exercise.

We point out that the structure of $\Omega_{P}^{\bullet}$-Module on each $\widetilde{\mathcal{J}^{\bullet}, q}$ is given by

$$
\alpha \underset{\tilde{\mathscr{f}}, q}{\dot{\bullet}, q}{ }^{(-1)^{i q} \alpha \underset{\mathscr{J} \bullet, q}{ } u, ~}
$$

if $\alpha \in \Omega^{i}$ and $u \in \mathscr{J}^{\bullet, q}$.
2.3.2. Corollary. Let $\mathscr{F} \bullet$ be a $\mathscr{C}_{\dot{P}}$-Module and $\mathscr{J} \bullet \bullet$ a complex of $\mathscr{C}_{\dot{P}}$-Modules as before. Then we have canonical isomorphisms of complexes of $\mathscr{C}_{P}^{\bullet}$-Modules
where $\sigma_{\mathscr{F} *}=\mathscr{H} o m_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathrm{id} \mathscr{F}_{\bullet}, \sigma_{\mathscr{F}}\right), \sigma_{\mathscr{H} o m}^{p, q}=\sigma_{\mathscr{H} m_{\Omega_{P}^{\bullet}}^{p}}(\mathscr{F} \bullet, \tilde{\mathcal{F}} \cdot q)$ and

$$
\left.\left.{\widetilde{\mathscr{H} O} m_{\Omega_{P}^{\bullet}}^{p}}^{\mathscr{F}^{\bullet}}, \widetilde{\mathscr{J}}^{\bullet}, q\right)\right)_{p, q}=\left(\left({\mathscr{H} O m_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{F}^{\bullet}, \widetilde{\mathscr{J}}^{\bullet}, q\right.}_{)}\right)_{p, q}\right) \sim
$$

[^0]Proof. This is a consequence of the previous lemma.
We make explicit the differentials in the objects of the corollary in view of the next result. The differentials in the first term are given by $d^{\prime}$ and $d^{\prime \prime}$ defined as usual:

$$
d^{\prime p, q}: \mathscr{H} o m_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{F} \bullet, \mathscr{J}^{\bullet, q}\right) \longrightarrow \mathscr{H} o m_{\Omega_{P}^{p}}^{p+1}\left(\mathscr{F} \bullet, \mathscr{J}^{\bullet, q}\right) \quad d^{\prime p, q}(\Phi)=d_{\mathscr{F}}^{\prime \bullet, q} \circ \Phi-(-1)^{p} \Phi \circ d_{\mathscr{F}}^{\bullet}
$$

and

$$
d^{\prime \prime p, q}: \mathscr{H} o m_{\Omega_{P}}^{p}\left(\mathscr{F} \bullet, \mathscr{J}^{\bullet, q}\right) \longrightarrow \mathscr{H} o m_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{F} \bullet, \mathscr{J}^{\bullet, q+1}\right) \quad d^{\prime \prime p, q}(\Phi)=d_{\mathscr{F}}^{\prime \prime \bullet, q} \circ \Phi .
$$

The second term also has the usual differentials, but using the differentials of $\widetilde{\mathscr{J}}$ instead of $\mathscr{J}$. So we have
and

$$
\tilde{d}^{\prime \prime p, q}: \mathscr{H} o m_{\Omega_{P}^{\bullet}}^{p}(\mathscr{F} \bullet, \widetilde{\mathcal{J}} \cdot q) \longrightarrow \mathscr{H} m_{\Omega_{P}^{\prime}}^{p}(\mathscr{F} \bullet, \widetilde{\mathcal{J}} \cdot q+1) \quad \tilde{d}^{\prime \prime p, q}(\Psi)_{a}=(-1)^{p+a} d_{\mathscr{F}}^{\prime \prime \bullet, q} \circ \Psi_{a}
$$

where $\Psi=\left(\Psi_{a}\right)$, with $\Psi_{a}: \mathscr{F}^{a} \rightarrow \mathscr{J}^{a+p, q}$.
Finally the differentials of the last object are given by

$$
\delta^{\prime p, q}: \widetilde{\mathscr{H} o m}_{\Omega_{P}^{\bullet}}^{p}(\mathscr{F} \bullet, \widetilde{\mathscr{J}} \cdot q) \longrightarrow \widetilde{\mathscr{H} o m}_{\Omega_{P}^{\bullet}}^{p+1}\left(\mathscr{F} \bullet, \widetilde{J}^{\bullet}, q\right) \quad \delta^{\prime p, q}(\Psi)=d_{\mathscr{J}}^{\bullet}, q \circ \Psi-(-1)^{p+q} \Psi \circ d_{\mathscr{F}}^{\bullet}
$$

and

$$
\delta^{\prime \prime} p, q:{\widetilde{\mathscr{H} o m_{\Omega_{P}^{\bullet}}}}_{p}\left(\mathscr{F} \bullet \widetilde{\mathcal{J}^{\bullet}, q}\right) \longrightarrow{\widetilde{\mathscr{H} o m_{\Omega_{P}^{\bullet}}^{\bullet}}}_{p}\left(\mathscr{F} \cdot \widetilde{\mathcal{J}^{\bullet}, q+1}\right) \quad \delta^{\prime \prime p, q}(\Psi)_{a}=(-1)^{a} d_{\mathscr{J}}^{\prime \prime \bullet, q} \circ \Psi_{a}
$$

We note that the hyperext functors are defined by [HL] using the third bicomplex of the corollary, and by [B.74] using the first one; the corollary proves therefore that the definitions are equivalent.
2.3.3. Proposition. Using the previous notation, we have a canonical identification of $\mathscr{C}_{\mathfrak{P}}$-Modules

$$
\left.\left(\left(\widetilde{\mathscr{H} o m}_{\Omega_{P}^{\bullet}}^{p}(\mathscr{F} \cdot, \widetilde{\mathscr{J}}, q)\right)_{p, q}\right)_{\mathrm{tot}} \cong \mathscr{H}_{\mathrm{H}}^{\boldsymbol{\Omega _ { P }}}{ }_{(\mathscr{F} \cdot}, \mathscr{J}_{\text {tot }}^{\bullet}\right)
$$

 indicates the total complex associated to $\mathscr{J}^{\bullet \bullet}$.

Proof. We first point out that the total complex associated to a complex of $\mathscr{C}_{P}^{\bullet}$-Modules is canonically a $\mathscr{C}_{P}$-Module. Notice that, for any $r$, the $r$-th level of any of the two complexes appearing in the statement is described as collection of $\mathscr{O}_{P}$-linear morphisms ( $\Psi_{a}^{p, q}$ ) (for varying $a, p$ and $q$ with $p+q=r$ ) where $\Psi_{a}^{p, q}: \mathscr{F}^{a} \rightarrow \mathscr{J}^{a+p, q}$ and satisfying the following linearity w.r.t. sections $\alpha \in \Omega_{P}^{i}$

$$
\Psi_{a+i}^{p, q}(\alpha u)=(-1)^{i(p+q)} \alpha \Psi_{a}^{p, q}(u)
$$

for any $u$ section of $\mathscr{F}^{a}$. Therefore we only have to prove that the differentials in the two complexes coincide. We can prove that they are given by

$$
D^{r}(\Psi)_{a}^{p, q}=d_{\mathscr{F}}^{\prime a+p-1, q} \circ \Psi_{a}^{p-1, q}+(-1)^{r+1} \Psi_{a+1}^{p-1, q} \circ d_{\mathscr{F}}^{a}+(-1)^{p+a+1} d_{\mathscr{J}}^{\prime \prime a+p, q-1} \circ \Psi_{a}^{a+p, q-1}
$$

where $\Psi=\left(\Psi_{a}^{p, q}\right)$ with $\Psi_{a}^{p, q}: \mathscr{F}^{a} \rightarrow \mathscr{J}^{a+p, q}($ for $p+q=r)$, and $D^{r}(\Psi)_{a}^{p, q}: \mathscr{F}^{a} \rightarrow \mathscr{J}^{a+p, q}($ for $p+q=r+1)$. In fact, using the previous notation, the total differential $\Delta^{r}$ in the first case is defined by

$$
\begin{aligned}
\Delta^{r}(\Psi)_{a}^{p, q} & =\left(\delta^{\prime} \Psi\right)_{a}^{p, q}+(-1)^{p+1}\left(\delta^{\prime \prime} \Psi\right)_{a}^{p, q} \\
& =d_{\mathscr{F}}^{\prime a+p-1, q} \circ \Psi_{a}^{p-1, q}-(-1)^{r} \Psi_{a+1}^{p-1, q} \circ d_{\mathscr{F}}^{a}+(-1)^{p+1}(-1)^{a} d_{\mathscr{F}}^{\prime \prime a+p, q-1} \circ \Psi_{a}^{a+p, q-1}
\end{aligned}
$$

while the differential $D^{r}$ in the second complex is given by

$$
\begin{aligned}
D^{r}(\Psi)_{a}^{p, q} & =\left(d_{\mathscr{F} \text { tot }}^{\bullet} \circ \Psi\right)_{a}^{p, q}-(-1)^{r}\left(\Psi \circ d_{\mathscr{F}}^{\bullet}\right)_{a}^{p, q} \\
& =d_{\mathscr{F}}^{\prime a+p-1, q} \circ \Psi_{a}^{p-1, q}+(-1)^{p+a+1} d_{\mathscr{J}}^{\prime \prime a+p, q-1} \circ \Psi_{a}^{a+p, q-1}-(-1)^{r} \Psi_{a+1}^{p-1, q} \circ d_{\mathscr{F}}^{a}
\end{aligned}
$$

2.4. If $P$ is a smooth scheme of dimension $n$, then the sheaves of differentials $\Omega_{P}^{i}$ are locally free $\mathscr{O}_{P}$-Modules, so that we have the following isomorphisms

$$
\begin{align*}
& \mathscr{H} m_{\Omega_{P}^{\bullet}}\left(\mathscr{F}, E^{r}\left(\Omega_{P}^{\bullet}\right)\right) \cong \mathscr{H}_{\Omega_{\Omega_{P}^{\bullet}}^{\bullet}}\left(\mathscr{F}^{\bullet}, E^{r}\left(\mathscr{O}_{P}\right) \otimes_{\mathscr{O}_{P}} \Omega_{P}^{\bullet}\right) \\
& \cong \mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{F}^{\bullet}, E^{r}\left(\mathscr{O}_{P}\right) \otimes_{\mathscr{O}_{P}} \mathscr{H}^{\circ} m_{\mathscr{O}_{P}}\left(\Omega_{P}^{\bullet}, \Omega_{P}^{n}[-n]\right)\right) \\
& \cong \mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{F}^{\bullet}, \mathscr{H o m}_{\mathfrak{O}_{P}}\left(\Omega_{P}^{\bullet}, E^{r}\left(\mathscr{O}_{P}\right) \otimes_{\mathscr{O}_{P}} \Omega_{P}^{n}[-n]\right)\right)  \tag{2.4.1}\\
& \cong \mathscr{H o m}_{\mathscr{O}_{P}}\left(\mathscr{F} \cdot E^{r}\left(\mathscr{O}_{P}\right) \otimes_{\mathscr{O}_{P}} \Omega_{P}^{n}[-n]\right) \\
& \cong \mathscr{H o m}_{\mathscr{O}_{P}}\left(\mathscr{F} \cdot, E^{r}\left(\Omega_{P}^{n}\right)[-n]\right)
\end{align*}
$$

(using once more the contravariant ring change formula). Similarly,

$$
\begin{equation*}
\operatorname{Hom}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{F} \bullet, E^{r}\left(\Omega_{P}^{\bullet}\right)\right) \cong \operatorname{Hom}_{\mathscr{O}_{P}}^{\bullet}\left(\mathscr{F}^{\bullet}, E^{r}\left(\Omega_{P}^{n}\right)[-n]\right) \tag{2.4.2}
\end{equation*}
$$

and, in particular,

$$
\operatorname{Hom}_{\Omega_{P}^{\bullet}}\left(\mathscr{F}^{\bullet}, E^{r}\left(\Omega_{P}^{\bullet}\right)\right) \cong \operatorname{Hom}_{\mathscr{O}_{P}}\left(\mathscr{F} \bullet, E^{r}\left(\Omega_{P}^{n}\right)[-n]\right)
$$

Since $E^{r}\left(\Omega_{P}^{n}\right)$ is an injective $\mathscr{O}_{P}$-Module for any $r$, we conclude that $E^{r}\left(\Omega_{P}^{\bullet}\right)$ is an injective $\Omega_{P}^{\bullet}$-Module.
2.4.3. Proposition. We have a canonical isomorphism

$$
\mathscr{H o m}_{\Omega_{P}^{\bullet}}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right) \longrightarrow \mathscr{H}^{\bullet} \dot{\mathscr{O}}_{P}\left(\mathscr{M}^{\bullet}, E^{\bullet}\left(\Omega_{P}^{n}\right)[-n]\right)
$$

for any graded $\Omega_{P}^{\bullet}$-Module $\mathscr{M}^{\bullet}$.
Proof. Follows immediately from 2.3.3 and 2.4.2.
2.5. Definition. Let $P$ be a smooth scheme of pure dimension $n$. For any $\mathscr{C}_{\dot{P}}$-Module $\mathscr{M}^{\bullet}$, we define its dual $\mathscr{C}_{\dot{P}}$-Module as

$$
\left(\mathscr{M}^{\bullet}\right)^{*}:=\mathscr{H o m}_{\Omega_{P}^{\bullet}}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right)[2 n]
$$

Notice that, as a graded $\mathscr{O}_{P}$-Module, $\left(\mathscr{M}^{\bullet}\right)^{*} \cong \mathscr{H}^{\circ} \boldsymbol{m}_{\mathscr{O}_{P}}\left(\mathscr{M}^{\bullet}, E^{\bullet}\left(\Omega_{P}^{n}\right)[n]\right)$, so that this notion of dual is compatible with duality in the derived category of $\mathscr{O}_{P}$-Modules. If $\mathscr{N}^{\bullet}=\left(\mathscr{M}^{\bullet}\right)^{*}$ is a dual, then for any $i$, the $\mathscr{O}_{P}$-Module $\mathscr{N}^{i}=\bigoplus_{j=0}^{n} \mathscr{H}^{\prime} m_{\mathscr{O}_{P}}\left(\mathscr{M}^{j-n-i}, E^{j}\left(\Omega_{P}^{n}\right)\right)$ is flabby, since $E^{j}\left(\Omega_{P}^{n}\right)$ is an injective $\mathscr{O}_{P}$-Module. The functor

$$
\begin{aligned}
\mathscr{C}(P)^{\circ} & \longrightarrow \mathscr{C}(P) \\
\mathscr{M}^{\bullet} & \longmapsto\left(\mathscr{M}^{\bullet}\right)^{*}
\end{aligned}
$$

is exact and naturally extends to a functor

$$
(\operatorname{Pro} \mathscr{C}(P))^{\circ} \longrightarrow \operatorname{Ind} \mathscr{C}(P)
$$

2.6. De Rham Homology. Coming back to the notation of 1.1, we rewrite the definition of De Rham homology for an easier construction of the duality morphism in the following section.
2.6.1. Proposition. Let $j_{W}: W \rightarrow P$ be the open immersion of $W$ in $P$, and assume that $P$ is smooth. Then

$$
\left(\underset{M \geqslant N}{\lim } " \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right)^{*} \cong j_{W *} \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)
$$

and we have canonical isomorphisms

$$
H_{\bullet}^{\mathrm{DR}}(X) \cong \mathbf{H}^{-\bullet}(X,(\underset{M \geqslant N}{\overbrace l i m^{\approx}} \mathscr{I}_{C}^{N-} \bullet \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet})^{*}) .
$$

Proof. We compute

$$
\begin{aligned}
& j_{W *} \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right) \cong \underset{N}{\lim _{M}} \mathscr{I}_{C}^{-N} \Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right) \\
& \cong \underset{M, N}{\lim _{C}} \mathscr{I}_{C}^{-N} \mathscr{H}_{\boldsymbol{m}}^{\dot{\mathscr{O}}_{P}}\left(\mathscr{O}_{P} / \mathscr{I}_{\bar{X}}^{M}, E\left(\Omega_{P}^{\bullet}\right)\right) \\
& \cong \underset{M, N}{\lim } \mathscr{H} m_{\mathscr{O}_{P}}\left(\mathscr{I}_{C}^{N}, \mathscr{H} o m_{\mathscr{O}_{P}}\left(\mathscr{O}_{P} / \mathscr{I}_{\bar{X}}^{M}, E\left(\Omega_{P}^{\cdot}\right)\right)\right) \\
& \cong \underset{M \geqslant N}{\lim _{M}} \mathscr{H}_{\operatorname{O}_{P}}^{\cdot}\left(\mathscr{I}_{C}^{N} / \mathscr{I}_{\bar{X}}^{M}, E\left(\Omega_{P}^{\bullet}\right)\right) \\
& \cong \mathscr{H o m}_{\mathscr{O}_{P}}\left(\lim _{M \geqslant N} \mathscr{I}_{C}^{N} / \mathscr{I}_{\bar{X}}^{M}, E\left(\Omega_{P}^{\bullet}\right)\right) \\
& \cong \mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\lim _{M \geqslant N} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right) .
\end{aligned}
$$

Since $\Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)$ is a complex of flabby abelian sheaves (the objects are injective $\mathscr{O}_{W}$-Modules), we have, in the derived category of abelian sheaves on $P, j_{W *} \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right) \cong \mathbf{R} j_{W *} \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)$ and

$$
\mathbf{H}_{X}^{\bullet}\left(W, \Omega_{W}^{\bullet}\right) \cong \mathbf{H}^{\bullet}\left(W, \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)\right) \cong \mathbf{H}^{\bullet}\left(P, j_{W *} \Gamma_{X} E\left(\Omega_{W}^{\bullet}\right)\right)
$$

2.6.2. Corollary. In the previous notation, let $\mathscr{M} \cdot$ be a $\mathscr{C}_{\dot{P}}$-Module. There are canonical isomorphisms

$$
\stackrel{\mathscr{E} x t}{ }_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \cong H^{p}\left(\mathscr{H}_{\operatorname{Com}_{\Omega_{P}^{\bullet}}^{\bullet}}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right)\right)
$$

and

$$
\underline{\operatorname{Ext}}_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \cong H^{p}\left(\operatorname{Hom}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right)\right)
$$

Moreover,

Proof. By (2.4.1) and [B.74, II.5.4.8], the Cousin resolution $E^{\bullet}\left(\Omega_{P}^{\bullet}\right)$ of $\Omega_{P}^{\bullet}$ permits the computation of the (local and global) hyperExt functors of [HL]. So we have

$$
\underline{\mathscr{E x t}}_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \cong H^{p}\left(\left(\mathscr{H}_{\mathrm{Com}}^{\Omega_{P}^{\bullet}}\left(\mathscr{M}^{\bullet}, E^{\bullet}\left(\Omega_{P}^{\bullet}\right)\right)\right)_{\mathrm{tot}}\right)
$$

Proposition 2.3.3 gives

$$
H^{p}\left(\left(\mathscr{H}_{\mathrm{om}}^{\Omega_{P}^{\bullet}},\left(\mathscr{M}^{\bullet}, E^{\bullet}\left(\Omega_{P}^{\bullet}\right)\right)\right)_{\mathrm{tot}}\right) \cong H^{p}\left(\left(\mathscr{H}_{\mathrm{om}}^{\Omega_{\dot{P}}}\left(\mathscr{M}^{\bullet},\left(E^{\bullet}\left(\Omega_{P}^{\bullet}\right)\right)_{\mathrm{tot}}\right)\right)\right.
$$

which proves the first formula. The second formula follows immediately from the first. The last assertion of the statement combines the first one with proposition 2.6.1.

## 3. Algebraic Poincaré Duality.

Our first result is the construction of a duality morphism, which will be compatible with the duality morphisms constructed in the proper case in [H.75, II.5]. As already said, in the body of this section, we suppose that the scheme $P$ of the setting 1.1 be smooth. The Poincaré duality theorem will be proved using the long exact sequences for a closed subset in De Rham cohomology with compact supports and De Rham homology.

We also give an alternative proof, with a strategy similar to the original one of [H.75] for the proof of the duality theorem in the proper case: the canonical duality morphism is induced by a morphism of spectral sequences, and we can use the results of [H.72] (cohomology with compact supports for coherent sheaves) to prove that it is an isomorphism. For the construction of the morphism of spectral sequences we make
explicit the (equivalent) point of view of pairings of spectral sequences used in [HL]. In fact the alternative proof could also be rewritten adapting to the open case the proof of the duality theorem [HL, 5.7] for proper smooth spaces.
3.0. In this section we will discuss duality results for not necessarily finite dimensional $K$-vector spaces, so we recall some important notions which will be freely used. Let $\mathscr{V}$ be the category of finite dimensional $K$-vector spaces. The category of all $K$-vector spaces is equivalent to Ind $\mathscr{V}$. On the other hand, Pro $\mathscr{V}$ is equivalent, via the inverse limit functor, to the category of linearly topologized and linearly compact $K$-vector spaces (with continuous maps as morphisms). Notice that it is in fact an abelian category.

The usual duality for finite dimensional $K$-vector spaces, sending $V$ to $V^{\prime}=\operatorname{Hom}_{K}(V, K)$, extends to contravariant functors $\operatorname{Ind} \mathscr{V} \longrightarrow \operatorname{Pro} \mathscr{V}$ and $\operatorname{Pro} \mathscr{V} \longrightarrow \operatorname{Ind} \mathscr{V}$ which are (anti-)equivalences inverse to each other. In fact we have canonical equivalences $\operatorname{Ind}\left(\mathscr{V}^{\circ}\right) \cong \operatorname{Pro}(\mathscr{V})^{\circ}$ and $\operatorname{Pro}\left(\mathscr{V}^{\circ}\right) \cong \operatorname{Ind}(\mathscr{V})^{\circ}$; so the Indextension of $\mathscr{V}^{\circ} \xrightarrow{\prime} \mathscr{V}$, again denoted with the prime apex, is $\operatorname{Pro}(\mathscr{V})^{\circ} \cong \operatorname{Ind}\left(\mathscr{V}^{\circ}\right) \xrightarrow{\prime} \operatorname{Ind}(\mathscr{V})$. Similarly the Pro-extension of $\mathscr{V}^{\circ} \xrightarrow{\prime} \mathscr{V}$ is $\operatorname{Ind}(\mathscr{V})^{\circ} \cong \operatorname{Pro}\left(\mathscr{V}^{\circ}\right) \xrightarrow{\prime} \operatorname{Pro}(\mathscr{V})$.

We explicitly point out that, given a spectral sequence $E: E_{r}^{p, q} \Rightarrow E^{p+q}$ in $\mathscr{V}$ (resp. Ind $\mathscr{V}$ ), the dual spectral sequence $E^{\prime}$ in $\mathscr{V}$ (resp. Pro $\mathscr{V}$ ) can be defined in the following way. We put $E_{r}^{\prime p, q}=\left(E_{r}^{-p,-q}\right)^{\prime}$ and $E^{\prime n}=\left(E^{n}\right)^{\prime}$ where the prime apex indicates the dual $K$-vector space (resp. topological $K$-vector space), in the above sense. The limit object is endowed with the filtration given by orthogonality, that is $F_{i}\left(E^{\prime n}\right):=\left(F_{i} E^{n}\right)^{\perp}=\operatorname{ker}\left(\left(E^{n}\right)^{\prime} \rightarrow\left(F_{i} E^{n}\right)^{\prime}\right)$. The exactness of the duality functor and the relation $\left(F_{i} / F_{j}\right)^{\prime} \cong F_{j}^{\perp} / F_{i}^{\perp}$ permit to endow $E^{\prime}$ with a well defined structure of spectral sequence, where the differentials at the $r$-th level are defined by duality:

$$
d_{r}^{\prime p, q}=\left(d_{r}^{-p-r,-q-r+1}\right)^{\prime}
$$

3.1. Construction of the duality morphism. Let $X, \bar{X}$ and $P$ be as in 1.1 , and suppose $P$ smooth (see remark 1.9); let $p=p_{X}: X \rightarrow \operatorname{Spec} K$ the canonical morphism (we call $p$ also the corresponding morphisms for $\bar{X}$ and $P$ ) and let $n=n_{P}$ be the dimension of $P$. Let $\mathscr{M}^{\bullet}=" \varliminf_{\lim _{M}}{ }_{M} \mathscr{M}^{(M) \bullet}$ where $\mathscr{M}^{(M) \bullet}$ is an object of $\mathscr{C}_{c}\left(\bar{X}_{P}^{(M)}\right)$; so, $\mathscr{M} \bullet$ is a pro-object of $\mathscr{C}_{c}(P)$, indexed by $\mathbb{N}$. Then

$$
\left(\mathscr{M}^{\bullet}\right)^{*}:=\mathscr{H} m_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right)[2 n] \cong \mathscr{H}_{\mathscr{O}_{P}}^{\bullet}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{n}\right)[n]\right)=\Gamma_{\bar{X}} \mathscr{H}_{0} m_{\mathscr{O}_{P}}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{n}\right)[n]\right)
$$

has support in $\bar{X}$. On the other hand, for any $j$, the $\mathscr{O}_{P}$-modules $\left(\left(\mathscr{M}^{\bullet}\right)^{*}\right)^{j}$ and $\left(\left(\mathscr{M}^{\bullet}\right)^{* *}\right)^{j}$ are flabby sheaves.
From the evaluation morphism

$$
\begin{equation*}
\eta=\eta_{\mathscr{M}}: \mathscr{M}^{\bullet} \longrightarrow\left(\mathscr{M}^{\bullet}\right)^{* *} \cong \mathscr{H}_{0} m_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{H o m}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{M}^{\bullet}, E\left(\Omega_{P}^{\bullet}\right)\right), \Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right)\right) \tag{3.1.1}
\end{equation*}
$$

clearly a morphism of complexes for the natural structure of complex of the second term, and the trace morphism

$$
\operatorname{Tr}=\operatorname{Tr}_{\bar{X}}: p_{*} \Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right)=\mathbf{R} p_{*} \Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right) \longrightarrow K[-2 n]
$$

of [H.75, II.2.3], which is a morphism of complexes since $\bar{X}$ is proper, we obtain a canonical morphism

$$
\tau=\tau_{\mathscr{M}}: \mathbf{R} p_{*} \mathscr{M}^{\bullet} \longrightarrow\left(\mathbf{R} p_{*}\left(\mathscr{M}^{\bullet}\right)^{*}\right)^{\prime}
$$

This is explained by the following diagram

where a primed object indicates the dual (topological) $K$-vector space.
We have therefore canonical morphisms of hypercohomology

$$
\begin{equation*}
\tau_{\mathscr{M}}^{\bullet}: \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{M}^{\bullet}\right) \longrightarrow\left(\mathbf{H}^{-\bullet}\left(\bar{X},\left(\mathscr{M}^{\bullet}\right)^{*}\right)\right)^{\prime} \tag{3.1.2}
\end{equation*}
$$

which we will call the canonical duality morphisms (for $\mathscr{M}^{\bullet}$ ).
3.2. Functoriality of the duality morphism. Let $h: X_{1} \rightarrow X_{2}$ be a proper morphism fitting in a diagram like (1.10.1), with $P_{1}$ and $P_{2}$ smooth. We first examine the case where $\mathscr{M}_{\boldsymbol{i}}$ is an object of
$\mathscr{C}_{c}\left(\left(\bar{X}_{i}\right)_{P_{i}}^{(M)}\right) \subset \mathscr{C}_{c}\left(P_{i}\right)$, for a fixed $M$ and $i=1,2$. We assume to be given a morphism $\gamma: \mathscr{M}_{2} \longrightarrow f_{*}\left(\mathscr{M}_{1}\right)$ in $\mathscr{C}_{c}\left(\left(\bar{X}_{i}\right)_{P_{2}}^{(M)}\right)$. Then $\gamma$ induces a morphism $\bar{\gamma}: \mathscr{M}_{2} \longrightarrow \mathbf{R} f_{*}\left(\mathscr{M}_{\mathbf{1}}\right)$ in the derived category of complexes of abelian sheaves on $P_{2}$. There exists a canonical commutative diagram of duality morphisms of finite dimensional $K$-vector spaces

where we set $\tau_{i}^{*}=\tau_{\mathscr{M}_{i}}^{\bullet}$. This is obtained from the following commutative diagram

where $d=n_{1}-n_{2}$ is the relative dimension of $f$ and the morphism $\tau: \mathbf{R} f_{*}\left(\mathscr{M}_{1}\right)^{*} \longrightarrow\left(\mathscr{M}_{2}^{\bullet}\right)^{*}$ is induced by compositions with $\gamma$ and the relative trace morphism $\operatorname{Tr}_{f}: f_{*} \Gamma_{\bar{X}_{1}} E\left(\Omega_{P_{1}}^{\bullet}\right) \longrightarrow \Gamma_{\bar{X}_{2}} E\left(\Omega_{P_{2}}^{\bullet}\right)[-2 d]$ as follows

$$
\begin{align*}
\mathbf{R} f_{*} \mathscr{H o m}_{\Omega_{P_{1}}^{\bullet}}^{\bullet}\left(\mathscr{M}_{1}^{\bullet}, \Gamma_{\bar{X}_{1}} E\left(\Omega_{P_{1}}^{\bullet}\right)\left[2 n_{1}\right]\right) \cong f_{*} \mathscr{H o m}_{\Omega_{P_{1}}^{\bullet}}^{\bullet}\left(\mathscr{M}_{1}^{\bullet}, \Gamma_{\bar{X}_{1}} E\left(\Omega_{P_{1}}^{\bullet}\right)\left[2 n_{1}\right]\right) \xrightarrow{\text { can }}  \tag{3.2.2}\\
\mathscr{H o m}_{\Omega_{P_{2}}}^{\bullet}\left(f_{*} \mathscr{M}_{1}^{\bullet}, f_{*} \Gamma_{\bar{X}_{1}} E\left(\Omega_{P_{1}}^{\bullet}\right)\left[2 n_{1}\right]\right) \longrightarrow \operatorname{Hom}_{\Omega_{P_{2}}^{\bullet}}\left(\mathscr{M}_{2}^{\bullet}, \Gamma_{\bar{X}_{2}} E\left(\Omega_{P_{2}}^{\bullet}\right)\left[2 n_{2}\right]\right) .
\end{align*}
$$

The commutativity of the diagram now follows from the functorial properties of the trace map for scheme morphisms, that is $\operatorname{Tr}_{\bar{X}_{1}}=\operatorname{Tr}_{\bar{X}_{2}} \circ \mathbf{R} p_{2 *}\left(\operatorname{Tr}_{f}\right)$, see [H.75, II.2]. The general case of $\mathscr{M}_{i}="{ }_{\text {lim }}{ }^{\prime}{ }_{M} \mathscr{M}_{i}^{(M)} \bullet$ where $\mathscr{M}_{i}^{(M)} \bullet$ is an object of $\mathscr{C}_{c}\left(\left(\bar{X}_{i}\right)_{P_{i}}^{(M)}\right)$ and of a morphism $\gamma=" \lim _{\longleftrightarrow}{ }_{M} \gamma_{M}$, with $\gamma_{M}: \mathscr{M}_{2}^{(M)} \bullet \longrightarrow f_{*}\left(\mathscr{M}_{1}^{(M)}\right)$ a morphism in $\mathscr{C}_{c}\left(\left(\bar{X}_{2}\right)_{P_{2}}^{(M)}\right)$, is then deduced from the previous discussion applying $l_{\rightleftarrows}$ to a projective system of diagrams of the form (3.2.1). We obtain diagram (3.2.1) in the category of linearly compact topological $K$-vector spaces.

An easier discussion gives the functoriality of the duality morphisms w.r.t. open immersions.
3.3. Application to De Rham coefficents. In particular, in our general setting 1.1, we can take as $\mathscr{M} \cdot$ any of the following three complexes:

$$
\begin{equation*}
" \underset{M}{\lim } " \Omega_{\bar{X}_{P}^{(M)}} \cong " \underset{M}{\lim _{M}} " \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}, \quad \underset{N}{\lim _{N}} " \Omega_{C_{P}^{(N)}} \cong " \lim _{N}^{\lim } \Omega_{P}^{\bullet} / \mathscr{I}_{C}^{N-\bullet} \tag{3.3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
" \varliminf_{M}^{\lim } " j_{!}^{(M)} \Omega_{X_{W}^{(M)}}^{(M)} \cong " \varliminf_{M \geqslant N}^{\lim } " \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \tag{3.3.2}
\end{equation*}
$$

In the first two cases we obtain the canonical duality (iso)morphisms

$$
\begin{equation*}
H_{\mathrm{DR}}^{\bullet}(\bar{X}) \longrightarrow\left(H_{\bullet}^{\mathrm{DR}}(\bar{X})\right)^{\prime} \quad \text { and } \quad H_{\mathrm{DR}}^{\bullet}(C) \longrightarrow\left(H_{\bullet}^{\mathrm{DR}}(C)\right)^{\prime} \tag{3.3.3}
\end{equation*}
$$

of Hartshorne. In fact, the first term of (3.1.2) is by definition the De Rham cohomology of $\bar{X}$ and $C$ respectively, while the second term can be reinterpreted using 2.5 as the dual of the De Rham homology of $\bar{X}$ (and the same is true for $C$ ). So the construction of the duality morphism in this case is just that of Hartshorne in the proof of the duality theorem [H.75, II.5.1] in the proper case.

In the third case, when $\mathscr{M} \bullet=" \varliminf_{M \geqslant N} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}$, proposition 2.6.1 of the previous paragraph identifies the second term of (3.1.2) with the dual of the De Rham homology $H_{\bullet}^{\mathrm{DR}}(X)$ of $X$, so that we have canonical morphisms

$$
\begin{equation*}
H_{\mathrm{DR}, c}^{\bullet}(X) \longrightarrow\left(H_{\bullet}^{\mathrm{DR}}(X)\right)^{\prime} \tag{3.3.4}
\end{equation*}
$$

3.4. Theorem. The duality morphism (3.3.4) is an isomorphism for any scheme $X$ as in our setting 1.1, and it is compatible with the functoriality of the two terms w.r.t. proper map and open immersions; more precisely we have for any proper morphism $p: X_{1} \rightarrow X_{2}$ (resp. any open immersion $j: X_{1} \rightarrow X_{2}$ ) commutative diagrams
where the horizontal maps are the duality isomorphisms, and the vertical ones are induced by functoriality.
Proof. We consider the open inclusion of $X$ in $\bar{X}$, with complement $C$. Then we have the long exact sequences of the closed subset $C$ for the De Rham homology given by [H.75, II.3.3] and for the De Rham cohomology with compact supports given in 1.11.2. Moreover the duality (iso)morphisms of (3.3.3) in the case of $\bar{X}$ and $C$, and (3.3.4) for $X$, give rise to the commutative diagram with exact rows

$$
\begin{aligned}
& \begin{array}{lcccccc}
\cdots & H_{\mathrm{DR}}^{i-1}(C) & \longrightarrow & H_{\mathrm{DR}, c}^{i}(X) & \longrightarrow & H_{\mathrm{DR}}^{i}(\bar{X}) & \longrightarrow \\
\cdots & \downarrow \cong & \downarrow & \downarrow \cong & H_{\mathrm{DR}}^{i}(C) & \downarrow & H_{\mathrm{DR}, c}^{i+1}(X)
\end{array} \longrightarrow \cdots \\
& \cdots \longrightarrow\left(H_{i-1}^{\mathrm{DR}}(C)\right)^{\prime} \longrightarrow\left(H_{i}^{\mathrm{DR}}(X)\right)^{\prime} \longrightarrow\left(H_{i}^{\mathrm{DR}}(\bar{X})\right)^{\prime} \longrightarrow\left(H_{i}^{\mathrm{DR}}(C)\right)^{\prime} \longrightarrow\left(H_{i+1}^{\mathrm{DR}}(X)\right)^{\prime} \longrightarrow \cdots
\end{aligned}
$$

so that we can apply the five lemma, since the schemes $\bar{X}$ and $C$ are proper and the duality theorem of [H.75, II.5.1] applies. The functoriality of the isomorphism is clear, in view of 3.2.
3.5. Pairing of Complexes and Spectral Sequences. We keep here the notation of (3.1). The evaluation morphism (3.1.1) can be seen as a pairing of complexes

$$
\begin{equation*}
\left(\mathscr{M}^{\bullet}\right)^{*} \otimes_{K} \mathscr{M}^{\bullet} \longrightarrow\left(\mathscr{M}^{\bullet}\right)^{*} \otimes_{\Omega_{P}^{\bullet}} \mathscr{M}^{\bullet} \longrightarrow \Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right)[2 n] . \tag{3.5.1}
\end{equation*}
$$

Let $\mathscr{M}^{(M)} \bullet \longrightarrow \mathscr{I}^{(M)} \bullet \bullet$ be an injective resolution of of $\mathscr{M}^{(M) \bullet}$ in the category of $\mathscr{C}_{\dot{X}_{P}^{(M)}}$-Modules. We obtain a resolution

$$
\begin{equation*}
\mathscr{M}^{\bullet} \cong "{\underset{M}{\lim }}_{M_{M}}^{\mathscr{M}^{(M)} \bullet} \longrightarrow "_{M}^{\lim " \mathscr{I}^{(M)} \bullet \bullet}=: \mathscr{I} \bullet \tag{3.5.2}
\end{equation*}
$$

of $\mathscr{M} \cdot$ in $\operatorname{Pro} \mathscr{C}(P)$, and the pairing (3.5.1) extends to a pairing of bicomplexes

$$
\mathscr{H o m}_{\Omega_{P}^{\bullet}}\left(\mathscr{I}^{\bullet \bullet}, E^{\bullet}\left(\Omega_{P}^{\bullet}\right)\right) \otimes \mathscr{I}^{\bullet \bullet} \longrightarrow \Gamma_{\bar{X}} E^{\bullet}\left(\Omega_{P}^{\bullet}\right)
$$

(see [HL, 4.2] for the notation). The trace morphism of [H.75, §2] for the structural morphism $p: P \rightarrow$ Spec $K$ and the subscheme $\bar{X}$, is actually a morphism of bicomplexes

$$
\begin{equation*}
\operatorname{Tr}_{\bar{X}}: \Gamma_{\bar{X}}\left(P, E^{\bullet}\left(\Omega_{P}^{\bullet}\right)\right) \longrightarrow K[-n,-n] \tag{3.5.3}
\end{equation*}
$$

and composing with the previous pairing, we obtain a pairing of bicomplexes with values in $K$.
We then have a pairing of associated spectral sequences
(see [HL, 4.1] for the definition of a pairing of spectral sequences), where ${\underline{\underline{\operatorname{Ext}^{\bullet}}} \Omega_{P}^{\bullet}}_{\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right)\left(\text { resp. } \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{M}^{\bullet}\right) \text {, }, \text {, }\right.}$ resp. $\left.\mathbf{H}_{\bar{X}}^{\bullet}\left(P, \Omega_{P}^{\bullet}\right)\right)$ stands for the full spectral sequence

$$
R^{q} \operatorname{Hom}_{\Omega_{P}^{\bullet}}^{p}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \cong \operatorname{Ext}_{\mathscr{O}_{P}}^{q}\left(\mathscr{M}^{n-p}, \Omega_{P}^{n}\right) \Longrightarrow \underline{\underline{\operatorname{Ext}}}_{\Omega_{P}^{\bullet}}^{p+q}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right)
$$

(resp. $H^{q}\left(\bar{X}, \mathscr{M}^{p}\right) \Longrightarrow \mathbf{H}^{p+q}\left(\bar{X}, \mathscr{M}^{\bullet}\right)$, resp. $H_{\bar{X}}^{q}\left(P, \Omega_{P}^{p}\right) \Longrightarrow \mathbf{H}_{\bar{X}}^{p+q}\left(P, \Omega_{P}^{\bullet}\right)$ ).
On the other hand (3.5.3) induces a morphism of first spectral sequences

$$
E_{1}^{p, q}=H_{\frac{q}{X}}^{q}\left(P, \Omega_{P}^{p}\right) \longrightarrow C_{1}^{p, q}= \begin{cases}K & \text { if } p=q=n  \tag{3.5.5}\\ 0 & \text { otherwise }\end{cases}
$$

Composing the pairing of spectral sequences (3.5.4) with the morphism of spectral sequence (3.5.5), we obtain another pairing of spectral sequences

$$
\begin{equation*}
\underline{\underline{\operatorname{Ext}}}_{\Omega_{P}^{\bullet}}^{\bullet}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \times \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{M}^{\bullet}\right) \longrightarrow C^{\bullet} \tag{3.5.6}
\end{equation*}
$$

with values in a "constant spectral sequence" $C \bullet \bullet$ (the first spectral sequence of $K[-n,-n]$ ) with $C^{\bullet}=$ $K[-2 n]$. In particular, for any $p$ there is a canonical pairing

$$
\begin{equation*}
\underline{\underline{\operatorname{Ext}}}^{2 n-p}\left(\mathscr{M}^{\bullet}, \Omega_{P}^{\bullet}\right) \otimes \mathbf{H}^{p}\left(\bar{X}, \mathscr{M}^{\bullet}\right) \longrightarrow K \tag{3.5.7}
\end{equation*}
$$

which corresponds to the duality morphisms (3.1.2) (via the definition 2.5). To the pairing (3.5.6) is associated a morphism of spectral sequences from $E_{\bullet \bullet}$ to $F_{\bullet \bullet}$ where

$$
E_{1}^{p, q}=H^{q}\left(P, \mathscr{M}^{p}\right) \Longrightarrow E^{p+q}=\mathbf{H}^{p+q}\left(P, \mathscr{M}^{\bullet}\right)
$$

and
where the prime apex means dual (topological) $K$-vector space.
3.6. Functoriality of the pairing. As we did in 3.2 for the duality morphisms, and in the same setting, we discuss the functoriality of the pairing (3.5.4) (and consequently also of (3.5.6) and (3.5.7)) with respect to proper morphisms. We place ourselves in the situation of 3.2. We have the following diagram of pairings of spectral sequences

such that

$$
\operatorname{Tr}_{f}^{i+j}\left\langle\varphi, \gamma^{j} \mu\right\rangle=\left\langle\tau^{i} \varphi, \mu\right\rangle
$$

for any $\varphi \in \underline{\underline{\operatorname{Ext}}}_{\Omega_{P_{1}}^{\bullet}}^{i}\left(\mathscr{M}_{1}^{\bullet}, \Omega_{P_{1}}^{\bullet}\right)$ and $\mu \in \mathbf{H}^{j}\left(\bar{X}_{2}, \mathscr{M}_{2}^{\bullet}\right)$, where $\langle-,-\rangle$ indicates the pairings (this corresponds to the commutative diagram (3.2.1)). In fact we can start with resolutions of the form (3.5.2) $\mathscr{J}_{1} \bullet$ and $\mathscr{J}_{2}^{\bullet}$ of $\mathscr{M}_{1}^{\bullet}$ and $\mathscr{M}_{2}^{\bullet}$, respectively. We have a canonical morphism $\gamma=" \varliminf_{\ddagger} " \gamma_{M}: \mathscr{J}_{2}^{\bullet \bullet} \longrightarrow f_{*} \mathscr{J}_{1}^{\bullet}$, with $\gamma_{M}: \mathscr{J}_{2}^{(M) \bullet \bullet} \longrightarrow f_{*} \mathscr{J}_{1}^{(M) \bullet \bullet}$ a morphism in $\mathscr{C}\left(\left(\bar{X}_{2}\right)_{P_{2}}^{(M)}\right)$. We observe that the trace map for $f$ induces a morphism of bicomplexes $\operatorname{Tr}_{f}: f_{*} \Gamma_{\bar{X}_{1}} E^{\bullet}\left(\Omega_{P_{1}}^{\bullet}\right) \longrightarrow \Gamma_{\bar{X}_{2}} E^{\bullet}\left(\Omega_{P_{2}}^{\bullet}\right)[-d,-d]$. Since for any $\mathscr{C}_{P_{i}}^{\bullet}$-Modules $\mathscr{E}_{i}^{\bullet}$, $\mathscr{F}_{i}$, with $\mathscr{E}_{i}$ supported in $\bar{X}_{i}, \mathscr{H} o m_{\Omega_{P_{i}}^{\bullet}}^{\bullet}\left(\mathscr{E}_{i}, \mathscr{F}_{i}^{\bullet}\right)=\mathscr{H} 0 m_{\Omega_{P_{i}}^{\bullet}}\left(\mathscr{E}_{i}^{\bullet}, \Gamma_{\bar{X}_{i}} \mathscr{F}_{i}^{\bullet}\right)$, we obtain a diagram of pairings of bicomplexes

(see [HL, 4.2] for the notation). From this we deduce the above functoriality with respect to proper morphisms. We omit the easier discussion of open immersions.
3.7. Application to De Rham coefficients: alternative proof (of the duality theorem 3.4). If we replace $\mathscr{M} \cdot$ by the coefficient (3.3.2), we end up with the pairings

$$
\begin{equation*}
H_{p}^{\mathrm{DR}}(X) \otimes H_{\mathrm{DR}, c}^{p}(X) \longrightarrow K \tag{3.7.1}
\end{equation*}
$$

obviously compatible with the duality morphisms (3.3.4). By 3.4 these pairings are perfect: we now provide a second proof of this fact.

From the pairing of spectral sequences (3.5.6), we deduce a morphism of spectral sequences $E_{\bullet \bullet \bullet}^{\bullet} \longrightarrow F_{\bullet}^{\bullet \bullet \bullet}$, where

$$
\begin{equation*}
E_{1}^{p, q}=\varliminf_{M \geqslant N} H^{q}\left(\bar{X}, \mathscr{I}_{C}^{N} \Omega_{P}^{p} / \mathscr{I}_{\bar{X}}^{M}\right) \Longrightarrow E^{p+q}=\varliminf_{M \geqslant N} \mathbf{H}^{p+q}\left(\bar{X}, \mathscr{I}_{C}^{N-} \cdot \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right)=H_{\mathrm{DR}, c}^{p+q}(X) \tag{3.7.2}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{1}^{p, q}=H_{X}^{n-q}\left(W, \Omega_{W}^{n-p}\right)^{\prime} \Longrightarrow F^{p+q}=\mathbf{H}_{X}^{2 n-(p+q)}\left(W, \Omega_{W}^{\bullet}\right)^{\prime}=H_{p+q}^{\mathrm{DR}}(X)^{\prime} \tag{3.7.3}
\end{equation*}
$$

It is sufficient to prove that the above morphism of spectral sequences is an isomorphism. The level one terms of $E_{\bullet \bullet \bullet}$ may be rewritten as $E_{1}^{p, q}=\varliminf_{M} H_{c}^{q}\left(X, \Omega_{W}^{p} / \mathscr{I}_{X}^{M}\right)$ where one uses the definition of
cohomology with compact supports for coherent sheaves given in [H.72]. On the other hand the level one terms of $F_{\bullet}^{\bullet \bullet}$ are $F_{1}^{p, q}=\left(H_{X}^{n-q}\left(W,\left(\Omega_{W}^{p}\right)^{\vee} \otimes_{\mathscr{O}_{W}} \Omega_{W}^{n}\right)\right)^{\prime}$, since $W$ is smooth.

Now the duality theorem for the cohomology with compact supports for coherent sheaves, see [H.72, 3.1], gives isomorphisms

$$
H_{c}^{q}\left(W, \Omega_{W}^{p} / \mathscr{I}_{X}^{M}\right) \longrightarrow \operatorname{Ext}_{W}^{n-q}\left(\Omega_{W}^{p} / \mathscr{I}_{X}^{M}, \Omega_{W}^{n}\right)^{\prime}=\operatorname{Ext}_{W}^{n-q}\left(\mathscr{O}_{W} / \mathscr{I}_{X}^{M},\left(\Omega_{W}^{p}\right)^{\vee} \otimes_{\mathscr{O}_{W}} \Omega_{W}^{n}\right)^{\prime} .
$$

Taking projective limits on $M$ we obtain isomorphisms

$$
\varliminf_{M}^{\lim } H_{c}^{q}\left(W, \Omega_{W}^{p} / \mathscr{I}_{X}^{M}\right) \longrightarrow\left(\underset{M}{\left(\lim _{\vec{W}}\right.} \operatorname{Ext}_{W}^{n-q}\left(\mathscr{O}_{W} / \mathscr{I}_{X}^{M},\left(\Omega_{W}^{p}\right)^{\vee} \otimes_{\mathscr{O}_{W}} \Omega_{W}^{n}\right)\right)^{\prime}=\left(H_{X}^{n-q}\left(W,\left(\Omega_{W}^{p}\right)^{\vee} \otimes_{\mathscr{O}_{W}} \Omega_{W}^{n}\right)\right)^{\prime}
$$

where the last equality is [H.RD, V.4.2]. So the morphism $E_{\bullet \bullet \bullet}^{\bullet} \longrightarrow F_{\bullet \bullet \bullet}$ of spectral sequences is an isomorphism at level one, and therefore also on the limits, by regularity of both spectral sequences.
3.8. REMARK: THE (SMOOTH) PROPER CASE. In the construction of the duality morphism we use the fact that the morphism (3.1.1) takes its values in $\Gamma_{\bar{X}} E\left(\Omega_{P}^{\bullet}\right)$. This is needed because in the non proper case the trace map $\operatorname{Tr}: p_{*} E\left(\Omega_{P}^{\bullet}\right) \rightarrow K$ is not a morphism of complexes and, in order to make it commute with differentials, we have to use a proper support. When $P$ is a proper scheme this is not needed. In this case the pairing (3.7.1) factors as

$$
H_{\bullet}^{\mathrm{DR}}(X) \otimes H_{\mathrm{DR}, c}^{\bullet}(X) \longrightarrow H_{\mathrm{DR}}^{2 n}(P) \cong H_{\mathrm{DR}, c}^{2 n}(W) \xrightarrow{\operatorname{Tr}_{P}} K
$$

## 4. Künneth formulae.

4.1. Construction of Künneth morphisms for De Rham functors. Let $X_{1} X_{2}$ be schemes. We consider the product $X_{1} \times X_{2}$ and we assume to be given immersions as in our setting 1.1 for $i=1,2$ (their products give automatically similar immersions for the product scheme). The canonical isomorphism

$$
p_{1}^{-1} \Omega_{W_{1}}^{\bullet} \otimes p_{2}^{-1} \Omega_{W_{2}}^{\bullet}=: \Omega_{W_{1}}^{\bullet} \boxtimes \Omega_{W_{2}}^{\bullet} \cong \Omega_{W_{1} \times W_{2}}^{\bullet}
$$

induces a canonical isomorphism between the completions on $W_{1 / X_{1}} \times W_{1 / X_{1}} \cong\left(W_{1} \times W_{2}\right)_{/ X_{1} \times X_{2}}$

$$
\left(\Omega_{W_{1}}^{\bullet}\right)_{/ X_{1}} \boxtimes\left(\Omega_{W_{2}}^{\bullet}\right)_{/ X_{2}} \xrightarrow{\cong}\left(\Omega_{W_{1} \times W_{2}}^{\bullet}\right)_{/ X_{1} \times X_{2}}
$$

and therefore canonical morphisms of hypercohomology groups

$$
\begin{gathered}
\mathbf{H} \cdot\left(W_{1},\left(\Omega_{W_{1}}^{\bullet}\right) / X_{1}\right) \otimes \mathbf{H} \cdot\left(W_{2},\left(\Omega_{W_{2}}^{\bullet}\right) / X_{2}\right) \\
\downarrow \\
\mathbf{H} \cdot\left(W_{1} \times W_{2},\left(\Omega_{W_{1}}^{\stackrel{1}{2}) / X_{1}} \otimes\left(\Omega_{W_{2}}^{\cdot}\right) / X_{2}\right)\right. \\
\downarrow \cong \\
\mathbf{H} \bullet\left(W_{1} \times W_{2},\left(\Omega_{W_{1} \times W_{2}}^{\bullet}\right) / X_{1} \times X_{2}\right)
\end{gathered}
$$

called the Künneth morphisms for De Rham cohomology

$$
\begin{equation*}
\kappa_{\mathrm{DR}}^{\bullet}\left(X_{1}, X_{2}\right): H_{\mathrm{DR}}^{\bullet}\left(X_{1}\right) \otimes H_{\mathrm{DR}}^{\bullet}\left(X_{2}\right) \longrightarrow H_{\mathrm{DR}}^{\bullet}\left(X_{1} \times X_{2}\right) \tag{4.1.1}
\end{equation*}
$$

In the same way, taking hypercohomologies with supports, we have canonical morphisms

$$
\mathbf{H}_{X_{1}}^{\bullet}\left(W_{1}, \Omega_{W_{1}}^{\bullet}\right) \otimes \mathbf{H}_{X_{2}}^{\bullet}\left(W_{2}, \Omega_{W_{2}}^{\bullet}\right) \longrightarrow \mathbf{H}_{X_{1} \times X_{2}}^{\bullet}\left(W_{1} \times W_{2}, \Omega_{W_{1}}^{\bullet} \boxtimes \Omega_{W_{2}}^{\bullet}\right) \xrightarrow{\cong} \mathbf{H}_{X_{1} \times X_{2}}^{\bullet}\left(W_{1} \times W_{2}, \Omega_{W_{1} \times W_{2}}^{\bullet}\right),
$$

and the Künneth morphisms for De Rham homology

$$
\begin{equation*}
\kappa_{\bullet}^{\mathrm{DR}}\left(X_{1}, X_{2}\right): H_{\bullet}^{\mathrm{DR}}\left(X_{1}\right) \otimes H_{\bullet}^{\mathrm{DR}}\left(X_{2}\right) \longrightarrow H_{\bullet}^{\mathrm{DR}}\left(X_{1} \times X_{2}\right) \tag{4.1.2}
\end{equation*}
$$

Finally, starting with the isomorphism

$$
\Omega_{P_{1}}^{\cdot} \boxtimes \Omega_{P_{2}}^{\cdot} \xrightarrow{\cong} \Omega_{P_{1} \times P_{2}}^{\cdot}
$$

we find a canonical isomorphism
where $C$ is $\bar{X}_{1} \times \bar{X}_{2} \backslash X_{1} \times X_{2}=C_{1} \times \bar{X}_{2} \cup \bar{X}_{1} \times C_{2}$. The pro-objects are in fact cofinal, since the Ideal $\mathscr{I}_{\bar{X}_{1} \times \bar{X}_{2}}$ of $\bar{X}_{1} \times \bar{X}_{2}$ is given by $\mathscr{I}_{\bar{X}_{1}} \boxtimes \mathscr{O}_{P_{2}}+\mathscr{O}_{P_{1}} \boxtimes \mathscr{I}_{\bar{X}_{2}}$, while the Ideal $\mathscr{I}_{C}$ of $C$ is given by $\mathscr{I}_{C_{1}} \boxtimes \mathscr{I}_{C_{2}}$. Taking hypercohomology we obtain canonical morphisms

$$
\begin{aligned}
& \mathbf{H} \cdot\left(P_{1}, \text { "lim" }_{M \geqslant N} \mathscr{I}_{C_{1}}^{N} \Omega_{P_{1}}^{\bullet} / \mathscr{I}_{\bar{X}_{1}}^{M}\right) \otimes \mathbf{H} \cdot\left(P_{2}, \lim _{\rightleftarrows}^{\leftrightarrows}{ }_{M \geqslant N} \mathscr{I}_{C_{2}}^{N} \Omega_{P_{2}}^{\bullet} / \mathscr{I}_{\overline{X_{2}}}^{M}\right) \\
& \downarrow \\
& \mathbf{H} \bullet\left(P_{1} \times P_{2}, \text { "lim " }_{M \geqslant N} \mathscr{I}_{C_{1}}^{N} \Omega_{P_{1}}^{\bullet} / \mathscr{I}_{\bar{X}_{1}}^{M} \boxtimes \lim ^{\lim }{ }_{M \geqslant N} \mathscr{I}_{C_{2}}^{N} \Omega_{P_{2}}^{\bullet} / \mathscr{I}_{\bar{X}_{2}}^{M}\right) \\
& \downarrow \cong \\
& \mathbf{H} \cdot\left(P_{1} \times P_{2}, \text { "lim" }_{M \geqslant N} \mathscr{I}_{C}^{N} \Omega_{P_{1} \times P_{2}}^{\bullet} / \mathscr{I} \bar{X}_{1} \times \bar{X}_{2}\right)
\end{aligned}
$$

which will be called the Künneth morphisms for De Rham cohomology with compact supports

$$
\begin{equation*}
\kappa_{\mathrm{DR}, c}^{\bullet}\left(X_{1}, X_{2}\right): H_{\mathrm{DR}, c}^{\dot{\circ}}\left(X_{1}\right) \otimes H_{\dot{\mathrm{DR}}, c}^{\dot{\circ}}\left(X_{2}\right) \longrightarrow H_{\mathrm{DR}, c}^{\dot{ }}\left(X_{1} \times X_{2}\right) . \tag{4.1.4}
\end{equation*}
$$

In order to prove that the morphisms $\kappa_{\mathrm{DR}, c}^{\bullet}$ and $\kappa_{\bullet}^{\mathrm{DR}}$ are isomorphisms we need to extend the Künneth formula of [EGA III, 6.7.8] to complexes of $\mathscr{O}$-Modules with differential operators as differentials.
4.2. Lemma. Let $W_{i}$ with $i=1,2$ be two separated schemes of finite type over $K$. For each pair of bounded below complexes $\mathscr{F}_{i}(i=1,2)$ whose terms are quasi-coherent $\mathscr{O}_{W_{i}}$-Modules and whose differentials are $K$-linear, we have that the canonical morphisms

$$
\mathbf{H}^{\bullet}\left(W_{1}, \mathscr{F}_{1}^{\bullet}\right) \otimes_{K} \mathbf{H}^{\bullet}\left(W_{2}, \mathscr{F}_{2}^{\bullet}\right) \longrightarrow \mathbf{H}^{\bullet}\left(W_{1} \times W_{2}, \mathscr{F}_{1}^{\bullet} \boxtimes \mathscr{F}_{2}^{\bullet}\right)
$$

are isomorphisms.
Proof. Theorem 6.7.8 of [EGA III] proves the claim when the $\mathscr{F}_{i}$ 's are complexes of quasi-coherent $\mathscr{O}_{W_{i}}$-Modules (with $\mathscr{O}_{W_{i}}$-linear differentials). In fact any object in $\mathscr{F}_{i}$ is $K$-flat and so is $\mathscr{H}^{\bullet}\left(W_{i}, \mathscr{F}_{i}\right)$. The proof of this theorem is based on the definition of $\mathscr{T}_{\operatorname{or}_{n}^{K}\left(\mathscr{F}_{1}, \mathscr{F}_{2}\right) \text { ([EGA III, 6.4.1]) and on the results of }}$ [EGA III, 6.7.6-7]. We observe that all of these results could be extended to the case of $K$-linear differentials.
4.3. Theorem. The canonical morphisms $\kappa_{\mathrm{DR}, c}^{\bullet}$ and $\kappa_{\bullet}^{\mathrm{DR}}$ are isomorphisms compatible with the Poincaré duality.

Proof. We suppose that $P_{i}$ are proper (and smooth). The compatibility with the Poincaré duality is obvious by construction, and therefore it is sufficient to show that the Künneth morphisms for De Rham cohomology with compact supports are isomorphisms. By the previous Lemma we obtain the following isomorphism

$$
\begin{gathered}
\lim _{M, N}\left(\mathbf{H} \cdot\left(P_{1}, \mathscr{I}_{C_{1}}^{N-\bullet} \Omega_{P_{1}}^{\bullet} / \mathscr{I}_{\bar{X}_{1}}^{M-\bullet}\right) \otimes \mathbf{H} \cdot\left(P_{2}, \mathscr{I}_{C_{2}}^{N-\bullet} \Omega_{P_{2}}^{\bullet} / \mathscr{I}_{\bar{X}_{2}}^{M-\bullet}\right)\right) \\
\downarrow \\
\lim _{M, N} \mathbf{H} \bullet\left(P_{1} \times P_{2}, \mathscr{I}_{C_{1}}^{N-} \cdot \Omega_{P_{1}}^{\bullet} / \mathscr{I}_{\bar{X}_{1}}^{M-\bullet} \boxtimes \mathscr{I}_{C_{2}}^{N-} \cdot \Omega_{P_{2}}^{\bullet} / \mathscr{I}_{\bar{X}_{2}}^{M-\bullet}\right) .
\end{gathered}
$$

The first term is isomorphic to

$$
\varliminf_{M, N} \mathbf{H}^{\bullet}\left(P_{1}, \mathscr{I}_{C_{1}}^{N-\bullet} \Omega_{P_{1}}^{\bullet} / \mathscr{I}_{\bar{X}_{1}}^{M-\bullet}\right) \otimes \lim _{M, N} \mathbf{H}^{\bullet}\left(P_{2}, \mathscr{I}_{C_{2}}^{N-\bullet} \Omega_{P_{2}}^{\bullet} / \mathscr{I}_{\bar{X}_{2}}^{M-\bullet}\right)=: H_{\mathrm{DR}, c}^{\bullet}\left(X_{1}\right) \otimes H_{\mathrm{DR}, c}^{\bullet}\left(X_{2}\right) .
$$

In fact, since the $P_{i}$ are proper, the $K$-vector spaces

$$
\mathbf{H}^{\bullet}\left(P_{i}, \mathscr{I}_{C_{i}}^{N-\bullet} \Omega_{P_{i}}^{\bullet} / \mathscr{I}_{\bar{X}_{i}}^{M-\bullet}\right) \quad \text { and } \quad \varliminf_{M, N} \mathbf{H}^{\bullet}\left(P_{i}, \mathscr{I}_{C_{i}}^{N-\bullet} \Omega_{P_{i}}^{\bullet} / \mathscr{I}_{\bar{X}_{i}}^{M-\bullet}\right)
$$

are finite dimensional. The second term is isomorphic to

$$
\varliminf_{M, N} \mathbf{H}^{\bullet}\left(P_{1} \times P_{2}, \mathscr{I}_{C}^{N-\bullet} \Omega_{P_{1} \times P_{2}}^{\bullet} / \mathscr{I}_{\bar{X}_{1} \times \bar{X}_{2}}^{M-}\right)=: H_{\mathrm{DR}, c}^{\bullet}\left(X_{1} \times X_{2}\right)
$$

by (4.1.3).

## 5. Classical comparison theorems.

In this section we keep the notation of our setting 1.1, and assume $K=\mathbb{C}$. We compare the algebraic De Rham cohomology with compact supports of the algebraic variety $X$, with the singular cohomology with
compact supports of the corresponding complex analytic space $X^{\text {an }}$. We refer to chapter IV of [H.75] for the analogous statements for De Rham cohomology without supports (or with supports in a closed subvariety $Z$ of $X)$ and for De Rham homology.

We recall that in the construction of the algebraic pairing of (3.1) we assumed for simplicity $P$ to be smooth. The algebraic pairing was then shown to be independent of the choice of $P$. When comparing the algebraic and analytic Poincaré duality pairings we will make the same assumption. We then construct canonical morphisms relating the spectral sequence of algebraic De Rham cohomology with compact supports (resp. algebraic De Rham homology) and the analytic one. These morphisms will be compatible with the natural pairings of spectral sequences of section 3 and of [HL].
5.0. Results from [H.75, IV]. Let $T$ be a complex analytic space and $S$ be a closed analytic subspace defined by the coherent $\mathscr{O}_{T}$-Ideal $\mathscr{I}_{S}$. Then $T_{/ S}$ will denote the formal completion of $T$ along $S$, namely the ringed space with underlying topological space $S$ and structural sheaf $\mathscr{O}_{T / S}:=\lim _{N}\left(\mathscr{O}_{T} / \mathscr{I}_{S}^{N}\right)_{\mid S}$. Similarly, for a coherent $\mathscr{O}_{T}$-Module $\mathscr{F}, \mathscr{F} / S$ will denote the $\mathscr{O}_{T_{/ S}}$-Module $\varliminf_{\varliminf_{N}}\left(\mathscr{F} / \mathscr{J}_{S}^{N} \mathscr{F}\right)_{\mid S}$, which coincides with the inverse image of the $\mathscr{O}_{T}$-Module $\mathscr{F}$ on $T_{/ S}$. If $\kappa: U \hookrightarrow T$ is an open immersion of analytic spaces, or formal completions of such, the functor

$$
\kappa_{!}: \mathscr{A} b(U) \longrightarrow \mathscr{A} b(T),
$$

left adjoint to $\kappa^{-1}: \mathscr{A} b(T) \longrightarrow \mathscr{A} b(U)$, is the usual topological extension by zero.
5.0.1. The formal analytic Poincaré lemma [H.75, IV.2.1]. The complex $\left(\Omega_{W^{\text {an }}}\right)_{/ X^{\text {an }}}$ is a resolution of the constant sheaf $\mathbb{C}_{X^{\text {an }}}$ in the category of abelian sheaves on $X^{\text {an }}$.
5.0.2. [H.75, IV.1.1]. The canonical morphism

$$
\beta^{i}: H_{X^{\text {an }}}^{i}\left(W^{\text {an }}, \mathbb{C}\right) \longrightarrow \mathbf{H}_{X^{\text {an }}}^{i}\left(W^{\text {an }}, \Omega_{W^{\text {an }}}^{\circ}\right)
$$

is an isomorphism for any $i$. We recall that, if $\operatorname{dim} W=n, H_{X^{\text {an }}}^{2 n-i}\left(W^{\text {an }}, \mathbb{C}\right) \cong H_{i}^{B M}\left(X^{\text {an }}, \mathbb{C}\right)$, the Borel-Moore homology of $X^{\text {an }}$. We set

$$
H_{i}^{D R}\left(X^{\mathrm{an}}\right):=\mathbf{H}_{X^{\text {an }}}^{2 n-i}\left(W^{\mathrm{an}}, \Omega_{W^{\text {an }}}\right) .
$$

5.0.3. [H.75, loc. cit.]. The canonical morphism

$$
\alpha^{i}: H_{D R}^{i}(X) \longrightarrow H_{D R}^{i}\left(X^{\mathrm{an}}\right)
$$

is an isomorphism for any $i$ (apply $\alpha^{i}$ of Hartshorne with $X, X, W$ as $Z, X, Y$ ).
5.0.4. [H.75, loc. cit.]. The canonical morphism

$$
\alpha_{i}: H_{i}^{D R}(X) \longrightarrow H_{i}^{D R}\left(X^{\mathrm{an}}\right)
$$

is an isomorphism for any $i$ (apply $\alpha^{i}$ of Hartshorne with $X, W, W$ as $Z, X, Y$ ).
5.1. Analytic De Rham cohomology with compact supports. We define

$$
H_{D R, c}^{\bullet}\left(X^{\mathrm{an}}\right):=\mathbf{H}_{c}^{\bullet}\left(\bar{X}^{\mathrm{an}},\left(\Omega_{W^{\mathrm{an}}}^{\bullet}\right) / X^{\mathrm{an}}\right) \cong \mathbf{H}^{\bullet}\left(\bar{X}^{\mathrm{an}}, j_{!}^{\mathrm{an}}\left(\Omega_{W^{\mathrm{an}}}^{\bullet}\right) / X^{\mathrm{an}}\right)
$$

and we recall that

$$
H_{c}^{i}\left(X^{\mathrm{an}}, \mathbb{C}\right) \cong H^{i}\left(\bar{X}^{\mathrm{an}}, j_{!}^{\mathrm{an}} \mathbb{C}_{X^{\mathrm{an}}}\right)
$$

By 5.0.1 and the exactness of $j_{!}^{\text {an }}$, the canonical morphism

$$
\beta_{c}^{i}: H_{c}^{i}\left(X^{\mathrm{an}}, \mathbb{C}\right) \longrightarrow H_{D R, c}^{i}\left(X^{\mathrm{an}}\right)
$$

is an isomorphism for any $i$.
5.1.1. Lemma. Let $\kappa: U \hookrightarrow T$ be an open immersion of complex analytic spaces and $\mathscr{J}$ be a coherent sheaf of $\mathscr{O}_{T}$-Ideals such that the support of $\mathscr{O}_{T} / \mathscr{J}$ is $T \backslash U$. Let $\mathscr{F}$ be a coherent $\mathscr{O}_{U}$-Module and let $\overline{\mathscr{F}}$ be any coherent extension of $\mathscr{F}$ to $T$. Then the canonical morphism

$$
\kappa!\mathscr{F} \longrightarrow \varliminf_{N} \lim ^{N} \stackrel{\rightharpoonup}{\mathscr{F}}
$$

is an isomorphism.
Proof. The assertion is easily checked on the fibers. In fact, if $x$ is a point of $T \backslash U, \mathscr{J}_{x}$ is a proper ideal of the noetherian ring $\mathscr{O}_{T, x}$. So, $\varliminf_{N} \mathscr{J}_{x}^{N} \overline{\mathscr{F}}_{x}=\bigcap_{N} \mathscr{J}_{x}^{N} \overline{\mathscr{F}}_{x}=0$. On the other hand, for any $M$, the


$$
\left(\varliminf_{N} \mathscr{J}^{N} \overline{\mathscr{F}}\right)_{x} \subseteq{\underset{N}{N}}^{\lim _{x}} \mathscr{J}_{x}^{N \overline{\mathscr{F}}_{x}=0}
$$

is also zero. On the other hand, both sheaves restrict to $\mathscr{F}$ on $U$.
5.1.2. Lemma. Let $\mathscr{F}$ be a coherent $\mathscr{O}_{P^{\text {an }}}$-Module. The canonical morphism

$$
j_{!}^{\text {an }}\left(\left(\mathscr{F}_{\mid W^{\text {an }}}\right) / X^{\text {an }}\right) \longrightarrow{\underset{N}{\leftrightarrows}}_{\lim _{C^{\text {an }}}}^{N} / \bar{X}^{\text {an }}
$$

is an isomorphism.
Proof. We have to show that for any $x \in P^{\text {an }}$, the fiber

If $x \in P^{\mathrm{an}} \backslash \bar{X}^{\mathrm{an}}$ the assertion is clear: on open neighborhoods $U$ of $x, x \in U \subseteq P^{\mathrm{an}} \backslash \bar{X}^{\mathrm{an}}$
since $\left(\mathscr{I}_{C^{\mathrm{an}}}\right)_{\mid P^{\mathrm{an}}} \backslash \bar{X}^{\mathrm{an}}=\left(\mathscr{I}_{\bar{X}^{\mathrm{an}}}\right)_{\mid P^{\mathrm{an}}} \backslash \bar{X}^{\mathrm{an}}=\left(\mathscr{O}_{P^{\mathrm{an}}}\right)_{\mid P^{\mathrm{an}}} \backslash \bar{X}^{\mathrm{an}}$. If $x \in X^{\mathrm{an}}$, the assertion is also clear, since
where $U$ varies among open neighborhoods of $x$ contained in $W^{\text {an }}$. But on $W^{\text {an }},\left(\mathscr{I}_{C^{\text {an }}}\right)_{W^{\text {an }}}=\mathscr{O}_{W^{\text {an }}}$, so that

We are left to show that, for $x \in C^{\text {an }},\left(\lim _{M \geqslant N} \mathscr{I}_{C^{\text {an }}}^{N} \mathscr{F} / \mathscr{I}_{\bar{X}^{\text {an }}}^{N} \mathscr{F}\right)_{x}=0$. To check this, we write
for the open immersions, and apply (5.1.1) to obtain the exact sequence

$$
\begin{aligned}
& 0 \longrightarrow\left(j _ { P ^ { \text { an } } \backslash \overline { X } ^ { \text { an } } ) ! \mathscr { F } _ { | P ^ { \text { an } } } \backslash \overline { X } ^ { \text { an } } } ^ { \longrightarrow } \left(j_{\left.P^{\text {an }} \backslash C^{\text {an }}\right)!\mathscr{F}} \mid P^{\text {an }} \backslash C^{\text {an }} \longrightarrow \lim _{M_{M}} \mathscr{I}_{C^{\text {an }}}^{N} \mathscr{F} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M} \mathscr{F} \longrightarrow\right.\right. \\
& \operatorname{coker}\left(\mathscr{F} \rightarrow \mathscr{F} / \bar{X}^{\text {an }}\right) \longrightarrow \operatorname{coker}\left(\mathscr{F} \rightarrow \mathscr{F} / C^{\text {an }}\right) \longrightarrow \cdots
\end{aligned}
$$

Taking fibers at $x \in C^{\text {an }}$, we obtain the exact sequence

$$
\begin{aligned}
& 0 \longrightarrow\left(\lim _{M \geqslant N} \mathscr{I}_{C^{\text {an }}}^{N} \mathscr{F} / \mathscr{I} \bar{X}^{M} \frac{\operatorname{Fi}}{}\right)_{x} \longrightarrow \operatorname{coker}\left(\mathscr{F} \rightarrow \mathscr{F} / \bar{X}^{\mathrm{an}}\right)_{x}= \\
& =\operatorname{coker}\left(\mathscr{F}_{x} \rightarrow\left(\mathscr{F}_{/ X^{\mathrm{an}}}\right)_{x}\right) \longrightarrow \operatorname{coker}\left(\mathscr{F} \rightarrow \mathscr{F}_{/ C^{\mathrm{an}}}\right)_{x}=\operatorname{coker}\left(\mathscr{F}_{x} \rightarrow\left(\mathscr{F}_{/ C^{\mathrm{an}}}\right)_{x}\right) .
\end{aligned}
$$

Now, for a Stein semianalytic compact neighbourhood $K$ of $x$ in $P^{\text {an }}, \Gamma(K, \mathscr{F})$ is a module of finite type over the noetherian ring $\Gamma\left(K, \mathscr{O}_{P \text { an }}\right)$. For a coherent $\mathscr{O}_{P \text { an }}$-Ideal $\mathscr{J}$, and $K$ as before, we denote by $\widehat{\Gamma(K, \mathscr{F})_{\Gamma(K, \mathscr{J})}}$ the $\Gamma(K, \mathscr{J})$-adic completion of $\Gamma(K, \mathscr{F})$. Then, by [BS, VI.2.2 (i)],

Then

$$
\begin{aligned}
& \operatorname{ker}\left(\operatorname{coker}\left(\mathscr{F}_{x} \rightarrow\left(\mathscr{F}_{X^{\mathrm{an}}}^{\text {an }}\right)_{x}\right) \longrightarrow \operatorname{coker}\left(\mathscr{F}_{x} \rightarrow\left(\mathscr{F}_{/ C^{\text {an }}}\right)_{x}\right)\right)= \\
& \quad \underset{K}{\text { lim }} \operatorname{ker}\left(\Gamma(K, \mathscr{F})_{\Gamma\left(K, \mathscr{I}_{\left.\bar{X}^{\text {an }}\right)} / \Gamma(K, \mathscr{F}) \longrightarrow \Gamma(\widehat{(K, \mathscr{F}})_{\Gamma\left(K, \mathscr{\mathscr { C }}_{C^{\text {an }}}\right)} / \Gamma(K, \mathscr{F})\right)=0 .} .\right.
\end{aligned}
$$

5.2. The canonical isomorphism

$$
j_{!}^{\mathrm{an}}\left(\Omega_{W^{\text {an }}}\right) / X^{\text {an }} \longrightarrow \varliminf_{M \geqslant N}\left(\mathscr{I}_{C^{\text {an }}}^{N-} \cdot \Omega_{P^{\text {an }}} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M-\bullet}\right)
$$

induces a canonical morphism of hypercohomology groups

$$
H_{\mathrm{DR}, c}^{i}\left(X^{\mathrm{an}}\right) \longrightarrow \lim _{M \geqslant N} \mathbf{H}^{i}\left(\bar{X}^{\mathrm{an}}, \mathscr{I}_{C^{\mathrm{an}}}^{N-\bullet} \Omega_{P^{\mathrm{an}}}^{\bullet} / \mathscr{I}_{\bar{X}^{\mathrm{an}}}^{M-\bullet}\right) .
$$

Via the GAGA isomorphisms

$$
\varliminf_{M \geqslant N} \mathbf{H}^{\bullet}\left(\bar{X}^{\mathrm{an}}, \mathscr{I}_{C^{\text {an }}}^{N-\bullet} \Omega_{P \mathrm{an}}^{\bullet} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M-\bullet}\right) \cong \varliminf_{M \geqslant N} \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{I}_{C}^{N-} \cdot \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right)
$$

we obtain a canonical morphism

$$
\begin{equation*}
\alpha_{c}^{i}: H_{\mathrm{DR}, c}^{i}\left(X^{\mathrm{an}}\right) \longrightarrow H_{\mathrm{DR}, c}^{i}(X) . \tag{5.2.1}
\end{equation*}
$$

5.3. Theorem. The canonical morphism $\alpha_{c}^{i}$ is an isomorphism for any $i$.

Proof.

$$
\begin{aligned}
& H_{\mathrm{DR}, c}^{\bullet}\left(X^{\mathrm{an}}\right)=\mathbf{H}^{\bullet}\left(\bar{X}^{\text {an }}, j_{!}^{\text {an }}\left(\Omega_{W^{\text {an }}}^{\bullet}\right) / X^{\text {an }}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \xrightarrow{\cong} \mathbf{H}^{\bullet}\left(\bar{X}^{\mathrm{an}}, \varliminf_{M \geqslant N}^{\varliminf_{\overparen{2}}}\left(\Omega_{P^{\text {an }}}^{\bullet} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M-} \rightarrow \Omega_{P^{\text {an }}}^{\bullet} / \mathscr{I}_{C^{\text {an }}}^{N-\bullet}\right)_{\mathrm{tot}}\right) \\
& \underset{(1.3 .1)}{\cong} \underset{M \geqslant N}{\lim _{\leftrightarrows} \mathbf{H}^{\bullet}\left(\bar{X}^{\text {an }},\left(\Omega_{P^{\text {an }}}^{\bullet} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M-\bullet} \rightarrow \Omega_{P^{\text {an }}}^{\bullet} / \mathscr{I}_{C^{\text {an }}}^{N-\bullet}\right)_{\mathrm{tot}}\right), ~\left(\Omega^{\circ}\right.} \\
& \cong \varliminf_{M \geqslant N} \mathbf{H}^{\bullet}\left(\bar{X}^{\text {an }}, \mathscr{I}_{C^{\text {an }}}^{N-} \cdot \Omega_{P^{\text {an }}}^{\bullet} / \mathscr{I}_{\bar{X}^{\text {an }}}^{M-\bullet}\right) \\
& \stackrel{\cong}{\cong} \lim _{\text {GAGA } \left.^{( }\right)} \mathbf{H}^{\bullet}\left(\bar{X}, \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet}\right)=H_{\mathrm{DR}, c}^{\bullet}(X)
\end{aligned}
$$

5.3.1. We point out the following alternative proof of theorem 5.3 , valid if $P$ is supposed to be smooth. We have a morphism of exact sequences of abelian sheaves on $\bar{X}^{\text {an }}$

where as usual $\mathbb{C}_{X^{\text {an }}}=j_{!}^{\text {an }} \mathbb{C}_{X^{\text {an }}}$ and $\mathbb{C}_{C^{\text {an }}}=h_{*}^{\text {an }} \mathbb{C}_{C^{\text {an }}}$, as sheaves on $\bar{X}^{\text {an }}$. All vertical arrows are quasiisomorphism by the Formal Analytic Poincaré lemma and exactness of $j_{!}^{\text {an }}$ and $h_{*}^{\text {an }}$. In particular

$$
H_{\mathrm{DR}, c}^{\bullet}\left(X^{\mathrm{an}}\right)=\mathbf{H}^{\bullet}\left(\bar{X}^{\mathrm{an}}, j_{!}^{\mathrm{an}}\left(\Omega_{W^{\text {an }}}^{\bullet}\right) / X^{\mathrm{an}}\right) \xrightarrow{\cong} \mathbf{H}^{\bullet}\left(\bar{X}^{\mathrm{an}},\left(\left(\Omega_{P^{\mathrm{an}}}^{\bullet}\right) / \bar{X}^{\mathrm{an}} \rightarrow h_{*}^{\mathrm{an}}\left(\Omega_{P^{\mathrm{an}}}^{\bullet}\right)_{/ \bar{C}^{\mathrm{an}}}\right)_{\mathrm{tot}}\right)
$$

which considerably simplifies the above proof.
5.3.2. We remark moreover that in the analytic case
while in the algebraic case this is false; in fact in general $\lim _{M \geqslant N}^{(1)} \mathscr{I}_{C}^{N-\bullet} \Omega_{P}^{\bullet} / \mathscr{I}_{\bar{X}}^{M-\bullet} \neq 0$.
5.4. Comparison of algebraic and analytic Poincaré dualities. We will show that the pairings (3.5.4) (and therefore also the duality morphisms (3.1.2)) are compatible with the algebraic-analytic comparison maps. Notation is as in (3.1); in particular, $P$ is smooth.
5.4.1. Proposition. Under the assumptions of (3.1), we have a canonical morphism of pairings of spectral sequences

 of $\mathscr{M}^{(M) \cdot \text { an }}$ in the category of $\mathscr{C}_{\left(\bar{X}^{\mathrm{an}}\right)_{P \mathrm{an}}^{(M)}}$-Modules. We obtain a resolution

$$
\begin{equation*}
\mathscr{M}^{\bullet \text { an }} \cong "{\underset{M}{\lim } "}_{M^{(M)} \bullet \text { an }}^{\longrightarrow} \breve{" l i m}_{\lim _{M}} \mathscr{J}^{(M) \cdot \bullet}=: \mathscr{J} \cdot \bullet \tag{5.4.2}
\end{equation*}
$$

of $\mathscr{M}^{\bullet a n}$ in $\operatorname{Pro} \mathscr{C}\left(P^{\text {an }}\right)$. There is a canonical morphism of complexes of objects of $\operatorname{Pro} \mathscr{C}\left(P^{\text {an }}\right), \mathscr{I} \cdot \bullet$ an $\longrightarrow \mathscr{J} \bullet \bullet$ (which is an isomorphism of $\mathbf{D}\left(\operatorname{Pro} \mathscr{C}\left(P^{\mathrm{an}}\right)\right)$ ). On the other hand, if $I^{\bullet}\left(\Omega_{P^{\text {an }}}^{\bullet}\right)$ is an injective resolution of $\Omega_{P^{\text {an }}}^{\text {an }}$ as a $\mathscr{C}_{P}^{\text {an }}$-Module, then we have a canonical morphism of complexes of objects of $\mathscr{C}\left(P^{\text {an }}\right)$, $E^{\bullet}\left(\Omega_{P}^{\bullet}\right)^{\text {an }} \longrightarrow I^{\bullet}\left(\Omega_{P^{\text {an }}}^{\bullet}\right)$ (again an isomorphism of $\mathbf{D}\left(\mathscr{C}\left(P^{\text {an }}\right)\right)$ ). Therefore we have the following diagram of (pairings of) double complexes


We notice that the canonical morphisms of bicomplexes
induce isomorphisms of the associated spectral sequences (see for example [HL, 4.2]). Therefore, taking the associated diagram of pairings of spectral sequences and composing with the canonical GAGA morphism of pairings of spectral sequences induced by

we complete the proof.
5.4.3. Theorem. For any $i$, the canonical comparison isomorphisms $\alpha_{i}$ of (5.0.4) and $\alpha_{c}^{i}$ of (5.2.1) are compatible with Poincaré duality, i.e. they fit in a commutative diagram

$$
\begin{array}{cc}
H_{i}^{\mathrm{DR}}(X) \otimes H_{\mathrm{DR}, c}^{i}(X) & \longrightarrow \mathbb{C} \\
\alpha_{i} \| & \| \alpha_{c}^{i} \\
H_{i}^{\mathrm{DR}}\left(X^{\mathrm{an}}\right) \otimes H_{\mathrm{DR}, c}^{i}\left(X^{\mathrm{an}}\right) \longrightarrow & \| \\
\mathbb{C} .
\end{array}
$$

Proof. We apply the previous proposition, with $\mathscr{M} \bullet="{ }_{\leftrightarrows}^{l_{m}}{ }_{M} j_{!}^{(M)} \Omega_{X_{W}^{(M)}}$. We obtain a commutative diagram

$$
\begin{aligned}
& H_{i}^{\mathrm{DR}}(X) \otimes H_{\mathrm{DR}, c}^{i}(X) \longrightarrow \mathbf{H}_{\bar{X}}^{2 n}\left(P, \Omega_{P}^{\bullet}\right)=H_{0}^{D R}(\bar{X}) \\
& 2 \| \alpha_{c}^{i} \\
& \alpha_{i} \| \\
& H_{i}^{D R}\left(\bar{X}^{\mathrm{an}}\right) \otimes H_{\mathrm{DR}, c}^{i}\left(X^{\mathrm{an}}\right) \longrightarrow \mathbf{H}_{\overline{X^{\mathrm{an}}}}^{2 n}\left(P^{\mathrm{an}}, \Omega_{P^{\mathrm{an}}}^{\bullet}\right)= H_{0}^{D R}\left(\bar{X}^{\mathrm{an}}\right) .
\end{aligned}
$$

We consider the algebraic trace map $\operatorname{Tr}: \mathbf{H}_{\bar{X}}^{2 n}\left(P, \Omega_{P}^{\bullet}\right) \longrightarrow \mathbb{C}$ and the map $\operatorname{Tr}^{\text {an }}: \mathbf{H}_{X^{\text {an }}}^{2 n}\left(P^{\text {an }}, \Omega_{P^{\text {an }}}\right) \longrightarrow \mathbb{C}$ uniquely defined by fitting in the commutative diagram


Composing with $\operatorname{Tr}$ and $\operatorname{Tr}^{\text {an }}$, we obtain the diagram of the statement.

## 6. Compatibility of rigid and algebraic Poincaré duality.

6.1. Setting. Let $\mathscr{V}$ be a discrete valuation ring of mixed characteristics $(0, p)$. As usual, let $K$ and $k$ be, respectively, the fraction field and the residue field of $\mathscr{V}$. All $\mathscr{V}$-schemes will be meant to be
separated and of finite type, and all morphisms between them will be assumed to be $\mathscr{V}$-morphisms. Let $X$ be a $\mathscr{V}$-scheme, and let $j: X \hookrightarrow \bar{X}$ be an open immersion in a proper $\mathscr{V}$-scheme. Let $i: \bar{X} \hookrightarrow P$ be a closed immersion of $\bar{X}$ in a $\mathscr{V}$-scheme $P$. We assume that there is an open smooth $\mathscr{V}$-subscheme $W$ of $P$ containing $X$ as a closed subscheme. Let $h: C \hookrightarrow \bar{X}$ be a closed $\mathscr{V}$-subscheme of $\bar{X}$, whose support is exactly $|\bar{X}| \backslash|X|$.
6.2. Notation. For a $\mathscr{V}$-scheme $T$, we will use the standard notation $T_{k}$ (resp. $T_{K}$ ) for the special (resp. the generic) fiber. The formal $\mathscr{V}$-scheme completion of $T$ along a closed $\mathscr{V}$-subscheme $S$, will be denoted by $T_{/ S}$. In the special case of $S=T_{k}$, we will write $\widehat{T}$ for $T_{/ T_{k}}$. We freely use the notions and notation of [B.96]. In particular, the (Raynaud) generic fiber of a formal scheme $\mathscr{T}$ over $\operatorname{Spf} \mathscr{V}$ (for the $p$-adic topology on $\mathscr{V}$ ) is denoted by $\mathscr{T}_{K}$. We point out that we take this notion in the extended sense of [B.96, 0.2.6]. If $T$ is a $\mathscr{V}$-scheme, the tube of a locally closed subset $U$ of $T_{k}$ in $(\widehat{T})_{K}=: \widehat{T}_{K}$ will be denoted by $] U{ }_{\widehat{T}}$. We use as much as possible a functorial notation for these constructions. For example, for $C \stackrel{h}{\hookrightarrow} \bar{X} \stackrel{i}{\hookrightarrow} P$ as before, we get

$$
\begin{gather*}
] h_{k}[\widehat{P}:] C_{k}[\widehat{P} \longleftrightarrow] \bar{X}_{k}\left[\widehat{P} \quad, \quad h_{P /}: P_{/ C_{k}} \longleftrightarrow P_{/ \bar{X}_{k}}\right.  \tag{6.2.1}\\
\left(h_{P /}\right)_{K}:\left(P_{/ C_{k}}\right)_{K} \longleftrightarrow\left(P_{/ \bar{X}_{k}}\right)_{K} \quad, \quad \text { and } \quad\left(i_{/ \bar{X}_{k}}\right)_{K}: \hat{\bar{X}}_{K} \longleftrightarrow\left(P_{/ \bar{X}_{k}}\right)_{K} \tag{6.2.2}
\end{gather*}
$$

Notice that, by [B.96, 0.2.7], ] $h_{k}\left[\widehat{P}\right.$ identifies with $\left(h_{P /}\right)_{K}$.
For a $K$-scheme of finite type $T$, we denote by $T^{\text {an }}$ the associated rigid analytic space and let

$$
\begin{equation*}
\varepsilon=\varepsilon_{T}: T^{\mathrm{an}} \longrightarrow T \tag{6.2.3}
\end{equation*}
$$

be the natural morphism of locally ringed $G$-spaces. For a coherent sheaf $\mathscr{E}$ on $T$ we set $\mathscr{E}{ }^{\text {an }}=\varepsilon_{T}^{*}(\mathscr{E})$. When $T$ is a proper $\mathscr{V}$-scheme, $\widehat{T}_{K} \cong\left(T_{K}\right)^{\text {an }}$. For example, the map $\left(i_{/ \bar{X}_{k}}\right)_{K}$ in (6.2.2) identifies with the natural closed immersion of rigid $K$-analytic spaces $\bar{X}_{K}^{\text {an }}$ in $] \bar{X}_{k}[\widehat{P}$.
6.3. СоhOMOLOGY WITH COMPACT SUPPORTS. The data in 6.1 permit to simultaneously calculate the algebraic De Rham cohomology with compact supports of $X_{K} / K, H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)$, and the rigid cohomology with compact supports of $X_{k} / K, H_{\text {rig }, c}^{\bullet}\left(X_{k} / K\right)$.

The algebraic De Rham cohomology with proper supports of $X_{K} / K$ may be calculated from the complex of abelian sheaves on $\bar{X}_{K}$

$$
\begin{equation*}
\underset{M \geqslant N}{\mathbf{R} \lim _{\overparen{C}}} \mathscr{I}_{C_{K}}^{N-\bullet} \cdot \Omega_{P_{K}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}}^{M-\bullet} \cong\left(\left(\Omega_{P_{K}}^{\bullet}\right)_{/ \bar{X}_{K}} \rightarrow\left(\Omega_{P_{K}}^{\bullet}\right)_{/ C_{K}}\right)_{\mathrm{tot}} \tag{6.3.1}
\end{equation*}
$$

so that

$$
\begin{equation*}
H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)=\mathbf{H} \bullet\left(\bar{X}_{K},\left(\left(\Omega_{P_{K}}^{\bullet}\right) / \bar{X}_{K} \rightarrow\left(\Omega_{P_{K}}^{\bullet}\right)_{/ C_{K}}\right)_{\mathrm{tot}}\right) \tag{6.3.2}
\end{equation*}
$$

6.4. The complex of coherent $\mathscr{O}_{P_{K}}$-Modules

$$
\left(\left(\Omega_{P_{K}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}}^{N-\bullet}\right)^{\text {an }} \rightarrow\left(\Omega_{P_{K}}^{\bullet} / \mathscr{I}_{C_{K}}^{N-\bullet}\right)^{\text {an }}\right)_{\mathrm{tot}} \cong\left(\left(\Omega_{P_{K}^{\text {an }}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}^{\mathrm{an}}}^{N-\bullet}\right) \rightarrow\left(\Omega_{P_{K}^{\bullet a n}}^{\bullet} / \mathscr{I}_{C_{K}^{\mathrm{an}}}^{N-\bullet}\right)\right)_{\mathrm{tot}}
$$

where $\mathscr{I}_{\bar{X}_{K}^{\text {an }}}\left(\right.$ resp. $\left.\mathscr{I}_{C_{K}^{\text {an }}}\right)$ is the $\mathscr{O}_{P_{K}^{\text {an }}}$-Ideal corresponding to the closed analytic subspace $\bar{X}_{K}^{\text {an }}$ (resp. $\left.C_{K}^{\text {an }}\right)$ of $P_{K}^{\text {an }}$ may also be regarded as a complex of abelian sheaves on $\bar{X}_{K}^{\text {an }}$. By GAGA over $K$ and (1.3.1) on $\bar{X}_{K}$, we then have

$$
\begin{equation*}
H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)={\underset{N}{\overleftarrow{N}}}_{\lim _{N}} \mathbf{H}^{\bullet}\left(\bar{X}_{K}^{\mathrm{an}},\left(\left(\Omega_{P_{K}^{\text {an }}} / \mathscr{I}_{\bar{X}_{K}^{\text {an }}}^{N-\bullet}\right) \longrightarrow\left(\Omega_{P_{K}^{\text {an }}}^{\bullet} / \mathscr{I}_{C_{K}^{\text {an }}}^{N-\bullet}\right)\right)_{\mathrm{tot}}\right) \tag{6.4.1}
\end{equation*}
$$

6.5. We have canonical flat morphisms of $\mathscr{V}$-formal schemes

$$
\begin{equation*}
P_{/ C_{k}} \xrightarrow{h_{P /}} P_{/ \bar{X}_{k}} \xrightarrow{i_{P /}} P_{/ P_{k}}=: \widehat{P} \tag{6.5.1}
\end{equation*}
$$

and open immersions of rigid $K$-analytic spaces

$$
\begin{equation*}
\left(P_{/ C_{k}}\right)_{K} \xrightarrow{\left(h_{P /}\right)_{K}}\left(P_{/ \bar{X}_{k}}\right)_{K} \xrightarrow{\left(i_{P /}\right)_{K}}\left(P_{/ P_{k}}\right)_{K}=\widehat{P}_{K} . \tag{6.5.2}
\end{equation*}
$$

The last sequence identifies via $[B .96,0.2 .7]$ with the natural sequence of open immersions of rigid $K$-analytic spaces
(6.5.3)

Let us consider the diagram

$$
] C_{k}[\widehat{P} \longleftrightarrow] \bar{X}_{k}\left[\widehat{P} \longleftrightarrow \widehat{P}_{K} .\right.
$$

$$
\begin{gathered}
P_{/ C_{k}} \xrightarrow{h_{P /}} P_{/ \bar{X}_{k}} \\
\searrow \swarrow \\
P
\end{gathered}
$$

We have $\Omega_{P}^{\bullet}$ on $P$ and denote by $\Omega_{P_{/ C_{k}}}^{\bullet}$ (resp. $\Omega_{P_{/ \bar{X}_{k}}}^{{ }^{\prime}}$ ), its inverse image on $P_{/ C_{k}}$ (resp. $P_{/ \bar{X}_{k}}$ ). By analytification in the sense of [B.96, 0.2.6], and using [B.96, 0.2.7], we obtain a complex of coherent sheaves on $\left.\left(P_{/ C_{k}}\right)_{K}=\right] C_{k}\left[\widehat{P}\right.$ (resp. on $\left.\left(P_{/ \bar{X}_{k}}\right)_{K}=\right] \bar{X}_{k}[\widehat{P})$, namely $\left(\Omega_{P_{/ C_{k}}}^{\bullet}\right)_{K}=\Omega_{j_{C_{k}} \widehat{\widehat{P}}^{\bullet}}\left(\operatorname{resp} .\left(\Omega_{P_{/ \bar{X}_{k}}}^{\bullet}\right)_{K}=\Omega_{]_{\bar{X}_{k}[\widehat{P}}}\right)$.

By [B.86, 3.2] we have

$$
\begin{equation*}
\mathbf{R} \Gamma_{] X_{k}\left[\widehat{P}_{P}\right.} \Omega_{]_{k}[\widehat{P}}^{\bullet}=\left(\Omega_{]_{\bar{X}_{k}}^{\bullet}} \rightarrow(]_{\widehat{P}}\left[\widehat{\widehat{P}}^{)_{*}} \Omega_{j C_{k}[\widehat{P}}^{\bullet}\right)_{\mathrm{tot}}=\left(\left(\Omega_{P_{/ \bar{X}_{k}}}^{\bullet}\right)_{K} \rightarrow\left(\left(h_{P /}\right)_{K}\right)_{*}\left(\Omega_{P_{/ C_{k}}}^{\bullet}\right)_{K}\right)_{\text {tot }}\right. \tag{6.5.4}
\end{equation*}
$$

where $\left(\mathscr{O}_{P_{/ \bar{X}}^{k}}\right)_{K}$ is taken in bidegree $(0,0)$. By definition [B.86, 3.3],

$$
\begin{equation*}
H_{\mathrm{rig}, c}^{\bullet \cdot}\left(X_{k} / K\right):=\mathbf{H}_{]_{k}\left[\left[_{\widehat{P}}\right.\right.}(] \bar{X}_{k}\left[_{\widehat{P}}, \Omega_{]_{\bar{X}_{k}[\widehat{P}}^{\bullet}}\right)=\mathbf{H}^{\bullet}(] \bar{X}_{k}\left[\widehat{P},\left(\Omega_{] \bar{X}_{k}[\widehat{P}}^{\bullet} \rightarrow(] h_{k}\left[\widehat{P}_{\widehat{P}}\right)_{*} \Omega_{\mathrm{C}_{k}[\widehat{P}}^{\bullet}\right)_{\text {tot }}\right) \tag{6.5.5}
\end{equation*}
$$

6.6. Theorem (Cospecialization morphism). Under the previous assumptions, there is a canonical functorial $K$-linear map

$$
\operatorname{cosp}{ }^{\bullet}: H_{c, \text { rig }}^{\bullet}\left(X_{k} / K\right) \longrightarrow H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)
$$

This map will be called cospecialization.
Proof. The proof is based on the following diagram of locally ringed $G$-spaces

We notice that for any $r$ and $N$ there are isomorphisms

$$
\begin{equation*}
(] h_{k}[\widehat{P})_{*}\left(\Omega_{] C_{k}[\widehat{P}}^{r} /\left(\mathscr{I}_{C_{K}^{\mathrm{an}}}^{N}\right)_{\mid]\left.C_{k}\right|_{\widehat{P}}}\right) \cong\left(\Omega_{P_{K}^{a \mathrm{an}}}^{r} / \mathscr{I}_{C_{K}^{\text {an }}}^{N}\right)_{\| \bar{X}_{k}\left[_{\widehat{P}}\right.} . \tag{6.6.1}
\end{equation*}
$$

We deduce the existence of maps

$$
(] h_{k}[\widehat{P})_{*} \Omega_{] C_{k}[\widehat{P}}^{r} \longrightarrow\left(\Omega_{P_{K}^{\mathrm{an}}}^{r} / \mathscr{I}_{C_{K}^{\mathrm{an}}}^{N}\right)_{\mid] \bar{X}_{k}[\widehat{P}} .
$$

(Since coherent $\mathscr{O}_{] C_{k}}\left[_{\widehat{P}}\right.$-Modules are (]$h_{k}[\widehat{P})_{*}$-acyclic, the previous maps are in fact surjective.) These we may view as a map

$$
\left(\left(i_{/ \bar{X}_{k}}\right)_{K}\right)^{-1}(] h_{k}[\widehat{P})_{*} \Omega_{\mathrm{j}_{k}[\widehat{P}}^{\bullet} \longrightarrow\left(\Omega_{P_{K}^{\mathrm{an}}}^{\bullet} / \mathscr{I}_{C_{K}^{\text {an }}}^{N-\bullet}\right)_{\mid \bar{X}_{K}^{\mathrm{an}}}
$$

of complexes on $\bar{X}_{K}^{\text {an }}$. Similarly, we get a map

$$
\left(\left(i_{/ \bar{X}_{k}}\right)_{K}\right)^{-1} \Omega_{]}^{\bullet} \bar{X}_{k} \hat{C}_{\widehat{P}} \longrightarrow\left(\Omega_{P_{K}^{\mathrm{an}}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}^{\mathrm{an}}}^{N-\boldsymbol{\bullet}}\right)_{\mid \bar{X}_{K}^{\mathrm{an}}}
$$

$$
\begin{aligned}
& C_{K}^{\mathrm{an}}=\left(C_{/ C_{k}}\right)_{K} \xrightarrow{h_{K}^{\mathrm{an}}} \bar{X}_{K}^{\mathrm{an}}=\left(\bar{X}_{/ \bar{X}_{k}}\right)_{K} \longleftrightarrow\left(P_{/ P_{k}}\right)_{K}=\widehat{P}_{K} \longleftrightarrow P_{K}^{\mathrm{an}} \\
& \left((i \circ h)_{/ C_{k}}\right)_{K} \quad\left(i / \bar{x}_{k}\right)_{K} \downarrow= \\
& ] C_{k}\left[\widehat{P}=\left(P_{/ C_{k}}\right)_{K} \underset{] h_{k}\left[\widehat{P}=\left(h_{P /}\right)\right.}{ }\right] \bar{X}_{k}\left[\widehat{P}=\left(P_{/ \bar{X}_{k}}\right)_{K} \longrightarrow\left(P_{/ P_{k}}\right)_{K} .\right.
\end{aligned}
$$

of complexes on $\bar{X}_{K}^{\mathrm{an}}$. Combining the two, we get morphisms

$$
\left(\left(i_{/_{X_{k}}}\right)_{K}\right)^{-1}\left(\Omega_{j \bar{X}_{k}\left[_{\widehat{P}}\right.}^{\bullet} \rightarrow(] h_{k}[\widehat{P})_{*} \Omega_{C_{k}[\widehat{P}}^{\bullet}\right)_{\text {tot }} \longrightarrow\left(\left(\Omega_{P_{K}^{\text {an }}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}^{\text {an }}}^{N-\bullet}\right)_{\mid \bar{X}_{K}^{\text {an }}} \rightarrow\left(\Omega_{P_{K}^{\text {an }}}^{\bullet} / \mathscr{I}_{C_{K}^{\text {an }}}^{N-\bullet}\right)_{\mid \bar{X}_{K}^{\text {an }}}\right)_{\text {tot }}
$$

and, passing to the limit, maps of complexes of abelian sheaves on $\bar{X}_{K}^{\mathrm{an}}$
(6.6.2)

$$
\left(\left(i / \bar{X}_{k}\right)_{K}\right)^{-1}\left(\left(\Omega_{P_{/ \bar{X}_{k}}^{\bullet}}^{\bullet}\right)_{K} \rightarrow\left(\left(h_{P /}\right)_{K}\right)_{*}\left(\Omega_{P_{/ C_{k}}}^{\bullet}\right)_{K}\right)_{\text {tot }} \longrightarrow\left(\left(\Omega_{P_{K}^{\text {an }}}^{\bullet}\right) / \bar{X}_{K}^{\mathrm{an}} \rightarrow\left(h_{K}^{\mathrm{an}}\right)_{*}\left(\Omega_{P_{K}^{\mathrm{an}}}^{\stackrel{\mathrm{an}}{ }}\right)_{/ C_{K}^{\text {an }}}^{\text {an }}\right)_{\text {tot }}
$$

We deduce morphisms of hypercohomology groups

$$
\begin{aligned}
& H_{c, \text { rig }}^{\bullet}\left(X_{k} / K\right):=\mathbf{H}^{\bullet}(] \bar{X}_{k}\left[\widehat{P}^{P},\left(\Omega_{\bar{X}_{k}\left[_{\widehat{P}}\right.} \rightarrow(] h_{k}[\widehat{P})_{*} \Omega_{\mathrm{C}_{k}[\widehat{P}}^{\boldsymbol{\bullet}}\right)_{\text {tot }}\right) \\
& \longrightarrow \mathbf{H}^{\bullet}\left(\bar{X}_{K}^{\text {an }},\left(\left(i_{/ \bar{X}_{k}}\right)_{K}\right)^{-1}\left(\Omega_{j \bar{X}_{k}\left[_{\widehat{P}}\right.}^{\bullet} \rightarrow(] h_{k}[\widehat{P})_{*} \Omega_{]_{k}[\widehat{P}}^{\bullet}\right)_{\text {tot }}\right) \\
& \xrightarrow{(6.6 .2)} \mathbf{H} \cdot\left(\bar{X}_{K}^{\mathrm{an}},\left(\Omega_{P_{K / \bar{X}_{K}^{\mathrm{an}}}^{\bullet}} \rightarrow\left(h_{K}^{\mathrm{an}}\right)_{*} \Omega_{P_{K / /_{K}^{\mathrm{an}}}^{\text {an }}}^{\bullet}\right)_{\mathrm{tot}}\right) \\
& \xrightarrow[N]{\cong} \stackrel{(1.3 .1)}{l_{N}} \mathbf{H}^{\bullet}\left(\bar{X}_{K}^{\mathrm{an}},\left(\left(\Omega_{P^{\mathrm{an}}{ }_{K}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}^{\mathrm{an}}}^{N-\bullet}\right)_{\mid \bar{X}_{K}^{\mathrm{an}}} \rightarrow\left(\Omega_{P^{\mathrm{an}}{ }_{K}}^{\bullet} / \mathscr{I}_{C_{K}^{\mathrm{an}}}^{N-\bullet}\right)_{\mid \bar{X}_{K}^{\mathrm{an}}}\right)_{\mathrm{tot}}\right) \\
& \underset{\cong}{\cong} \underset{N}{\text { GAGA }}{\underset{\mathrm{lim}}{ }}^{\mathbf{H}^{\bullet}}\left(\bar{X}_{K},\left(\left(\Omega_{P_{K}}^{\bullet} / \mathscr{I}_{\bar{X}_{K}}^{N-\bullet}\right)_{\left.\right|_{X_{K}}} \rightarrow\left(\Omega_{P_{K}}^{\bullet} / \mathscr{I}_{C_{K}}^{N-\bullet}\right)_{\mid \bar{X}_{K}}\right)_{\mathrm{tot}}\right)=: H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)
\end{aligned}
$$

which induce the cospecialization morphism.
6.7. The specialization morphism (From [BB]). For the convenience of the reader, we recall the construction given in $[\mathrm{BB}]$ of the specialization morphism between algebraic and rigid-analytic homologies (or cohomologies supported in a closed subset). In the notation of 6.1 , let $U$ be the open complement of $X$ in $W$ and $u$ be the open immersion $U \hookrightarrow W$. If we take an injective resolution $\mathscr{I} \bullet$ of $\Omega_{W_{K}}^{\bullet}$, and an injective resolution $\mathscr{J} \bullet$ of $\Omega_{W_{K}^{\bullet}}^{\text {an }}$, then we have a canonical morphism

$$
\begin{equation*}
\varepsilon_{W_{K}}^{-1}\left(\mathscr{I}^{\bullet} \rightarrow u_{K *} u_{K}^{-1} \mathscr{I}^{\bullet}\right)_{\mathrm{tot}} \longrightarrow\left(\mathscr{J}^{\bullet} \rightarrow u_{K *}^{\mathrm{an}} u_{K}^{\mathrm{an}-1} \mathscr{J}^{\bullet}\right)_{\mathrm{tot}} \tag{6.7.1}
\end{equation*}
$$

We observe that $W_{K}^{\text {an }}$ is a strict neighborhood of $\widehat{W}_{K}$ (resp. $\widehat{U}_{K}$ ) in $P_{K}^{\text {an }}$. We recall the definition of the (family of) functor $(\mathrm{s}) j_{W}^{\dagger}: \mathscr{A} b(V) \longrightarrow \mathscr{A} b(V)$ (resp. $j_{U}^{\dagger}: \mathscr{A} b(V) \longrightarrow \mathscr{A} b(V)$ ), for any fixed strict neighborhood $V$ of $\widehat{W}_{K}$ (resp. $\widehat{U}_{K}$ ) in $P_{K}^{\text {an }}$, namely

$$
\begin{equation*}
\mathscr{F} \longmapsto \underset{V^{\prime}}{\lim _{\vec{\prime}}} j_{V^{\prime} *} j_{V^{\prime}}^{-1} \mathscr{F} \tag{6.7.2}
\end{equation*}
$$

where $V^{\prime}$ runs over the strict neighborhoods of $\widehat{W}_{K}\left(\right.$ resp. $\left.\widehat{U}_{K}\right)$ in $V$, and $j_{V^{\prime}}: V^{\prime} \hookrightarrow V$ denotes the open immersion of $V^{\prime}$ in $V$. We take here $V=W_{K}^{\text {an }}$. Since $U_{K}^{\text {an }}$ is a strict neighborhood of $\widehat{U}_{K}$ in $W_{K}^{\text {an }}$, we have a canonical morphism

$$
\begin{equation*}
\left(\mathscr{J} \bullet \rightarrow u_{K *}^{\mathrm{an}} u_{K}^{\mathrm{an}-1} \mathscr{J}^{\bullet}\right)_{\mathrm{tot}} \longrightarrow\left(j_{W}^{\dagger} \mathscr{J}^{\bullet} \rightarrow j_{U}^{\dagger} \mathscr{J} \bullet\right)_{\text {tot }} \tag{6.7.3}
\end{equation*}
$$

We sum up the previous remarks into the following proposition.
6.7.4. Proposition. For any injective resolution $\mathscr{I} \bullet$ of $\Omega_{W_{K}}^{\bullet}$ and $\mathscr{J} \bullet$ of $\Omega_{W_{K}^{\bullet}}^{\text {an }}$, we have a canonical morphism

$$
\varepsilon_{W_{K}}^{-1}\left(\mathscr{I}^{\bullet} \rightarrow u_{K *} u_{K}^{-1} \mathscr{I}^{\bullet}\right)_{\text {tot }} \longrightarrow\left(j_{W}^{\dagger} \mathscr{J}^{\bullet} \rightarrow j_{U}^{\dagger} \mathscr{J}^{\bullet}\right)_{\text {tot }} .
$$

Via the functoriality map induced by $\varepsilon_{W_{K}}$, we deduce a morphism of complexes of $K$-vector spaces

$$
\begin{equation*}
\Gamma\left(W_{K},\left(\mathscr{J}^{\bullet} \rightarrow u_{K *} u_{K}^{-1} \mathscr{I} \bullet\right)_{\text {tot }}\right) \longrightarrow \Gamma\left(W_{K}^{\mathrm{an}},\left(j_{W}^{\dagger} \mathscr{J}^{\bullet} \rightarrow j_{U}^{\dagger} \mathscr{J} \bullet\right)_{\mathrm{tot}}\right) \tag{6.7.5}
\end{equation*}
$$

Since $\mathscr{J} \bullet$ is a flabby complex, $j_{W}^{\dagger} \mathscr{J} \bullet$ and $j_{U}^{\dagger} \mathscr{J} \bullet$ are $\Gamma\left(W_{K}^{\text {an }},-\right)$-acyclic, so that, taking cohomology, we have canonical morphisms

$$
\mathbf{H}^{\bullet}\left(W_{K},\left(\Omega_{W_{K}}^{\bullet} \rightarrow u_{K *} u_{K}^{-1} \Omega_{W_{K}}^{\bullet}\right)_{\mathrm{tot}}\right) \longrightarrow \mathbf{H}^{\bullet}\left(W_{K}^{\mathrm{an}},\left(j_{W}^{\dagger} \Omega_{W_{K}^{\mathrm{an}}}^{\bullet} \rightarrow j_{U}^{\dagger} \Omega_{W_{K}^{\mathrm{an}}}^{\bullet}\right)_{\mathrm{tot}}\right) .
$$

This is the specialization morphism

$$
\begin{equation*}
\mathrm{sp}^{\bullet}: H_{\mathrm{DR}, X_{K}}^{\bullet}\left(W_{K} / K\right) \longrightarrow H_{\mathrm{rig}, X_{k}}^{\bullet}\left(W_{k} / K\right) . \tag{6.7.6}
\end{equation*}
$$

6.8. Another construction of the cospecialization morphism. In order to compare Poincaré dualities in the algebraic and rigid contexts, it is convenient to exhibit a construction of the cospecialization morphism, closer to the previous definition of the specialization map.
6.8.1. We place ourselves in the previous setting, and let $\bar{W}$ be a proper $\mathscr{V}$-scheme containing $W$ as an open subscheme. We give here some structure of $\mathscr{V}$-scheme to $\bar{W} \backslash W$ (resp. to $\bar{W} \backslash U$ ) such that $\bar{W} \backslash W \hookrightarrow \bar{W} \backslash U \hookrightarrow \bar{W}$ be closed immersions; we denote by $l: \bar{W} \backslash W \hookrightarrow \bar{W}$ the composite immersion. Then

$$
\left(\bar{W}_{K} \backslash W_{K} \hookrightarrow \bar{W}_{K} \backslash U_{K} \hookrightarrow \bar{W}_{K}\right):=(\bar{W} \backslash W \hookrightarrow \bar{W} \backslash U \hookrightarrow \bar{W})_{K}
$$

and

$$
\left(\bar{W}_{K}^{\mathrm{an}} \backslash W_{K}^{\mathrm{an}} \hookrightarrow \bar{W}_{K}^{\mathrm{an}} \backslash U_{K}^{\mathrm{an}} \hookrightarrow \bar{W}_{K}^{\mathrm{an}}\right):=(\bar{W} \backslash W \hookrightarrow \bar{W} \backslash U \hookrightarrow \bar{W})_{K}^{\mathrm{an}}
$$

Let $j_{W_{K}}$ and $j_{U_{K}}$ be the open immersions of $W_{K}$ and $U_{K}$ in $\bar{W}_{K}$, respectively. We recall that

$$
H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)=\mathbf{H}^{\bullet}\left(P_{K},\left(\left(\Omega_{P_{K}}^{\bullet}\right) / \bar{X}_{K} \rightarrow\left(\Omega_{P_{K}}^{\bullet}\right) / C_{K}\right)_{\mathrm{tot}}\right) \cong \mathbf{H}^{\bullet}\left(\bar{W}_{K},\left(\mathbf{R} j_{U_{K}!} \Omega_{U_{K}}^{\bullet} \rightarrow \mathbf{R} j_{W_{K}!} \Omega_{W_{K}}^{\bullet}\right)_{\text {tot }}\right)
$$

where $\mathbf{R} j_{W_{K}}$ ! indicates the composite $\mathbf{R} \lim \circ j_{W_{K}}$ !. Then
and

$$
\left.\left.\mathbf{R} j_{U_{K}!} \Omega_{U_{K}} \cong\left(\Omega_{\bar{W}_{K}}^{\bullet} \rightarrow\left(\Omega_{\bar{W}_{K}}^{\bullet}\right) / \bar{W}_{K} \backslash U_{K}\right)\right)_{\mathrm{tot}} \cong{\underset{\lim }{N}}^{\left(\Omega_{\bar{W}_{K}}^{\bullet} \rightarrow \Omega_{\overline{\bar{W}}_{K}} / \mathscr{\mathscr { I }} \overline{\bar{W}}_{K}^{N-} \backslash U_{K}\right.}\right)_{\mathrm{tot}},
$$

respectively.
On the other hand, we may define

$$
H_{\mathrm{DR}, c}^{\bullet}\left(X_{K}^{\mathrm{an}} / K\right):=\mathbf{H}^{\bullet}\left(P_{K}^{\mathrm{an}},\left(\left(\Omega_{P_{K}^{\mathrm{an}}}^{\bullet}\right) / \bar{X}_{K}^{\mathrm{an}} \rightarrow\left(\Omega_{P_{K}^{\mathrm{an}}}^{\stackrel{a}{c}}\right) / C_{K}^{\mathrm{an}}\right)_{\mathrm{tot}}\right) \cong \mathbf{H}^{\bullet}\left(\bar{W}_{K}^{\mathrm{an}},\left(\mathbf{R} j_{U_{K}!}^{\mathrm{an}} \Omega_{U_{K}^{\mathrm{an}}}^{\bullet} \rightarrow \mathbf{R} j_{W_{K}!}^{\mathrm{an}} \Omega_{W_{K}^{\mathrm{an}}}^{\bullet}\right)_{\mathrm{tot}}\right)
$$

where
and
respectively.
By 1.3.1 we have

The canonical map $\varepsilon_{\bar{W}_{K}}: \bar{W}_{K}^{\text {an }} \rightarrow \bar{W}_{K}$ induces a canonical morphism

$$
\begin{equation*}
\varepsilon_{\bar{W}_{K}}^{-1}\left(\mathbf{R} j_{U_{K}!} \Omega_{U_{K}}^{\bullet} \rightarrow \mathbf{R} j_{W_{K}!} \Omega_{W_{K}}^{\bullet}\right)_{\mathrm{tot}} \longrightarrow\left(\mathbf{R} j_{U_{K}!}^{\mathrm{an}} \Omega_{\dot{U}_{K}^{\mathrm{an}}}^{\bullet} \rightarrow \mathbf{R} j_{W_{K}!}^{\mathrm{an}} \Omega_{W_{K}^{\mathrm{an}}}^{\bullet}\right)_{\mathrm{tot}} \tag{6.8.2}
\end{equation*}
$$

taking cohomology over $\bar{W}_{K}^{\text {an }}$ and composing with the canonical functorial map induced by $\varepsilon_{\bar{W}_{K}}$ we have the morphism

$$
\begin{equation*}
H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right) \longrightarrow H_{\mathrm{DR}, c}^{\bullet}\left(X_{K}^{\mathrm{an}}\right) \tag{6.8.3}
\end{equation*}
$$

which is an isomorphism, as already said, by a GAGA argument over $K$.
6.8.4. From the commutative diagram

$$
\begin{gathered}
\bar{W}_{K}^{\mathrm{an}} \backslash W_{K}^{\mathrm{an}} \xrightarrow{l_{K}^{\mathrm{an}}} \bar{W}_{K}^{\mathrm{an}}=\widehat{\bar{W}}_{K} \\
\quad \| \\
] \bar{W}_{k} \backslash W_{k}\left[\widehat{\bar{W}} \underset{l_{k}\left[l_{\widehat{W}}^{\longrightarrow}\right.}{\longrightarrow}\right] \bar{W}_{k}[\widehat{\bar{W}}
\end{gathered}
$$

as in (6.6.2), we have a canonical morphism

$$
\begin{equation*}
\left(\Omega_{]}^{\bullet} \bar{W}_{k}\left[\widehat{\bar{W}} \rightarrow(] l_{k}[\widehat{\widehat{W}})_{*} \Omega_{] \bar{W}_{k} \backslash W_{k}[\widehat{\widehat{W}}}\right)_{\text {tot }} \longrightarrow\left(\Omega_{\bar{W}_{K}^{\text {an }}} \rightarrow\left(\Omega_{\bar{W}_{K}^{\bullet}}^{\text {an }}\right) / \bar{W}_{K}^{\text {an }} \backslash W_{K}^{\text {an }}\right)_{\text {tot }}\right. \tag{6.8.5}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\mathbf{R} \Gamma_{] W_{k}[\widehat{\widehat{W}}} \Omega_{\bar{W}_{k}[\widehat{\widehat{W}}} \longrightarrow \mathbf{R} j_{W_{K}!}^{\mathrm{an}} \Omega_{W_{K}^{\mathrm{an}}}^{\bullet} . \tag{6.8.6}
\end{equation*}
$$

Applying this for $W$ and $U$, we obtain a morphism

Taking hypercohomology, we find a canonical morphism

$$
\begin{equation*}
H_{\mathrm{rig}, c}^{\bullet}\left(X_{k} / K\right) \longrightarrow H_{\mathrm{DR}, c}^{\bullet}\left(X_{K}^{\mathrm{an}}\right) ; \tag{6.8.8}
\end{equation*}
$$

finally, composing with the isomorphism $H_{\mathrm{DR}, c}^{\bullet}\left(X_{K}^{\mathrm{an}}\right) \cong H_{\mathrm{DR}, c}^{\bullet}\left(X_{K} / K\right)$ we obtain the cospecialization morphism of the theorem.
6.9. Theorem (Compatibility of algebraic and rigid Poincaré dualities). For $i=$ $0, \ldots, 2 n$, the canonical morphisms of specialization

$$
\mathrm{sp}^{2 n-i}: H_{\mathrm{DR}, X_{K}}^{2 n-i}\left(W_{K} / K\right) \longrightarrow H_{\mathrm{rig}, X_{k}}^{2 n-i}\left(W_{k} / K\right)
$$

of $[B B]$ and of cospecialization

$$
\operatorname{cosp}^{i}: H_{\mathrm{rig}, c}^{i}\left(X_{k} / K\right) \longrightarrow H_{\mathrm{DR}, c}^{i}\left(X_{K} / K\right)
$$

are compatible with Poincaré pairings

$$
\langle,\rangle: H_{\mathrm{DR}, X_{K}}^{2 n-i}\left(W_{K} / K\right) \otimes H_{\mathrm{DR}, c}^{i}\left(X_{K} / K\right) \longrightarrow K
$$

and

$$
\langle,\rangle: H_{\mathrm{rig}, X_{k}}^{2 n-i}\left(W_{k} / K\right) \otimes H_{\mathrm{rig}, c}^{i}\left(X_{k} / K\right) \longrightarrow K
$$

in the sense that, for $\alpha \in H_{\mathrm{DR}, X_{K}}^{2 n-i}\left(W_{K} / K\right)$ and $\beta \in H_{\mathrm{rig}, c}^{i}\left(X_{k} / K\right)$, one has $\left\langle\operatorname{sp}^{2 n-i} \alpha, \beta\right\rangle=\left\langle\alpha, \operatorname{cosp}^{i} \beta\right\rangle$.
Proof. In the notation of 6.1 , we may and will assume $P$ proper and $P_{K}$ smooth. So, in the second construction of the cospecialization morphism, we take $\bar{W}=P$. In this special case, by its very definition, the trace map $\operatorname{Tr}_{W_{k}}^{\text {rig }}: H_{\text {rig }, c}^{2 n}\left(W_{k} / K\right) \longrightarrow K$ of $[\mathrm{B} .97,2]$ is the composite

$$
H_{\mathrm{rig}, c}^{2 n}\left(W_{k} / K\right) \xrightarrow{\cong} H_{\mathrm{rig}}^{2 n}\left(P_{k} / K\right) \xrightarrow{\operatorname{Tr}_{P_{k}}^{\mathrm{rig}}} K
$$

and sits in a commutative diagram

$$
\begin{array}{ccc}
H_{D R, c}^{2 n}\left(W_{K} / K\right) & \cong & H_{D R}^{2 n}\left(P_{K} / K\right) \xrightarrow{\operatorname{Tr}_{P_{K}}} K \\
\operatorname{cosp}^{2 n} \uparrow \cong & \operatorname{cosp}^{2 n} \uparrow \cong & \| \\
H_{\text {rig }, c}^{2 n}\left(W_{k} / K\right) & \cong & H_{\text {rig }}^{2 n}\left(P_{k} / K\right) \xrightarrow[\operatorname{Tr}_{P_{k}}^{\text {rig }}]{\longrightarrow} K
\end{array}
$$

where $\operatorname{Tr}_{P_{K}}: H_{D R}^{2 n}\left(P_{K} / K\right) \longrightarrow K$ is the algebraic trace map. We observe that in this case $\operatorname{cosp}^{2 n}$ : $H_{\mathrm{rig}}^{2 n}\left(P_{k} / K\right) \longrightarrow H_{D R}^{2 n}\left(P_{K} / K\right)$ and $\mathrm{sp}^{2 n}: H_{D R}^{2 n}\left(P_{K} / K\right) \longrightarrow H_{\text {rig }}^{2 n}\left(P_{k} / K\right)$ are inverse isomorphisms.

The pairing in the rigid context is constructed by Berthelot in [B.97, 2.2] using the adjunction between the functors $j_{W}^{\dagger}$ and $\Gamma_{] W_{k}}$ ([B.97, 2.1]). Taking an injective resolution $\mathscr{J} \bullet$ of $\Omega_{P_{K}^{\text {an }}}^{\text {an }}$ as a complex of sheaves of $K$-vector spaces on $P_{K}^{\text {an }}$, and a pairing $\mathscr{J} \bullet \otimes \mathscr{J} \bullet \rightarrow \mathscr{J} \bullet$ extending the wedge product, one obtains a pairing

$$
j_{W}^{\dagger} \mathscr{J} \cdot \otimes \Gamma_{] W_{k}\left[\widehat{P}_{P}\right.} \mathscr{J}^{\bullet} \longrightarrow \mathscr{J}^{\bullet}
$$

This, applied to the smooth schemes $W_{k}$ and $U_{k}$, leads to the pairing

$$
\left(j_{W}^{\dagger} \mathscr{J}^{\bullet} \rightarrow j_{U}^{\dagger} \mathscr{J}^{\bullet}\right)_{\mathrm{tot}} \otimes\left(\Gamma_{] U_{k}\left[\left[_{P}\right.\right.} \mathscr{J}^{\bullet} \rightarrow \Gamma_{] W_{k}\left[_{\hat{P}}\right.} \mathscr{J}^{\bullet}\right)_{\mathrm{tot}} \longrightarrow \mathscr{J}^{\bullet}
$$

where $j_{W}^{\dagger} \mathscr{J}^{0}$ (resp. $\Gamma_{] W_{k}\left[\widehat{P}^{\prime}\right.} \mathscr{J}^{0}$ ) is placed in bidegree (0,0). Taking cohomology, we get the pairings

$$
\begin{equation*}
H_{p}^{\mathrm{rig}}\left(X_{k} / K\right) \otimes H_{\mathrm{rig}, c}^{p}\left(X_{k} / K\right) \longrightarrow H_{\mathrm{rig}}^{2 n}\left(P_{k} / K\right) \tag{6.9.1}
\end{equation*}
$$

In the present algebraic context, the pairing (3.5.1) becomes, via (2.6.1),

$$
j_{W_{K} *} E\left(\Omega_{W_{K}}^{\bullet}\right) \otimes \varliminf_{N}^{\lim } " \mathscr{I}_{P_{K} \backslash W_{K}}^{N-\cdot} \Omega_{P_{K}}^{\bullet} \longrightarrow E\left(\Omega_{P_{K}}^{\bullet}\right)
$$

This pairing is also induced by an extension $E\left(\Omega_{P_{K}}^{\bullet}\right) \otimes \Omega_{P_{K}}^{\bullet} \rightarrow E\left(\Omega_{P_{K}}^{\bullet}\right)$ of the wedge product. So, we get the canonical pairing

$$
j_{W_{K} *} E\left(\Omega_{W_{K}}^{\bullet}\right) \otimes \mathbf{R} j_{W_{K}!} \Omega_{W_{K}}^{\bullet} \longrightarrow E\left(\Omega_{P_{K}}^{\bullet}\right) .
$$

This, applied to the smooth schemes $W_{K}$ and $U_{K}$, leads to the pairing

$$
\left(j_{W_{K} *} E\left(\Omega_{W_{K}}^{\bullet}\right) \rightarrow j_{U_{K} *} E\left(\Omega_{U_{K}}^{\bullet}\right)\right)_{\text {tot }} \otimes\left(\mathbf{R} j_{U_{K}!} \Omega_{U_{K}}^{\bullet} \rightarrow \mathbf{R} j_{W_{K}!} \Omega_{W_{K}}^{\bullet}\right)_{\text {tot }} \longrightarrow E\left(\Omega_{P_{K}}^{\bullet}\right) .
$$

Taking cohomology, we obtain the pairings

$$
\begin{equation*}
H_{p}^{\mathrm{DR}}\left(X_{K} / K\right) \otimes H_{\mathrm{DR}, c}^{p}\left(X_{K} / K\right) \longrightarrow H_{\mathrm{DR}}^{2 n}\left(P_{K} / K\right) \tag{6.9.2}
\end{equation*}
$$

Now the compatibility of (6.9.2) and (6.9.1) is obvious via proposition 6.7.4, (6.8.2) and (6.8.7).

## References.

[AB] André Y. and Baldassarri F. De Rham cohomology of differential modules on algebraic varieties. Progress in Mathematics, 189 Birkhäuser Verlag, Basel, 2001.
[BB] Baldassarri F. and Berthelot P. On Dwork cohomology for singular hypersurfaces, work in progress.
[BC] Baldassarri F. and Chiarellotto B. Algebraic versus rigid cohomology with logarithmic coefficients. in Barsotti Memorial Symposium, Perspectives in Mathematics 15 (1994), 11-50.
[BS] Bănică C. and Stănăşilă O. Algebraic Methods in the Global Theory of Complex Spaces. John Wiley \& Sons, (1976), 296 pages.
[B.74] Berthelot P. Cohomologie cristalline des schémas de caractéristique $p>0$. Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974.
[B.86] Berthelot P. Géométrie rigide et cohomologie des variétés algébriques de caractéristique p. Journées d'analyse p-adique (1982), in Introduction aux Cohomologies p-adiques Bull. Soc. Math. France, Mémoire 23 (1986), 7-32.
[B.96] Berthelot P. Cohomologie rigide et cohomologie rigide à support propres. Première partie. Prépublication IRMAR 96-03, 89 pages (1996).
[B.97] Berthelot P. Dualité de Poincaré et formule de Künneth en cohomologie rigide. C.R.Acad.Sci. Paris Sr. I Math., 325 (1997), 493-498.
[B.00] Berthelot P. D-modules arithmétiques. II. Descente par Frobenius. Mém. Soc. Math. Fr., 81 (2000), $\mathrm{vi}+136 \mathrm{pp}$.
[BeO] Berthelot P. and Ogus A. Notes on crystalline cohomology. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.
[CLS] Chiarellotto B. and Le Stum B. A comparison theorem for weights. to appear in Comp. Math.
[DB.81] Du Bois Ph. Complexe de de Rham filtré d'une variété singulière. Bull. Soc. Math. France, 109 (1981), 41-81.
[DB.90] Du Bois Ph. Dualité dans la catégorie des complexes filtrés d'opérateurs differentiels d'ordre $\leqslant 1$. Collect. Math., 41 (1990), 89-121.
[EGA] Grothendieck A. and Dieudonné J. Eléments de géométrie algébrique. Inst. Hautes Études Sci. Publ. Math., $\mathbf{4}$ (1960), $\mathbf{8}$ (1961), $\mathbf{1 1}$ (1961), $\mathbf{1 7}$ (1963), 20 (1964), 24 (1965), 28 (1966), $\mathbf{3 2}$ (1967).
[H.RD] Hartshorne R. Residues and duality. Lecture Notes in Mathematics, Vol. 20, Springer-Verlag, BerlinNew York, 1966.
[H.75] Hartshorne R. On the De Rham cohomology of algebraic varieties. Inst. Hautes Études Sci. Publ. Math., 45 (1975), 5-99.
[H.72] Hartshorne R. Cohomology with compact supports for coherent sheaves on an algebraic variety. Math. Ann., 195 (1972), 199-207.
[HL] Herrera M. and Lieberman D. Duality and the de Rham cohomology of infinitesimal neighborhoods. Invent. Math., 13 (1971), 97-124.
[KL] Katz N.M. and Laumon G. Transformation de Fourier et majoration de sommes exponentielles. Inst. Hautes Études Sci. Publ. Math., 62 (1985), 361-418.
[M] Mebkhout Z. Le formalisme des six opérations de Grothendieck pour les $\mathscr{D}_{X}$-modules cohérents. Travaux en Cours, 35, Hermann, Paris, 1989.
[S] Saito M. Induced $\mathscr{D}$-modules and differential complexes. Bull. Soc. Math. France, 117 (1989), 361387.


[^0]:    ${ }^{(1)} c f$. the lines of loc. cit. preceding formula (3.1): ". . $\Omega^{\bullet}$-linearity, a property which is preserved by $d^{\prime}$ and $d^{\prime \prime "}$. This statement is false.

