Geometria Superiore Reference Cards

Spectral Sequences

Send comments and corrections to ⟨maurizio@math.unipd.it⟩

Generalities.

Definitions Let $\mathcal{A}$ be an abelian category. Define $\mathcal{A}_f$ the quasi-abelian category of filtered objects of $\mathcal{A}$: objects are objects A of $\mathcal{A}$ with a (decreasing) filtration, i.e. a family of subobjects $(F^p(A))_{p\in\mathbb{Z}}$ of A s.t. $F^{p+1}(A)\subseteq F^p(A)$ for every $p\in\mathbb{Z}$; morphisms $\varphi : A\longrightarrow B$ are morphisms of $\mathcal{A}$ respecting the filtrations, i.e. $\varphi(F^p(A))\subseteq F^p(B)$.

The filtration $(F^p(A))_{p\in\mathbb{Z}}$ of A is:

(i) separated if $\bigcap_{p\in\mathbb{Z}} F^p(A)=0$,

(ii) coseparated (or exhaustive) if $\bigcup_{p\in\mathbb{Z}} F^p(A)=A$,

(iii) discrete if there exists $p\in\mathbb{Z}$ s.t. $F^p(A)=0$ (so $F^q(A)=0$ for every $q\geqslant p$),

(iv) codiscrete if there exists $p\in\mathbb{Z}$ s.t. $F^p(A)=A$ (so $F^q(A)=A$ for every $q\leqslant p$),

(v) finite if it is discrete and codiscrete.

Remark that discrete (resp. codiscrete) filtrations are separated (resp. exhaustive).

For $A\in\mathcal{A}_f$: $\mathrm{gr}_p(A):=F^p(A)\big/F^{p+1}(A)\in\mathcal{A}$, and if $\mathcal{A}$ admits arbitrary direct sums, define: $\mathrm{gr}(A):=\bigoplus_{p\in\mathbb{Z}}\mathrm{gr}_p(A)\in\mathcal{A}$, the associated graded object. This gives a functor $\mathrm{gr}:\mathcal{A}_f\to\mathcal{A}$.

Let $\varphi : A\longrightarrow B$ be a morphism in $\mathcal{A}_f$; then:

(1) if $\mathrm{gr}\varphi$ is a monomorphism, and the filtration of A is exhaustive and separated, then φ is a monomorphism;

(2) if $\mathrm{gr}\varphi$ is an epimorphism, and the filtration of B is exhaustive and discrete, then φ is an epimorphism;

(2′) if $\mathrm{gr}\varphi$ is an epimorphism, the filtration of B is exhaustive and separated, the filtration of A induces a complete topology, then φ is an epimorphism;

(3) if $\mathrm{gr}\varphi$ is an isomorphism, the filtration of A is exhaustive and separated, and the filtration of B is exhaustive and discrete, then φ is an isomorphism.

$SpS(\mathcal{A})$: **the category of spectral sequences of** $\mathcal{A}$.

A spectral sequence $E=\left((E_r^{p,q})_{\substack{r\geqslant 2\\ p,q\in\mathbb{Z}}},(E^n)_{n\in\mathbb{Z}}\right)$ in $\mathcal{A}$ is the data of:

(a) a family $(E_r^{p,q})$ of objects of $\mathcal{A}$ defined for $r\geqslant 2$ and $p,q\in\mathbb{Z}$,

(b) a family of morphisms $(d_r^{p,q})$ where $d_r^{p,q}: E_r^{p,q}\to E_r^{p+r,q-r+1}$ and $d_r^{p,q}\circ d_r^{p-r,q+r-1}=0$ for all indices;

so that we can define: $Z_{r+1}(E_r^{p,q})=\ker(d_r^{p,q})$ and $B_{r+1}(E_r^{p,q})=\mathrm{im}(d_r^{p-r,q+r-1})$ for which $B_{r+1}(E_r^{p,q})\subseteq Z_{r+1}(E_r^{p,q})\subseteq E_r^{p,q}$ and we require:

(c) a family $(\alpha_r^{p,q})$ of isomorphisms
$$\alpha_r^{p,q}: Z_{r+1}(E_r^{p,q})\big/B_{r+1}(E_r^{p,q})\xrightarrow{\sim}E_{r+1}^{p,q}\ ;$$

we can define for every $p,q\in\mathbb{Z}$ two families of subobjects of $E_2^{p,q}$, $B_k(E_2^{p,q})$ and $Z_k(E_2^{p,q})$, in such a way that $E_k^{p,q}\simeq Z_k(E_2^{p,q})\big/B_k(E_2^{p,q})$ by induction:

for $k=2$ put $B_2(E_2^{p,q})=0$ and $Z_2(E_2^{p,q})=E_2^{p,q}$,

for $k=3$ already defined,

for $k>3$ we can choose $B_{k+1}(E_2^{p,q})$ and $Z_{k+1}(E_2^{p,q})$ as the subobjects of $Z_k(E_2^{p,q})$ containing $B_k(E_2^{p,q})$ and identified by means of the isomorphism of the previous step k to the subobjects $\mathrm{im}(d_k^{p-k,q+k-1})$ and $\ker(d_k^{p,q})$ of $E_k^{p,q}$ respectively; in this way we have
$$Z_{k+1}(E_2^{p,q})\big/B_{k+1}(E_2^{p,q})\simeq E_{k+1}^{p,q}.$$

So we have a long chain of inclusions:
$$E_2^{p,q}=Z_2(E_2^{p,q})\supseteq\cdots\supseteq Z_k(E_2^{p,q})\supseteq Z_{k+1}(E_2^{p,q})\supseteq\cdots\supseteq$$
$$\cdots\supseteq B_{k+1}(E_2^{p,q})\supseteq B_k(E_2^{p,q})\supseteq\cdots\supseteq B_2(E_2^{p,q})=0$$

and the definition continues with:

(d) for every $p,q\in\mathbb{Z}$ two subobjects $B_\infty(E_2^{p,q})$ and $Z_\infty(E_2^{p,q})$ of $E_2^{p,q}$ such that $Z_\infty(E_2^{p,q})\supseteq B_\infty(E_2^{p,q})$ and for every k
$$Z_k(E_2^{p,q})\supseteq Z_\infty(E_2^{p,q})\supseteq B_\infty(E_2^{p,q})\supseteq B_k(E_2^{p,q})$$
and define: $E_\infty^{p,q}=Z_\infty(E_2^{p,q})\big/B_\infty(E_2^{p,q})$;

(e) a family $(E^n)_{n\in\mathbb{Z}}$ of objects of $\mathcal{A}_f$;

(f) a family $(\beta^{p,q})_{p,q\in\mathbb{Z}}$ of isomorphisms $\beta^{p,q}: E_\infty^{p,q}\xrightarrow{\sim}\mathrm{gr}_p(E^{p+q})$.

Remarks on the definition of spectral sequences:

(1) the construction of chains of subobjects in (c) can be performed for every r in the spectral sequence, defining for every $k\geqslant r$ the subobjects $B_k(E_r^{p,q})$ and $Z_k(E_r^{p,q})$ of $E_r^{p,q}$.

(2) for a fixed $n\in\mathbb{Z}$, the family $\{E_\infty^{p,q}\}_{p+q=n}$ gives a measure of the skips in the filtration of E^n: in fact, if the terms exist, we have $\bigoplus_{p+q=n}E_\infty^{p,q}\simeq\mathrm{gr}(E^n)$.

A morphism of two spectral sequences $E=(E_r^{p,q},E^n)$ and $E'=(E'^{p,q}_r,E'^n)$ is the datum of $f=(f_r^{p,q},f^n)$ where:

(a) $f_r^{p,q}: E_r^{p,q}\longrightarrow E'^{p,q}_r$ are morphisms of $\mathcal{A}$,

(b) compatibility with the differentials: commutation of
$$\begin{tikzcd} E_r^{p,q} \arrow[r, "f_r^{p,q}"] \arrow[d, "d_r^{p,q}"'] & E'^{p,q}_r \arrow[d, "d'^{p,q}_r"] \\ E_r^{p+r,q-r+1} \arrow[r, "f_r^{p+r,q-r+1}"'] & E'^{p+r,q-r+1}_r \end{tikzcd}$$

(c) compatibility with the isomorphisms:
$$\begin{tikzcd} \dfrac{Z_{r+1}(E_r^{p,q})}{B_{r+1}(E_r^{p,q})} \arrow[r, "\bar f_r^{p,q}"] \arrow[d, "\alpha_r^{p,q}"'] & \dfrac{Z_{r+1}(E'^{p,q}_r)}{B_{r+1}(E'^{p,q}_r)} \arrow[d, "\alpha'^{p,q}_r"] \\ E_{r+1}^{p,q} \arrow[r, "f_{r+1}^{p,q}"'] & E'^{p,q}_{r+1} \end{tikzcd}$$

commutes, where the $\bar f_r^{p,q}$ define the morphisms in cohomology;

(d) compatibility with the ∞ level
$f_2^{p,q}\left(B_\infty(E_2^{p,q})\right)\subseteq B_\infty(E'^{p,q}_2)$, $f_2^{p,q}\left(Z_\infty(E_2^{p,q})\right)\subseteq Z_\infty(E'^{p,q}_2)$
in such a way that we induce morphisms: $\bar f_\infty^{p,q}: E_\infty^{p,q}\to E'^{p,q}_\infty$;

(e) $f^n: E^n\to E'^n$ morphisms of filtered objects;

(f) finally the compatibility with the limit
$$\begin{tikzcd} E_\infty^{p,q} \arrow[r, "\bar f_\infty^{p,q}"] \arrow[d, "\beta^{p,q}"'] & E'^{p,q}_\infty \arrow[d, "\beta'^{p,q}"] \\ \mathrm{gr}_p(E^{p+q}) \arrow[r, "\mathrm{gr}_p(f^{p+q})"'] & \mathrm{gr}_p(E'^{p+q}) \end{tikzcd}\ .$$

Remarks on the definition of morphisms of spectral sequences:

(1) if a morphisms f of spectral sequences has the property that for a fixed r all $f_r^{p,q}$ are isomorphisms, then by (c) for all $s\geqslant r$ the $f_s^{p,q}$ are also isomorphisms.

(2) a morphism of spectral sequences is an isomorphism if and only if: the $f_2^{p,q}$ are isomorphisms for all $p,q\in\mathbb{Z}$, in the condition (d) equalities hold, and the f^n are isomorphisms for all $n\in\mathbb{Z}$.

Convergence and regularity conditions:

(o) a spectral sequence is weakly convergent if

$$B_\infty(E_2^{p,q})=\bigcup_{k\geqslant 2} B_k(E_2^{p,q}) \qquad Z_\infty(E_2^{p,q})=\bigcap_{k\geqslant 2} Z_k(E_2^{p,q})$$

hold, i.e. if the data of (d) are determined by the previous points. A weakly convergent spectral sequence is:

(i) regular if

(1) for any p,q the decreasing sequence $\big(Z_k(E_2^{p,q})\big)_k$ becomes stationary,

(2) for any n the filtration of E^n is discrete and exhaustive.

This implies that for a fixed pair (p,q) we have $Z_\infty(E_2^{p,q})=Z_k(E_2^{p,q})$ for $k\gg 0$, but depending on p and q.

(ii) co-regular if:

(1) for any p,q the increasing sequence $\big(B_k(E_2^{p,q})\big)_k$ becomes stationary,

(2) for any n the filtration of E^n is codiscrete

This implies that for a fixed pair (p,q) we have $B_\infty(E_2^{p,q})=B_k(E_2^{p,q})$ for $k\gg 0$, but depending on p and q.

(iii) biregular if it is regular and co-regular, i.e. if:

(1) for any p,q the sequences $\big(Z_k(E_2^{p,q})\big)_k$ and $\big(B_k(E_2^{p,q})\big)_k$ became stationary,

(2) for any n the filtration of E^n is finite.

This implies that for a fixed pair (p,q) we have $E_\infty^{p,q}=E_k^{p,q}$ for $k\gg 0$, but depending on p and q.

Remarks on the regular spectral sequences:

(1) if E is regular, then $E_\infty^{p,q}=\varinjlim_{r\gg 0} E_r^{p,q}$ where the inductive system is given by:

$$E_r^{p,q}\simeq Z_\infty^{p,q}\big/B_r(E_2^{p,q})\longrightarrow Z_\infty^{p,q}\big/B_{r+1}(E_2^{p,q})\simeq E_{r+1}^{p,q}.$$

The result follows from $B_\infty^{p,q}=\varinjlim_{r\gg 0} B_r(E_2^{p,q})$.

(2) dually, if E is co-regular, then $E_\infty^{p,q}=\varprojlim_{r\gg 0} E_r^{p,q}$ where the projective system is given by:

$$E_r^{p,q}\simeq Z_r(E_2^{p,q})\big/B_\infty^{p,q}\longrightarrow Z_{r+1}(E_2^{p,q})\big/B_\infty^{p,q}\simeq E_{r+1}^{p,q}.$$

The result follows from $Z_\infty^{p,q}=\varprojlim_{r\gg 0} Z_r(E_2^{p,q})$.

(3) Let f be a morphism between spectral sequences E and E'; we have that $f_2^{p,q}$ isomorphisms for all (p,q) implies f is an isomorphism in the following two cases:

(i) E and E' are biregular;

(ii) E and E' are regular and $\mathcal{A}$ is an abelian category where the inductive filtered limits exist and are exact.

Degeneration of spectral sequences:

(0) (original sense: Godement, Grothendieck) a spectral sequence E degenerates if there exists an $r\geqslant 2$ s.t. for all $n\in\mathbb{Z}$ there exists $q(n)\in\mathbb{Z}$ with $E_r^{n-q,q}=0$ for all $q\neq q(n)$ (so $E_s^{n-q,q}=0$ for all $s\geqslant r$ and $q\neq q(n)$). Remarks on degenerate spectral sequences:

(1) if we have $q(n+1)>q(n)-r+1$ for all n (for example: $q(n)=k$ or $q(n)=n+k$ with constant k) we can easily see that $E_s^{n-q(n),q(n)}=E_r^{n-q(n),q(n)}$ for all $s\geqslant r$, so that $E_\infty^{n-q(n),q(n)}=E_r^{n-q(n),q(n)}$ if the spectral sequence is weakly convergent: the limit of the sequence can be read at level r.

(2) if moreover the filtrations on the limit are exhaustive and separated, then they are finite and $F^p(E^n)=0$ for $p>n-q(n)$ and $F^p(E^n)=E^n$ for $p\leqslant n-q(n)$; we have also an isomorphism $E^n\simeq E_r^{n-q(n),q(n)}$.

($0'$) (Deligne) a spectral sequence E is degenerated in E_{r_0}, for $r_0\geqslant 2$, if it is biregular and for any $r\geqslant r_0$ we have $d_r^{p,q}=0$ for all p and q; then $E_{r_0}=E_\infty$ and $\mathrm{gr}^pE^{p+q}=E_{r_0}^{p,q}$.

($1'$) in the category of R-modules of finite length with R a ring,

(o) $\sum_{p+q=n}\lg E_r^{p,q}\geqslant \lg E^n$;

(i) equality holds for $r=r_0$ if and only if E degenerates at r_0;

(ii) if $E^n=0$ except for a finite number of n, then for all $r\geqslant 2$ we have $\sum_{p,q}(-)^{p+q}\lg E_r^{p,q}=\sum_n(-)^n\lg E^n$.

Homological and Spherical Sp.Seq.

Cohomological spectral sequences: E is cohomological if $E_2^{p,q}=0$ whenever $p<0$ or $q<0$. Then under the same conditions $E_r^{p,q}=0$ for all r, and $E_\infty^{p,q}=0$. Properties of the cohomological spectral sequences:

(1) if the spectral sequence is weakly convergent and the filtrations of the limit are exhaustive and separated, then from the isomorphisms $\beta^{p,q}$ for $p<0$ or $q<0$ we deduce that

$$F^p(E^n)=\begin{cases}E^n & \text{if } p\leqslant 0\\ 0 & \text{if } p>n\ .\end{cases}$$

In particular we have an exact sequence

$$0\longrightarrow E_\infty^{1,0}\longrightarrow E^1\longrightarrow E_\infty^{0,1}\longrightarrow 0$$

coming from $0\to F^1(E^1)\to E^1\to E^1/F^1(E^1)\to 0$.

(2) In the previous situation we have the edge morphisms:

$$E_r^{n,0}\longrightarrow E^n \qquad E^n\longrightarrow E_r^{0,n}$$

given resp. by the compositions

$$E_r^{n,0}\twoheadrightarrow E_\infty^{n,0}\simeq \mathrm{gr}_n(E^n)=F^n(E^n)\hookrightarrow E^n$$

where the first map comes from $E_\infty^{n,0}=\varinjlim_r E_r^{n,0}$; and

$$E^n\twoheadrightarrow E^n/F^1(E^n)=\mathrm{gr}_0(E^n)\simeq E_\infty^{0,n}\hookrightarrow E_r^{0,n}$$

where the last map comes from $E_\infty^{0,n}=\varprojlim_r E_r^{0,n}=\bigcap_r E_r^{0,n}$.

(3) There exists an exact sequence

$$0\longrightarrow E_2^{1,0}\longrightarrow E^1\longrightarrow E_2^{0,1}\xrightarrow{d_2^{0,1}} E_2^{2,0}\longrightarrow E^2.$$

Homological spectral sequences: E is homological if $E_2^{p,q}=0$ whenever $p>0$ or $q>0$. Then under the same conditions $E_r^{p,q}=0$ for all r, and $E_\infty^{p,q}=0$. Properties of the cohomological spectral sequences:

(1) if the spectral sequence is weakly convergent and the filtrations of the limit are exhaustive and separated,

$$F^p(E^n)=\begin{cases}E^n & \text{if } p\leqslant n\\ 0 & \text{if } p>0\ .\end{cases}$$

In particular we have an exact sequence

$$0\longrightarrow E_\infty^{0,-1}\longrightarrow E^{-1}\longrightarrow E_\infty^{-1,0}\longrightarrow 0.$$

(2) In the previous situation we have the edge morphisms:

$$E_r^{0,-n}\longrightarrow E^{-n} \qquad\text{and}\qquad E^{-n}\longrightarrow E_r^{-n,0}\ .$$

(3) There exists an exact sequence

$$E^{-2}\longrightarrow E_2^{-2,0}\overset{d_2^{-2,0}}{\longrightarrow} E_2^{0,-1}\longrightarrow E^{-1}\longrightarrow E_2^{-1,0}\to 0.$$

Spectral sequences with spherical base: E is said to have spherical bases if there exists r and $n\geqslant r$ for which $E_r^{p,q}=0$ if $p\neq 0,n$. Properties of spherical base spectral sequences:

(1) if the spectral sequence is weakly convergent and the filtrations of the limit are exhaustive and separated, then the filtrations are finite and

$$F^p(E^i)=\begin{cases}E^i & \text{if } p\leqslant 0\\ E_\infty^{n,i-n} & \text{if } 0<p\leqslant n\\ 0 & \text{if } p>n.\end{cases}$$

(2) in the previous situation for every i there is the exact sequence:

$$0\longrightarrow E_\infty^{n,i-n}\longrightarrow E^i\longrightarrow E_\infty^{0,i}\longrightarrow 0$$

coming from the sequence $0\to F^n(E^i)\to E^i\to E^i/F^n(E^i)\to 0$ considering $E^i/F^n(E^i)=E^i/F^1(E^i)=\mathrm{gr}_0E^i\simeq E_\infty^{0,i}$. If in the definition we put $n<r$, then the spectral sequence is constant after the r level, and, under the usual hypothesis of weak convergence and filtrations of the limit exhaustive and separated, we can read the exact sequence with $\infty=r$.

(3) we find also the morphisms

$$E_r^{n,i-n}\longrightarrow E^i \qquad\text{and}\qquad E^i\longrightarrow E_r^{0,i}$$

given resp. by the compositions

$$E_r^{n,i-n}\twoheadrightarrow E_\infty^{n,i-n}\simeq \mathrm{gr}_n(E^i)=F^n(E^i)\hookrightarrow E^i$$

where the first morphism is given by $E_\infty^{n,i-n}=\varinjlim_r E_r^{n,i-n}$; and

$$E^i\twoheadrightarrow E^i/F^1(E^i)=\mathrm{gr}_0(E^i)\simeq E_\infty^{0,i}\hookrightarrow E_r^{0,i}$$

where the last morphism is given by $E_\infty^{0,i}=\varprojlim_r E_r^{0,i}$.

(4) There exists a long exact sequence

$$\cdots\longrightarrow E_r^{n,i-n}\longrightarrow E^i\longrightarrow E_r^{0,i}\longrightarrow E_r^{n,i+1-n}\longrightarrow E^{i+1}\longrightarrow\cdots$$

In fact for $r<n$ we can see that $E_r^{p,q}=E_{r+1}^{p,q}$ or all p and q, so that we can suppose $r=n$; moreover

$$E_\infty^{0,i}\simeq\ker(d_n^{0,i})\hookrightarrow E_n^{0,i}$$

$$E_n^{n,i-n+1}\twoheadrightarrow\frac{E_n^{n,i-n+1}}{\mathrm{im}(d_n^{0,i})}\simeq E_\infty^{n,i-n+1}\ .$$

Spectral sequences with spherical fiber: E is said to have spherical fiber if there exists r and $n\geqslant r-1$ for which $E_r^{p,q}=0$ if $q\neq 0,n$. Properties of spherical fiber spectral sequences:

(1) if the spectral sequence is weakly convergent and the filtrations of the limit are exhaustive and separated, then the filtrations are finite and

$$F^p(E^i)=\begin{cases}E^i & \text{if } p\leqslant i-n\\ E_\infty^{i-n,n} & \text{if } i-n<p\leqslant i\\ 0 & \text{if } p>i.\end{cases}$$

(2) in the previous situation for every i there is the exact sequence:

$$0\longrightarrow E_\infty^{i,0}\longrightarrow E^i\longrightarrow E_\infty^{i-n,n}\longrightarrow 0$$

coming from the sequence $0\to F^i(E^i)\to E^i\to E^i/F^i(E^i)\to 0$ considering $E^i/F^i(E^i)=\mathrm{gr}_{i-n}E^i\simeq E_\infty^{i-n,n}$.

(3) we find also the morphisms

$$E_r^{i,0}\longrightarrow E^i \qquad\text{and}\qquad E^i\longrightarrow E_r^{i-n,n}\ .$$

(4) There exists a long exact sequence

$$\cdots\longrightarrow E_r^{i,0}\longrightarrow E^i\longrightarrow E_r^{i-n,n}\longrightarrow E_r^{i+1,0}\longrightarrow E^{i+1}\longrightarrow\cdots\ .$$

Spectral sequence of a filtered complex.

Let $\mathbf{C}(\mathcal{A})_f$ be the category of filtered complexes, i.e. complexes $C^\cdot$ of objects of $\mathcal{A}$ endowed with a (decreasing) filtration $F^pC^\cdot$ compatible with the differentials ($dF^pC^m\subseteq F^pC^{m+1}$). Using $F^p(C^\cdot)=\left(C^n\cap F^p(C^\cdot)\right)_{n\in\mathbb{Z}}$ to define a filtration on every single object, we have an equivalence of categories with $\mathbf{C}(\mathcal{A}_f)$.

The filtration is regular (resp. strongly regular) if for every n there exists an integer $M(n)$ s.t. for $p>M(n)$ we have $H^n(F^pC)=0$ (resp. $F^pC^n=0$).

A homotopy between morphisms of filtered complexes is of order less than or equal to k if it sends F^p into F^{p-k}.

(1) We have a functor from $\mathbf{C}(\mathcal{A})_f$ to $SpS(\mathcal{A})$, defined also for the levels $r=0,1$, for which:

(r=0) $Z_0^{p,q}=F^p(C^{p+q})$, $Z_{-1}^{p+1,q-1}=F^{p+1}(C^{p+q})$ so that $E_0^{p,q}=F^p(C^{p+q})\big/F^{p+1}(C^{p+q})$ with the differentials $d_0^{p,q}$ induced by those of the complex C by quotient.

(r=1) $E_1^{p,q}=H\left(F^p(C^{p+q})\big/F^{p+1}(C^{p+q})\right)=H^{p+q}\left(F^p(C)/F^{p+1}(C)\right)$ with the differentials $d_1^{p,q}$ obtained by the short exact sequence

$$0\to F^{p+1}(C)/F^{p+2}(C)\to F^p(C)/F^{p+2}(C)\to F^p(C)/F^{p+1}(C)\to 0$$

using the coboundary morphism:

$$E_1^{p,q}=H^{p+q}\left(\mathrm{gr}_p(C)\right)\overset{\delta^{p+q}}{\longrightarrow}H^{p+q+1}\left(gr^{p+1}(C)\right)=E_1^{p+1,q}\ .$$

(2) Construction of the functor:

(a) Define $F^{-\infty}C=C$ and $F^\infty C=0$. Consider the sequence of indexes $(p+r,p+1,p,p-r+1)$; from the exact sequence

$$0\to F^{p+1}C\big/F^{p+r}C\to F^pC\big/F^{p+r}C\to F^pC\big/F^{p+1}C\to 0$$

we define

$$\begin{aligned}Z_r^{p,q}(C)&=\mathrm{im}\left(H^{p+q}(F^pC/F^{p+r}C)\longrightarrow H^{p+q}(F^pC/F^{p+1}C)\right)\\ &\cong\ker\left(H^{p+q}(F^pC/F^{p+1}C)\longrightarrow H^{p+q+1}(F^{p+1}C/F^{p+r}C)\right)\ ;\end{aligned}$$

then from the exact sequence

$$0\to F^pC\big/F^{p+1}C\to F^{p-r+1}C\big/F^{p+1}C\to F^{p-r+1}C\big/F^pC\to 0$$

we define

$$\begin{aligned}B_r^{p,q}(C)&=\mathrm{im}\left(H^{p+q-1}(F^{p-r+1}C/F^pC)\longrightarrow H^{p+q}(F^pC/F^{p+1}C)\right)\\ &\cong\ker\left(H^{p+q}(F^pC/F^{p+1}C)\longrightarrow H^{p+q}(F^{p-r+1}C/F^{p+1}C)\right)\ ;\end{aligned}$$

we find $B_r^{p,q}(C)\subseteq Z_r^{p,q}(C)$ and we can use

$$E_r^{p,q}(C)=Z_r^{p,q}(C)\big/B_r^{p,q}(C).$$

In particular $Z_1^{p,q}(C)=H^{p+q}(F^pC/F^{p+1}C)$ and $B_1^{p,q}(C)=0$ (remark that the general definition does not work for $r=0$).

(d) similarly ($r=\infty$) we define:

$$\begin{aligned}Z_\infty^{p,q}(C)&=\mathrm{im}\left(H^{p+q}(F^pC)\longrightarrow H^{p+q}(F^pC/F^{p+1}C)\right)\\ &\cong\ker\left(H^{p+q}(F^pC/F^{p+1}C)\longrightarrow H^{p+q+1}(F^{p+1}C)\right)\\ B_\infty^{p,q}(C)&=\mathrm{im}\left(H^{p+q-1}(C/F^pC)\longrightarrow H^{p+q}(F^pC/F^{p+1}C)\right)\\ &\cong\ker\left(H^{p+q}(F^pC/F^{p+1}C)\longrightarrow H^{p+q}(C/F^{p+1}C)\right)\end{aligned}$$

and $E_\infty^{p,q}(C)$ as the quotient; we have in fact

$$0=B_1\subseteq\cdots\subseteq B_s\subseteq\cdots\subseteq B_\infty\subseteq Z_\infty\subseteq\cdots\subseteq Z_r\subseteq\cdots\subseteq Z_1=E_1\ .$$

(b) the differentials $d_r^{p,q}:\ E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ are defined by the compositions:
$$Z_r/B_r \twoheadrightarrow Z_r/Z_{r+1} \xrightarrow[\cong]{\delta_r} B_{r+1}/B_r \hookrightarrow Z_r/B_r$$
so that we have $\ker d_r = Z_{r+1}/B_r$ and $\mathrm{im} d_r = B_{r+1}/B_r$.

(c) the isomorphisms $\alpha_r^{p,q}$ are induced by
$$H(E_r) = \big(Z_{r+1}/B_r\big) \big/ \big(B_{r+1}/B_r\big) \cong Z_{r+1}/B_{r+1} = E_{r+1}\ ;$$

(e) we consider $E_n(C) = H^n(C)$ the cohomology of C as a collection of objects with the filtrations defined by $F^p H^n(C) = \mathrm{im}(H^n(F^p(C)) \longrightarrow H^n(C))$ with the morphisms induced in cohomology from the inclusions $F^p(C) \hookrightarrow C$;

(f) the isomorphisms β_n are induced by $E_\infty^{p,q}(C) \cong \mathrm{gr}^p\left(H^{p+q}(C)\right)$.

(x) the key lemma for the points (b), (c), (f) is: given a diagram with exact row:
$$\begin{tikzcd} & P \arrow[d, "c"] \arrow[dr, "d"] & \\ L \arrow[ur] \arrow[r, "a"'] & M \arrow[r, "b"'] & N \end{tikzcd}$$
then b induces an isomorphism $\mathrm{im}(c)/\mathrm{im}(a) \to \mathrm{im}(b)$. The square of the lemma says that if we have a commutative diagram
$$\begin{tikzcd} & & L' \arrow[dl] \arrow[d, "a'"] \\ & P \arrow[r, "c'"] \arrow[d, "c"] & M' \arrow[d, "b'"] \\ L \arrow[ur] \arrow[r, "a"'] & M \arrow[r, "b"'] & N \end{tikzcd}$$
with exact row and column, then we have the isomorphisms
$$\mathrm{im}(c)/\mathrm{im}(a) \cong \mathrm{im}(bc) = \mathrm{im}(b'c') \cong \mathrm{im}(c')/\mathrm{im}(a')$$
induced respectively from b and b'.

(y) the key for the points (a), (d) is: let $K(p,q) = F^p/F^q$ for $p \leqslant q$; for $(p_1,q_1) \leqslant (p_2,q_2)$, i.e. $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$, we have the commutative diagram
$$\begin{tikzcd} K(p_2,q_2) \arrow[r] \arrow[d] \arrow[dr] & K(p_1,q_2) \arrow[d] \\ K(p_2,q_1) \arrow[r] & K(p_1,q_1) \end{tikzcd}$$
and calling the diagonal map u_{12}, we have for $(p_2,q_2) \leqslant (p_3,q_3)$ the compatibility $u_{13} = u_{12} \circ u_{23}$. Moreover, for $p_1 \leqslant p_2 \leqslant p_3$ we have the exact sequence
$$0 \longrightarrow K(p_2,p_3) \longrightarrow K(p_1,p_3) \longrightarrow K(p_1,p_2) \longrightarrow 0$$
and the associated long exact sequence of cohomology.

(z) The chain of inclusions in (d) can be seen using the sequences of index $(p,p+1,p+r,p+r+1)$ and $(p-r,p-r+1,p,p+1)$, and the commutative diagrams
$$\begin{tikzcd} K(p,p+r+1) \arrow[r, "u_{r+1}"] \arrow[d] & K(p,p+1) \arrow[d, "="] \\ K(p,p+r) \arrow[r, "u_r"] & K(p,p+1) \end{tikzcd}, \quad \begin{tikzcd} K(p,p+1) \arrow[r, "v_{r-1}"] \arrow[d, "v_r"'] & K(p-r+1,p+1) \arrow[d] \\ K(p-r+1,p+1) \arrow[r, "="] & K(p-r,p+1) \end{tikzcd}.$$
In fact we have $Z_r = \mathrm{im}(H^n u_r) \supseteq \mathrm{im}(H^n u_{r+1}) = Z_{r+1}$ and $B_r = \ker(H^n v_{r-1}) \subseteq \ker(H^n v_r) = B_{r+1}$; moreover considering the index sequence $(p-r+1,p,p+1,p+s)$, the zero composition (of coboundary morphisms)
$$H^{n-1}K(p-r+1,p) \xrightarrow{\partial_r} H^n K(p,p+1) \xrightarrow{\partial_s} H^{n+1}K(p+1,p+s)$$
gives $Z_s = \ker(\partial_s) \supseteq \mathrm{im}(\partial_r) = B_s$ for any r,s, in particular for $r = s = \infty$.

The basic isomorphism for the definition of d_r can be see starting with the exact diagram of complexes
$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & 0 \arrow[d] & 0 \arrow[d] & K(p+1,p+r) \arrow[d] & \\ 0 \arrow[r] & K(p+r,p+r+1) \arrow[r] \arrow[d] & K(p,p+r+1) \arrow[r] \arrow[d] & K(p,p+r) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K(p+1,p+r+1) \arrow[r] \arrow[d] & K(p,p+r+1) \arrow[r] \arrow[d] & K(p,p+1) \arrow[r] \arrow[d] & 0 \\ & K(p+1,p+r) \arrow[d] & 0 & 0 & \\ & 0 & & & \end{tikzcd}$$
then constructing the commutative diagram
$$\begin{tikzcd} & & H^n K(p+1,p+r) \arrow[dl] \arrow[d, "\epsilon'"] \\ & H^n K(p,p+r) \arrow[r, "\epsilon"] \arrow[d, "u"] & H^{n+1}K(p+r,p+r+1) \arrow[d] \\ H^n K(p,p+r+1) \arrow[ur] \arrow[r, "u'"'] & H^n K(p,p+1) \arrow[r] & H^{n+1}K(p+1,p+r+1) \end{tikzcd}$$
and finally applying the square lemma to $\mathrm{im}(u) = Z_r^{p,q}$, $\mathrm{im}(u') = Z_{r+1}^{p,q}$ and $\mathrm{im}(\epsilon) = B_{r+1}^{p+r,q-r+1}$, $\mathrm{im}(\epsilon') = B_r^{p+r,q-r+1}$.

In order to evaluate the terms E_∞ we start with the exact diagram of complexes
$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & K(p+1,\infty) \arrow[r] \arrow[d] & K(p+1,\infty) \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & K(p,\infty) \arrow[r] \arrow[d] & K(\infty,-\infty) \arrow[r] \arrow[d] & K(-\infty,p) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K(p,p+1) \arrow[r] \arrow[d] & K(-\infty,p+1) \arrow[r] \arrow[d] & K(-\infty,p) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$
then construct the commutative diagram
$$\begin{tikzcd} & & H^n K(p+1,\infty) \arrow[dl] \arrow[d, "v'"] \\ & H^n K(p,\infty) \arrow[r, "u'"] \arrow[d, "u"] & H^n K(-\infty,\infty) \arrow[d] \\ H^{n-1}K(-\infty,p) \arrow[ur] \arrow[r, "v"'] & H^n K(p,p+1) \arrow[r] & H^n K(-\infty,p+1) \end{tikzcd}$$
and finally apply the square lemma to $\mathrm{im}(u) = Z_\infty^{p,q}$, $\mathrm{im}(v) = B_\infty^{p,q}$ and $\mathrm{im}(u') = F^p E^n$, $\mathrm{im}(v') = F^{p+1}E^n$, where $n = p+q$.

(3) Starting with complexes with regular and exhaustive filtrations, if $\mathcal{A}$ admits exact filtered inductive limits, then we obtain regular spectral sequences.

(4) A family $(B^{p,q})_{p,q\in\mathbb{Z}}$ of objects of $\mathcal{A}$ is convergent to a family of objects $(D^n)_{n\in\mathbb{Z}}$, $B^{p,q} \Longrightarrow D^{p+q}$, if there exists a complex $X \in \mathbf{C}(\mathcal{A})$ with regular filtration s.t. $E_2^{p,q}(X) \simeq B^{p,q}$ and $H^n(X) \simeq D^n$ for every $p,q,n \in \mathbb{Z}$.

(5) Given two morphisms of filtered complexes $f,g:\ C \to C'$ with a homotopy s of order less than or equal than k, we have for the morphisms induced on the spectral sequences $f,g:\ E(C) \to E'(C')$ that $f^n = g^n$ for any $n \in \mathbb{Z}$ and also $f_r^{p,q} = g_r^{p,q}$ for every $r > k$. This is the main motivation for the definition of spectral sequence with constraint $r \geqslant 2$, instead of $r \in \mathbb{N}$.

Sp. sequences of a double complex.

Let $\mathbf{C}^2(\mathcal{A})$ be the category of bicomplexes with objects in $\mathcal{A}$, i.e. $\mathbf{C}(\mathbf{C}(\mathcal{A}))$ or more explicitly: objects are the collections
$$\left((X^{n,m})_{n,m\in\mathbb{Z}}, (d_{IX}^{n,m})_{n,m\in\mathbb{Z}}, (d_{IIX}^{n,m})_{n,m\in\mathbb{Z}}\right)$$
where $X^{n,m} \in \mathrm{ob}\mathcal{A}$,

$$
\begin{array}{ll}
X^{n,m} \xrightarrow{d_{IX}^{n,m}} X^{n+1,m} & d_{IX}^{n+1,m} \circ d_{IX}^{n,m} = 0 \\
X^{n,m} \xrightarrow{d_{IIX}^{n,m}} X^{n,m+1} , & d_{IIX}^{n,m+1} \circ d_{IIX}^{n,m} = 0
\end{array}
$$

and $d_{IIX}^{n+1,m} \circ d_{IX}^{n,m} = d_{IX}^{n,m+1} \circ d_{IIX}^{n,m}$. Morphisms between $X^{\cdot\cdot}$ and $Y^{\cdot\cdot}$ are collections $X^{n,m} \xrightarrow{f^{n,m}} Y^{n,m}$ for $n,m \in \mathbb{Z}$ s.t.

$$
\begin{array}{c}
d_{IY}^{n,m} \circ f^{n,m} = f^{n+1,m} \circ d_{IX}^{n,m} \\
d_{IIY}^{n,m} \circ f^{n,m} = f^{n,m+1} \circ d_{IIX}^{n,m} \ .
\end{array}
$$

Remark that $\mathbf{C}^2(\mathcal{A})$ has an internal automorphism (exchange of the indexes). Standard notions on the category of bicomplexes:

(1) Cocycles, coboundary and cohomology operators:

$$
\begin{array}{ll}
Z_{II}^{\cdot,m} := \ker(d_{II}^{\cdot,m}) & Z_{I}^{n,\cdot} := \ker(d_{I}^{n,\cdot}) \\
B_{II}^{\cdot,m} := \mathrm{im}(d_{II}^{\cdot,m-1}) & B_{I}^{n,\cdot} := \mathrm{im}(d_{I}^{n-1,\cdot}) \\
H_{II}^{\cdot,m} := Z_{II}^{\cdot,m} \big/ B_{II}^{\cdot,m} & H_{I}^{n,\cdot} := Z_{I}^{n,\cdot} \big/ B_{I}^{n,\cdot}
\end{array}
$$

these structures are complexes w.r.t. the pointed index (with differentials resp. $d_I^{\cdot,m}$ and $d_{II}^{n,\cdot}$) so that we define also:

$$
\begin{array}{ll}
Z_I^n H_{II}^m := \ker(H_{II}^{n,m} \to H_{II}^{n+1,m}) & Z_{II}^m H_I^n := \ker(H_I^{n,m} \to H_I^{n,m+1}) \\
B_I^n H_{II}^m := \mathrm{im}(H_{II}^{n-1,m} \to H_{II}^{n,m}) & B_{II}^m H_I^n := \mathrm{im}(H_I^{n,m-1} \to H_I^{n,m}) \\
H_I^n H_{II}^m := Z_I^n H_{II}^m \big/ B_I^n H_{II}^m & H_{II}^m H_I^n := Z_{II}^m H_I^n \big/ B_{II}^m H_I^n \ .
\end{array}
$$

(2) a morphism f of bicomplexes X and Y is homotopic to zero if there exist a pair of families of morphisms for $n,m \in \mathbb{Z}$

$$
\begin{array}{c}
s^{n,m} : \ X^{n,m} \to Y^{n-1,m} \\
t^{n,m} : \ X^{n,m} \to Y^{n,m-1}
\end{array}
$$

with the commutation properties

$$
\begin{array}{c}
s^{n,m+1} \circ d_{IIX}^{n,m} = d_{IIY}^{n-1,m} \circ s^{n,m} \\
t^{n+1,m} \circ d_{IX}^{n,m} = d_{IY}^{n,m-1} \circ t^{n,m}
\end{array}
$$

and reconstructing f: $f^{n,m}$ is

$$
s^{n+1,m} \circ d_{IX}^{n,m} + d_{IY}^{n-1,m} \circ s^{n,m} + t^{n,m+1} \circ d_{IIX}^{n,m} + d_{IIY}^{n,m-1} \circ t^{n,m} .
$$

Obviously s (resp. t) induces homotopies for the morphism induced by f between the complexes $H_{II}^{\cdot,m}(X)$ and $H_{II}^{\cdot,m}(Y)$ (resp. $H_I^{n,\cdot}(X)$ and $H_I^{n,\cdot}(Y)$) and so f induces the null map between the cohomologies $H_I^n H_{II}^m(X)$ and $H_I^n H_{II}^m(Y)$ (resp. $H_{II}^m H_I^n(X)$ and $H_{II}^m H_I^n(Y)$). Two maps of bicomplexes are homotopic if the difference is homotopic to zero.

(3) Under one of the following hypothesis:

(i) $\mathcal{A}$ admits inductive limits;

(ii) for objects of $\mathbf{C}^2(\mathcal{A})$ s.t. for any k the cardinality of $\{(n,m) |\ n+m=k,\ X^{n,m} \neq 0\}$ is finite;

we have a functor $s : \mathbf{C}^2(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A})$ acting on the objects by

$$
s(X)^k = \bigoplus_{n+m=k} X^{n,m} \quad \text{and} \quad d_{s(X)}^k : \ s(X)^k \longrightarrow s(X)^{k+1}
$$

where the differentials are induced by $X^{n,m} \longrightarrow s(X)^{k+1}$ defined for $n+m=k$ as $\iota_{n+1,m}^{k+1} \circ d_I^{n,m} + (-)^n \iota_{n,m+1}^{k+1} \circ d_{II}^{n,m}$ using $\iota_{i,j}^{k+1} : \ X^{i,j} \longrightarrow s(X)^{k+1}$ for $i+j=k+1$. By definition the (total) cohomology of a bicomplex is the cohomology of the associated simple complex.

(4) We have two different filtrations on $s(X)$ given by

$$
F_I^p\left(s(X)^k\right) = \bigoplus_{\substack{n+m=k \\ n \geqslant p}} X^{n,m} \qquad \text{and} \qquad F_{II}^p\left(s(X)^k\right) = \bigoplus_{\substack{n+m=k \\ m \geqslant p}} X^{n,m} \ ;
$$

they are exhaustive and separated; we use the symbols $F_I^p(X)$ and $F_{II}^p(X)$. Moreover: $F_I^p(X)$ is discrete (resp. codiscrete) and $F_{II}^p(X)$ codiscrete (resp. discrete) if there exists n' (resp. n'') s.t. for any $n \geqslant n'$ (resp. $n \leqslant n''$) we have $X^{n,m}=0$; or if there exists m'' (resp. m') s.t. for any $m \leqslant m''$ (resp. $m \geqslant m'$) we have $X^{n,m}=0$.

(5) We define two functors $\mathbf{C}^2(\mathcal{A}) \longrightarrow SpS(\mathcal{A})$: the two spectral sequences associated to a bicomplex X are the spectral sequences associated to the simple complex using the two filtrations: ${}^{I}E(X)$ and ${}^{II}E(X)$ respectively. They are the same limit (the total cohomology) and we can calculate the first terms:

$$
\begin{array}{ccccccc}
{}^{I}E_0^{p,q}(X) & & {}^{I}E_0^{p,q+1}(X) & & {}^{II}E_0^{q,p}(X) & & {}^{II}E_0^{q,p+1}(X) \\
\| & & \| & & \| & & \| \\
X^{p,q} & \xrightarrow[{}^{I}d_0^{p,q} = d_{II}^{p,q}]{} & X^{p,q+1} & & X^{p,q} & \xrightarrow[{}^{II}d_0^{q,p} = d_I^{p,q}]{} & X^{p+1,q} \\
{}^{I}E_1^{p,q}(X) & & {}^{I}E_1^{p+1,q}(X) & & {}^{II}E_1^{q,p}(X) & & {}^{II}E_1^{q+1,p}(X) \\
\| & & \| & & \| & & \| \\
H_{II}^{p,q}(X) & \xrightarrow[{}^{I}d_1^{p,q} = d_I^{p,q}]{} & H_{II}^{p+1,q}(X) & & H_I^{p,q}(X) & \xrightarrow[{}^{II}d_1^{q,p} = d_{II}^{p,q}]{} & H_I^{p,q+1}(X)
\end{array}
$$

$$
{}^{I}E_2^{p,q}(X) = H_I^p H_{II}^q(X) \qquad\qquad {}^{II}E_2^{q,p}(X) = H_{II}^q H_I^p(X) \ .
$$

(6) Two homotopic morphisms of double complexes induce the same morphism on the associated spectral sequence.

(7) Given a bicomplex X, if one of the following conditions hold:

(i) there exists n_0 and m_0 s.t. for $n < n_0$ or $m < m_0$ (or: $n > n_0$ or $m > m_0$) we have $X^{n,m}=0$;

(ii) there exists n_0 and n_1 (or: m_0 and m_1) s.t. for $n < n_0$ or $n > n_1$ (or: $m < m_0$ or $m > m_1$) we have $X^{n,m}=0$;

then the two associated spectral sequences are biregular (in particular, for $k > n_1 - n_0$ (or: $k > m_1 - m_0$) the sequences are constant).

(8) Given a bicomplex X in $\mathcal{A}$ category with exact filtrant inductive limits, then:

(i) if there exists n_0 s.t. for $n > n_0$ (or: there exists m_0 s.t. for $m < m_0$) we have $X^{n,m}=0$ then the sequence ${}^{I}E(X)$ is regular;

(ii) if there exists n_0 s.t. for $n < n_0$ (or: there exists m_0 s.t. for $m > m_0$) we have $X^{n,m}=0$ then the sequence ${}^{II}E(X)$ is regular.

(9) (Weyl's lemma) Given a bicomplex $C^{\cdot\cdot}$ with non negative degrees ($C^{n,m}=0$ if $n<0$ or $m<0$) with exact rows and columns, except for the first, there exists a canonical isomorphism between the cohomology of the first row and that of the first column.

Sp. seq. of a functor w.r.t. a finitely filtered object.

Let $\mathcal{A}_{ff}$ be the abelian category of finitely filtered objects of $\mathcal{A}$, i.e. objects of $\mathcal{A}$ endowed with a finite filtration; morphisms in this category are the morphisms of $\mathcal{A}$ (without reference to the filtrations). If $\mathcal{A}$ has sufficiently many injective objects, we have a functor of "filtered resolution": $\mathcal{A}_{ff} \stackrel{\cdot}{\longrightarrow} \mathbf{C}(\mathcal{I})_f$ with the properties that:

(1) $A^{\cdot}$ is an injective resolution of A,

(2) $F^i(A^{\cdot})$ is an injective resolution of $F^i(A)$,

(3) $F^i(A^{\cdot})=0$ (resp. $=A$) if $F^i(A)=0$ (resp. $=A$),

(4) given $F : A \to B$ and two morphisms $u^{\cdot}, v^{\cdot} : A^{\cdot} \longrightarrow B^{\cdot}$ extensions of f then they are homotopic: $u^{\cdot} \sim v^{\cdot}$,

(5) given $f : A \to B$ for which $f\left(F^i(A)\right) \subseteq F^{i+s}(B)$ for any $i \in \mathbb{Z}$ then $f^{\cdot} : A^{\cdot} \to B^{\cdot}$ verifies $f^{\cdot}\left(F^i(A^{\cdot})\right) \subseteq F^{i+s}(B^{\cdot})$.

Given an additive functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ between abelian categories, the first with suff. many injectives, and an object A of $\mathcal{A}_{ff}$, we take the filtered injective resolution $A^\cdot$ and we define the spectral sequence of F w.r.t. A as the spectral sequence associated to the filtered complex $F(A^\cdot)$ with filtration $F^i\left(F(A^\cdot)\right)=F\left(F^i(A^\cdot)\right)$. Limit for this sp.seq. is the cohomology $H^{p+q}\left(F(A^\cdot)\right)=R^{p+q}F(A)$ with the filtration

$$\begin{aligned}F^i\left(R^{p+q}F(A)\right)&=\mathrm{im}\left(H^{p+q}\left(F^i\left(F(A^\cdot)\right)\right)\longrightarrow H^{p+q}\left(F(A^\cdot)\right)\right)\\&=\mathrm{im}\left(R^{p+q}F\left(F^i(A)\right)\longrightarrow R^{p+q}F(A)\right)\\&=\ker\left(R^{p+q}F(A)\longrightarrow R^{p+q}F\left(A/F^i(A)\right)\right)\ .\end{aligned}$$

The terms in the first level are given by:

$$E_1^{p,q}=H^{p+q}\left(gr^p\left(F(A^\cdot)\right)\right)=R^{p+q}F(gr^p(A))\ .$$

Hypercohomology.

(0) Given a complex $C^\cdot$ in $\mathbf{C}(\mathcal{A})$, a Cartan-Eilenberg resolution of $C^\cdot$ is a bicomplex $L^{\cdot\cdot}$ with $L^{\cdot j}=0$ for $j<0$ with a morphism of complexes $C^\cdot\xrightarrow{\epsilon}L^{\cdot 0}$ s.t.

(*i*) for any i the sequence $0\longrightarrow C^i\xrightarrow{\epsilon^i}L^{i\cdot}$ is exact and induce exact sequences by the coboundary, cocycles and cohomology functors;

(*ii*) for any j the simple complex $L^{\cdot j}$ admits split short exact sequences

$$\begin{aligned}0\longrightarrow B^i(L^{\cdot j})\longrightarrow Z^i(L^{\cdot j})\longrightarrow H^i(L^{\cdot j})\longrightarrow 0\\0\longrightarrow Z^i(L^{\cdot j})\longrightarrow L^{i,j}\longrightarrow B^{i+1}(L^{\cdot j})\longrightarrow 0\end{aligned}$$

for any i.

The resolution is injective if every object $L^{i,j}$ is (so that by the second condition, also the coboundary, cocycles and cohomology objects are).

(1) If $\mathcal{A}$ has enough injectives then every complexes admits an injective Cartan Eilenberg resolution and a morphism between complexes $C^\cdot\xrightarrow{f}C'^\cdot$ can be lift to a morphism between the resolutions $L^{\cdot\cdot}\xrightarrow{F}L'^{\cdot\cdot}$, so that we obtain an exact "resolution functor" $CE:\ \mathbf{C}(\mathcal{A})\to\mathbf{C}^2(\mathcal{I})$. Moreover homotopic morphisms go to homotopic morphism between resolutions. Obviously, bounded complexes admit resolutions bounded in the same way.

(2) Let $F:\ \mathcal{A}\longrightarrow\mathcal{B}$ be a covariant functor between abelian categories; we (partially) define two functors $\mathbf{C}(\mathcal{A})\longrightarrow SpS(\mathcal{B})$ in the following way: starting with a complex $C^\cdot$ we take an injective CE resolution $L^{\cdot\cdot}$, if there exists the simple complex associated to $F(L^{\cdot\cdot})\in\mathbf{C}^2(\mathcal{B})$ we use the two spectral sequences associated to the bicomplex: they are the spectral sequences of hypercohomology of F w.r.t. the complex $C^\cdot$. The limit of the spectral sequences is the hypercohomology of F w.r.t. $C^\cdot$ indicated as $\mathbf{R}^nF(C^\cdot)=H^n\left(F(L^{\cdot\cdot})\right)$ and the terms in the first level are:

$${}^IE_2^{p,q}(FL^{\cdot\cdot})=H^p\left(R^qF(C^\cdot)\right)\qquad {}^{II}E_2^{p,q}(FL^{\cdot\cdot})=R^pF\left(H^q(C^\cdot)\right)$$

where $R^qF(C^\cdot)=\left(R^qF(C^i)\right)_{i\in\mathbb{Z}}$.

(3) the first spectral sequence does not require a CE resolution, and it starts with ${}^IE_1^{p,q}(FL^{\cdot\cdot})=R^qF(C^p)$.

Standard results:

(1) Given $A\in\mathcal{A}$ and an arbitrary resolution $C^\cdot$ (i.e. a positive degree complex s.t. $0\to A\to C^\cdot$ is exact) then $\mathbf{R}^pF(C^\cdot)\simeq R^pF(A)$ and $H^p\left(R^qF(C^\cdot)\right)\Longrightarrow R^{p+q}F(A)$. In fact the second sp.seq. degenerates: ${}^{II}E_2^{p,q}(C^\cdot)=R^pF\left(H^q(C^\cdot)\right)=0$ if $q>0$ and ${}^{II}E_2^{p,0}(C^\cdot)=R^pF\left(H^0(C^\cdot)\right)\simeq R^pF(A)$.

(2) Given a complex $X^\cdot$ of F-acyclic objects (i.e. $R^qF(X^i)=0$ for every $q>0$) then $\mathbf{R}^pF(X^\cdot)\simeq H^pF(X^\cdot)$ and moreover $R^pF\left(H^q(X^\cdot)\right)\Longrightarrow H^{p+q}F(X^\cdot)$. In fact the first sp.seq. degenerates in ${}^IE_2^{p,0}(X^\cdot)=H^p\left(R^0F(X^\cdot)\right)=H^pF(X^\cdot)$.

(3) By the previous two points, we can calculate the derived functors of F using F-acyclic resolutions of the objects of $\mathcal{A}$.

(4) Considering two functors $F:\ \mathcal{A}\longrightarrow\mathcal{B}$ and $G:\ \mathcal{B}\longrightarrow\mathcal{C}$ and $A\in\mathcal{A}$, if $\mathcal{A}$ has sufficiently many injectives we can resolve A with $I^\cdot\in\mathbf{C}(\mathcal{I})$, then apply the spectral sequences of G w.r.t. $F(I^\cdot)$; we have:

$$\begin{aligned}{}^IE_2^{p,q}(FI^\cdot)&=H^p\left(R^qG(FI^\cdot)\right) & {}^{II}E_2^{p,q}(FI^\cdot)&=R^pG\left(H^q(FI^\cdot)\right)\\&=H^p\left(R^qG\circ F(I^\cdot)\right) & &=R^pG(R^qF(A))\\&=R^p\left(R^qG\circ F\right)(A) & &=(R^pG\circ R^qF)(A)\end{aligned}$$

sequences with the same limit: the hypercohomology of $G\circ F$ w.r.t. A, $\mathbf{R}^\cdot G\left(F(I^\cdot)\right)$.

(5) *Leray sp.seq.:* if F sends injective objects of $\mathcal{A}$ into G-acyclic objects of $\mathcal{B}$ and G is left exact, then $\mathbf{R}^\cdot G\left(F(I^\cdot)\right)\simeq R^p(G\circ F)(A)$ and $R^pG\circ R^qF\Longrightarrow R^{p+q}(G\circ F)$. In fact the first sp.seq. degenerates in ${}^IE_2^{p,0}(FI^\cdot)=R^p\left(R^0G\circ F\right)(A)=R^p(G\circ F)(A)$ by the acyclicity assumption.

(6) In the general situation of (4) we can write the edge morphisms:

$$R^p\left(R^0G\circ F\right)(A)\longrightarrow\mathbf{R}^pG\left(F(I^\cdot)\right)\longrightarrow R^0\left(R^pG\circ F\right)(A)$$

$$(R^pG\circ R^0F)(A)\longrightarrow\mathbf{R}^pG\left(F(I^\cdot)\right)\longrightarrow(R^0G\circ R^pF)(A)$$

and by composition (first up and second down) we have a canonical morphism:

$$R^p\left(R^0G\circ F\right)\longrightarrow R^0G\circ R^pF.$$

If G is an exact functor, then this is an isomorphism and we write $R^p(G\circ F)\simeq G\circ R^pF$. In fact the exactness of G implies the degeneracy of both sp.seq. and the edge morphisms are isomorphisms.

(7) If F is an exact functor, without hypothesis on G, then the second sp.seq. degenerates in ${}^{II}E_2^{p,0}(FI^\cdot)=(R^pG\circ F)(A)\simeq\mathbf{R}^pG\left(F(I^\cdot)\right)$ and we have $R^p(R^qG\circ F)(A)\Longrightarrow(R^{p+q}G\circ F)(A)$ so that by the first sp.seq. which is cohomological we have the exact sequence of the low degree terms:

$$0\to R^1(R^0G\circ F)\to R^1G\circ F\to R^0(R^1G\circ F)\to R^2(R^0G\circ F)\to R^2G\circ F$$

but nothing more.

(8) Let $X^\cdot\in\mathbf{C}(\mathcal{A})$ and $I^{\cdot\cdot}\in\mathbf{C}^2(\mathcal{A})$ be a resolution of $X^\cdot$ with the second degree positive, not necessarily CE, with F-acyclic objects of $\mathcal{A}$ for a functor $F:\mathcal{A}\to\mathcal{B}$. We define a bicomplex $X^{\cdot\cdot}\in\mathbf{C}^2(\mathcal{A})$ by $X^{p,q}=\begin{cases}X^p & \text{se } q=0\\0 & \text{se } q\neq 0\end{cases}$. The first spectral sequences associated to $X^{\cdot\cdot}$ and $I^{\cdot\cdot}$ give

$${}^IE_1^{p,q}(I^{\cdot\cdot})=H_{II}^q(I^{p,\cdot})=\begin{cases}H_{II}^0(I^{p,\cdot})\simeq X^p & \text{if } q=0\\0 & \text{if } q\neq 0\end{cases}\simeq{}^IE_1^{p,q}(X^{\cdot\cdot})$$

wence the canonical $X^{\cdot\cdot}\to I^{\cdot\cdot}$ gives isomorphisms on the limit: $H^n(X^\cdot)=H^n_{tot}(X^{\cdot\cdot})\simeq H^n_{tot}(I^{\cdot\cdot})=H^n\left(s(I^{\cdot\cdot})\right)$ so that $X^\cdot$ and $s(I^{\cdot\cdot})$ are quasi-isomorphic. Moreover $RF(X^\cdot)\simeq RF\left(s(I^{\cdot\cdot})\right)\simeq F\left(s(I^{\cdot\cdot})\right)\simeq s\left(F(I^{\cdot\cdot})\right)$ and also $H^nRF(X^\cdot)\simeq H^n_{tot}\left(F(I^{\cdot\cdot})\right)$. For the first sp.seq. associated to $CE\left(F(X^\cdot)\right)$ we have (terms at level one) $R^qF(X^p)\Longrightarrow\mathbf{R}^{p+q}F(X^\cdot)$, and for the first sp.seq. associated to the bicomplex $F(I^{\cdot\cdot})$ we have ${}^IE_1^{p,q}\left(F(I^{\cdot\cdot})\right)=H^q_{II}\left(F(I^{p,\cdot})\right)\simeq R^qF(X^p)$ and so we have a canonical morphism at the limit: $\mathbf{R}^{p+q}F(X^\cdot)\simeq H^{p+q}_{tot}\left(F(I^{\cdot\cdot})\right)\simeq H^{p+q}RF(X^\cdot)$

(we are used $RF(X^\cdot)$ in the sense of derived categories, and the symbol $\simeq$ means quasi-isomorphic, i.e. isomorphism in the derived category).

Tricomplexes and Hypercohomology of bicomplexes. Let $\mathcal{A}$ be an abelian category and $X^{\cdots} \in \mathbf{C}^3(\mathcal{A})$ (category of tricomplexes of $\mathcal{A}$). Under the usual hypothesis we have three ways to reduce $X^{\cdots}$ to a bicomplex, associating the indices i,j,k as $i+j=m$, $k=n$ or $i+k=m$, $j=n$ or $i=m$, $j+k=n$. Accordingly, we have six sp.seq. converging to the same limit (the total cohomology):

$$\left.\begin{array}{l} H^m_{I,II}H^n_{III}(X^{\cdots}) \\ H^m_{III}H^n_{I,II}(X^{\cdots}) \\ H^m_{I,III}H^n_{II}(X^{\cdots}) \\ H^m_{II}H^n_{I,III}(X^{\cdots}) \\ H^m_{I}H^n_{II,III}(X^{\cdots}) \\ H^m_{II,III}H^n_{I}(X^{\cdots}) \end{array}\right\} \Longrightarrow H^{m+n}_{tot}(X^{\cdots}) \ .$$

Let $F:\mathcal{A}\to\mathcal{B}$ an additive functor between abelian categories, $\mathcal{A}$ having sufficiently many injectives. Then we define the hyperderived functors of F on a bicomplex $X^{\cdot\cdot} \in \mathbf{C}^2(\mathcal{A})$ as the total cohomology of the tricomplex $F(I^{\cdots})$ where $I^{\cdots}$ is an injective CE-resolution of $X^{\cdot\cdot}$. We have the following spectral sequences with limit the hypercohomology of F on $X^{\cdot\cdot}$:

$$\left.\begin{array}{l} H^m(R^nF(X^{\cdot\cdot})) \\ R^mF(H^n_{tot}(X^{\cdot\cdot})) \\ \mathbf{R}^mF(H^n_{II}(X^{\cdot\cdot})) \\ H^m_{II}(\mathbf{R}^n_{I}F(X^{\cdot\cdot})) \\ H^m_{I}(\mathbf{R}^n_{II}F(X^{\cdot\cdot})) \\ \mathbf{R}^m_{II}F(H^n_{I}(X^{\cdot\cdot})) \end{array}\right\} \Longrightarrow \mathbf{R}^{m+n}F(X^{\cdot\cdot}) \ .$$

Suppose now one has $G:\mathcal{B}\to\mathcal{C}$ another functor, with $\mathcal{B}$ having enough G-acyclic objects. We start with a complex $X^\cdot \in \mathbf{C}(\mathcal{A})$, we take an injective CE-resolution $I^{\cdot\cdot}$ and calculate the hypercohomology of G on the bicomplex $F(I^{\cdot\cdot}) \in \mathbf{C}^2(\mathcal{B})$. We have the following sp.seq.

$$\left.\begin{array}{l} \mathbf{R}^m(R^nG\circ F)(X^\cdot) \\ R^mG(\mathbf{R}^nF(X^\cdot)) \\ \mathbf{R}^mG(R^nF(X^\cdot)) \\ \mathbf{R}^m(\mathbf{R}^nG\circ F)(X^\cdot) \\ H^m_{I}(\mathbf{R}^n_{II}G(FI^{\cdot\cdot})) \\ \mathbf{R}^m_{II}F(H^n_{I}(I^{\cdot\cdot})) \end{array}\right\} \Longrightarrow \mathbf{R}^{m+n}G(FI^{\cdot\cdot}) \ .$$

Suppose that F sends injective objects of $\mathcal{A}$ to G-acyclic objects of $\mathcal{B}$; then we obtain the following situation:

$$\left.\begin{array}{l} R^mG(\mathbf{R}^nF(X^\cdot)) \\ \mathbf{R}^mG(R^nF(X^\cdot)) \\ \mathbf{R}^m(G\circ F)H^n(X^\cdot) \\ H^m(\mathbf{R}^n(G\circ F)(X^\cdot)) \end{array}\right\} \Longrightarrow \mathbf{R}^m(R^0G\circ F)(X^\cdot) \ .$$

Algebraic spectral sequences.

Kunneth spectral sequences. Let $L\in\mathbf{C}(\text{mod-}A)$ and $M\in\mathbf{C}(A\text{-mod})$; then we have two spectral sequences

$$\left.\begin{array}{l} \bigoplus\limits_{q'+q''=q}\mathrm{Tor}^A_p(H^{q'}(L),H^{q''}(M)) \\ H^p_{tot}(\mathrm{Tor}^A_q(L,M)) \end{array}\right\} \Longrightarrow \mathbf{Tor}^A_{p+q}(L,M)$$

and in case L or M is flat we have

$$\bigoplus_{q'+q''=q}\mathrm{Tor}^A_p(H^{q'}(L),H^{q''}(M)) \Longrightarrow H^p_{tot}(L\otimes_A M) \ .$$

For $M\in\text{mod-}R$, $T\in R\text{-mod-}S$, $N\in S\text{-mod}$, from the associativity property for $M\otimes_R T\otimes_S N$ we obtain two spectral sequences

$$\left.\begin{array}{l} \mathrm{Tor}^R_p(M,\mathrm{Tor}^S_q(T,N)) \\ \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(M,T),N) \end{array}\right\} \Longrightarrow H_{p+q}(M^\cdot\otimes_R T\otimes_S N^\cdot)$$

In case that M is R-flat we deduce that

$$M\otimes_R\mathrm{Tor}^S_q(T,N)\cong\mathrm{Tor}^S_p(M\otimes_R T,N) \ ;$$

and if N is S-flat we have

$$\mathrm{Tor}^R_p(M,T\otimes_S N)\cong\mathrm{Tor}^R_p(M,T)\otimes_S N \ .$$

In particular if we have a morphism $R\to S$, we can choose $T=S$ and we have the base change spectral sequence

$$\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(M,S),N) \Longrightarrow \mathrm{Tor}^R_{p+q}(M,N) \ .$$

Exchanging the roles of R and S we can start with $N\in\text{mod-}S$, $T\in S\text{-mod-}R$, $M\in R\text{-mod}$; from the associativity property for $N\otimes_S T\otimes_R M$ we obtain two spectral sequences

$$\left.\begin{array}{l} \mathrm{Tor}^S_p(N,\mathrm{Tor}^R_q(T,M)) \\ \mathrm{Tor}^R_p(\mathrm{Tor}^S_q(N,T),M) \end{array}\right\} \Longrightarrow H_{p+q}(N^\cdot\otimes_R T\otimes_S M^\cdot)$$

In case that N is S-flat we deduce that

$$N\otimes_S\mathrm{Tor}^R_q(T,M)\cong\mathrm{Tor}^R_p(N\otimes_S T,M) \ ;$$

and if M is R-flat we have

$$\mathrm{Tor}^S_p(N,T\otimes_R M)\cong\mathrm{Tor}^S_p(N,T)\otimes_R M \ .$$

In particular if we have a morphism $R\to S$, we can choose $T=S$ and we have the base change spectral sequence

$$\mathrm{Tor}^S_p(N,\mathrm{Tor}^R_q(S,M)) \Longrightarrow \mathrm{Tor}^R_{p+q}(N,M) \ .$$

Kunneth-dual spectral sequences. Let $L\in\mathbf{C}(\text{mod-}A)$ and $M\in\mathbf{C}(A\text{-mod})$; then we have two spectral sequences

$$\left.\begin{array}{l} \bigoplus\limits_{q'+q''=q}\mathrm{Ext}^p_A(H_{q'}(L),H^{q''}(M)) \\ H^p_{tot}(\mathrm{Ext}^q_A(L,M)) \end{array}\right\} \Longrightarrow \mathbf{Ext}^A_{p+q}(L,M)$$

and in case L is projective or M is injective we have

$$\bigoplus_{q'+q''=q}\mathrm{Ext}^p_A(H_{q'}(L),H^{q''}(M)) \Longrightarrow H^p_{tot}(\mathrm{Hom}(L\otimes_A M)) \ .$$

For $M\in R\text{-mod}$, $T\in S\text{-mod-}R$, $N\in S\text{-mod}$, from the adjoint property $\mathrm{Hom}_R(M,\mathrm{Hom}_S(T,N))\cong\mathrm{Hom}_S(T\otimes_R M,N)$ we obtain two spectral sequences

$$\left.\begin{array}{l} \mathrm{Ext}^p_R(M,\mathrm{Ext}^q_S(T,N)) \\ \mathrm{Ext}^p_S(\mathrm{Tor}^R_q(T,M),N) \end{array}\right\} \Longrightarrow \begin{array}{l} H^{p+q}(\mathrm{Hom}_R(M^\cdot,\mathrm{Hom}_S(T,N^\cdot)))\cong \\ \cong H^{p+q}(\mathrm{Hom}_S(M^\cdot\otimes_R T,N^\cdot)) \ . \end{array}$$

In case that M is R-projective we deduce that

$$\mathrm{Hom}_R(M,\mathrm{Ext}^q_S(T,N)\cong\mathrm{Ext}^q_S(T\otimes_R M,N) \ ;$$

and if N is S-injective we have

$$\mathrm{Ext}^p_R(M,\mathrm{Hom}_S(T,N)\cong\mathrm{Hom}_S(\mathrm{Tor}^R_p(T,M),N) \ .$$

In particular if we have a morphism $R\to S$, we can choose $T=S$ and we have the Hom-base change spectral sequence

$$\mathrm{Ext}^p_S(\mathrm{Tor}^R_q(S,M),N) \Longrightarrow \mathrm{Ext}^{p+q}_R(M,N) \ .$$

Exchanging the roles of R and S we can start with $N\in S\text{-mod}$, $T\in R\text{-mod-}S$, $M\in R\text{-mod}$; from the usual adjoint property $\mathrm{Hom}_S(N,\mathrm{Hom}_R(T,M))\cong\mathrm{Hom}_R(T\otimes_S N,M)$ we obtain two spectral sequences

$$\left.\begin{array}{l} \mathrm{Ext}^p_S(N,\mathrm{Ext}^q_R(T,M)) \\ \mathrm{Ext}^p_R(\mathrm{Tor}^S_q(T,N),M) \end{array}\right\} \Longrightarrow \begin{array}{l} H^{p+q}(\mathrm{Hom}_S(N^\cdot,\mathrm{Hom}_R(T,M^\cdot)))\cong \\ \cong H^{p+q}(\mathrm{Hom}_R(T\otimes_S N^\cdot,M^\cdot)) \ . \end{array}$$

In case that N is S-projective we deduce that

$$\mathrm{Hom}_S(N,\mathrm{Ext}^q_R(T,M)\cong\mathrm{Ext}^p_R(T\otimes_S N,M) \ ;$$

and if M is R-projective we have

$$\mathrm{Ext}^p_S(N,\mathrm{Hom}_R(T,M))\cong\mathrm{Hom}_R(\mathrm{Tor}^S_p(T,N),M) \ .$$

In particular if we have a morphism $R\to S$, we can choose $T=S$ and we have the Hom-base change spectral sequence

$$\mathrm{Ext}^p_S(N,\mathrm{Ext}^q_R(S,M)) \Longrightarrow \mathrm{Ext}^{p+q}_R(N,M) \ .$$

Duality spectral sequences. In the situation $M\in R$-mod, $T\in R$-mod-S, $N\in$ mod-S, from the canonical morphism

$$\mathrm{Hom}_S(T,N)\otimes_R M \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(M,T),N)$$

we obtain canonical morphisms

$$\mathrm{Tor}^R(\mathrm{Hom}_S(T,N),M)\longrightarrow \mathrm{Hom}_S(\mathrm{Ext}_R(M,T),N)\ .$$

Let be R left noetherian, M of finite type over R; then if N is injective, the morphisms are iso. In particular we have the following spectral sequences

$$\left.\begin{array}{l}\mathrm{Tor}^R_{-q}(\mathrm{Ext}^q_S(T,N),M)\\ \mathrm{Ext}^p_S(\mathrm{Ext}^{-q}_R(M,T),N)\end{array}\right\}\Longrightarrow \begin{array}{l}H^{p+q}(\mathrm{Hom}_S(T,N^{\cdot})\otimes_R M.)\cong\\ \cong H^{p+q}(\mathrm{Hom}_S(\mathrm{Hom}_R(M.,T),N^{\cdot}))\ .\end{array}$$

Other spectral sequences. In the case $M\in R$-mod, $T\in R$-mod-S, $N\in S$-mod, we have a canonical morphism

$$\mathrm{Hom}_R(M,T)\otimes_S N\longrightarrow \mathrm{Hom}_R(M,T\otimes_S N)$$

and, in case R is left noetherian and M of finite type over R, we can write the spectral sequences

$$\left.\begin{array}{l}\mathrm{Tor}^S_p(\mathrm{Ext}^q_R(M,T),N)\\ \mathrm{Ext}^p_R(M,\mathrm{Tor}^S_q(T,N))\end{array}\right\}\Longrightarrow \begin{array}{l}H^{p+q}_{tot}(\mathrm{Hom}_R(M.,T)\otimes_S N.)\cong\\ \cong H^{p+q}_{tot}\mathrm{Hom}_R(M.,T\otimes_S N.)\end{array}$$

Spectral sequences of group cohomology.

Let G be a group and $M\in\mathbb{Z}[G]$-mod (resp. $M\in$ mod-$\mathbb{Z}[G]$). The cohomology (resp. homology) groups of G with values in M is $H^n(G,M)=\mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},M)$ (resp. $H_n(G,M)=\mathrm{Tor}^{\mathbb{Z}[G]}_n(M,\mathbb{Z})$).

We have that $H^n(G,M)=H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(B.(\mathbb{Z}[G],\mathbb{Z}),M))$ and $H_n(G,M)=H_n(M\otimes_{\mathbb{Z}[G]}B.(\mathbb{Z}[G],\mathbb{Z}))$ where $B.(\mathbb{Z}[G],\mathbb{Z})$ is the standard resolution of $\mathbb{Z}$ as (trivial) $\mathbb{Z}[G]$-module, i.e. $B_i(\mathbb{Z}[G],\mathbb{Z})=\mathbb{Z}\otimes_{\mathbb{Z}[G]}\mathbb{Z}[G]^{\otimes i+2}$ with projection $\epsilon: B_0(\mathbb{Z}[G],\mathbb{Z})\longrightarrow\mathbb{Z}$.

The cohomology of G with values in M is determined also as $H^n(G,M)=H^n(C^{\cdot}(G,M))$ where $C^{\cdot}(G,M)$ is the complex given by $C^n(G,M)=\mathrm{Hom}(G^n,M)$ with differentials

$$\begin{aligned}d^n(f)(g_0,\dots,g_n)&=g_0f(g_1,\dots,g_n)\\&+\sum_{i=1}^{n}(-)^i f(g_0,\dots,g_{i-1}g_i,\dots,g_n)+(-)^{n+1}f(g_0,\dots,g_{n-1})\ .\end{aligned}$$

The homology of G with values in M is determined also as $H_n(G,M)=H_n(C.(G,M))$ where $C.(G,M)$ is the complex given by $C_n(G,M)=M\otimes_{\mathbb{Z}}\mathbb{Z}[G^n]$ with differentials

$$\begin{aligned}d_n(m\otimes e_{g_1,\dots,g_n})&=mg_1\otimes e_{g_2,\dots,g_n}\\&+\sum_{i=1}^{n-1}(-)^i m\otimes e_{g_0,\dots,g_ig_{i+1},\dots,g_n}+(-)^n m\otimes e_{g_1,\dots,g_{n-1}}\ .\end{aligned}$$

We have $H^0(G,M)=M^G$ and $H_0(G,M)=M/\langle mg-m|m\in M,g\in G\rangle$. If G acts trivially on M we have $H^0(G,M)=M=H_0(G,M)$, $H^1(G,M)=\mathrm{Hom}_{gp}(G,M)$ and $H_1(G,M)=M\otimes_{\mathbb{Z}}(G/(G,G))$.

Let now H be a normal subgroup of G; then G/H acts by conjugation on $H^q(H,M)$. From the spectral sequence

$$\mathrm{Ext}^p_{\mathbb{Z}[G/H]}\left(\mathbb{Z},\mathrm{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}[G/H],M)\right)\Longrightarrow\mathrm{Ext}^{p+q}_{\mathbb{Z}[G]}(\mathbb{Z},M)$$

since $\mathrm{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}[G/H],M)\cong\mathrm{Ext}^q_{\mathbb{Z}[H]}(\mathbb{Z},M)$, we have the following spectral sequence

$$E_2^{p,q}=H^p(G/H,H^q(H,M))\Longrightarrow H^{p+q}(G,M)\ .$$

Dually G/H acts by conjugation on $H_q(H,M)$. Then from the spectral sequence

$$\mathrm{Tor}^{\mathbb{Z}[G/H]}_p\left(\mathrm{Tor}^{\mathbb{Z}[G]}_q(M,\mathbb{Z}[G/H]),\mathbb{Z}\right)\Longrightarrow\mathrm{Tor}^{\mathbb{Z}[G]}_{p+q}(M,\mathbb{Z})$$

since $\mathrm{Tor}^{\mathbb{Z}[G]}_q(M,\mathbb{Z}[G/H])\cong\mathrm{Tor}^{\mathbb{Z}[H]}_q(M,\mathbb{Z})$, we have the following spectral sequence

$$E_2^{p,q}=H_p(G/H,H_q(H,M))\Longrightarrow H_{p+q}(G,M)\ .$$

Spectral sequences in sheaf theory.

Cohomology and Hypercohomology of (global sections of) Sheaves:

(1) $H^n(X,F)$ is the n-th cohomology group of the sheaves F:

$$H^n(X,-)=R^n\Gamma(X,-):\ \mathbf{Ab}(X)\longrightarrow\mathbf{Ab}$$

(2) the hypercohomology groups of the global sections on $K^{\cdot}$:

$$\mathbf{H}^n(X,K^{\cdot})=\mathbf{R}^n\Gamma(X,-)(K^{\cdot})$$

is a complex for which the two sequences

$$\begin{aligned}{}^IE_2^{p,q}&=H^p\left(R^q\Gamma(X,-)(K^{\cdot})\right) & {}^{II}E_2^{p,q}&=R^p\Gamma(X,-)\left(H^q(K^{\cdot})\right)\\&=H^p\left(H^q(X,K^{\cdot})\right) &&=H^p\left(X,H^q(K^{\cdot})\right)\end{aligned}$$

converge (the first starts with: ${}^IE_1^{p,q}=C^p(H^q(X,K^p))$).

Cohomology and Hypercohomology of Coverings:

(0) Given a topological space X, a presheaf F on X and an open covering of X $\mathbf{U}=(U_i)_{i\in\mathcal{I}}$, or $I=(I_0,\cdots,I_q)\in\mathcal{I}^{q+1}$ we will write $U_I=\bigcap_{\ell=0}^{q}U_{I_\ell}$ and also $\varphi_I\in F(U_I)$. The complex of alternating q-cochains with values in F is given by:

$$C^q(\mathbf{U},F)=\left\{\varphi\in\prod_{I\in\mathcal{I}^{q+1}}F(U_I)\middle|\ \ \varphi_I=sgn(\sigma)\varphi_{\sigma(I)}\quad\forall I,\forall\sigma\in\Sigma(I)\right\}$$

with differentials $d^q:\ C^q(\mathbf{U},F)\longrightarrow C^{q+1}(\mathbf{U},F)$ sending $\varphi=(\varphi_I)_{I\in\mathcal{I}^{q+1}}$ in $d^q(\varphi)$ defined for $J\in\mathcal{I}^{q+2}$ by

$$d^q(\varphi)_J=\sum_{\ell=0}^{q+1}(-)^\ell\varphi_{(J_0,\cdots,\hat{J}_\ell,\cdots,J_{q+1})}|_{U_J}$$

If I is ordered we can write $C^q(\mathbf{U},F)=\prod_{I_0<\cdots<I_q}F(U_{(I_0,\dots,I_q)})$.

(0′) The presheaf of q-cochains w.r.t. a covering $\mathbf{U}$ is defined by $\mathcal{C}^q(\mathbf{U},F)$ sending an open set U of X to $C^q(\mathbf{U}|_U,F|_U)$ here $\mathbf{U}|_U=\{U_i\cap U\}_{i\in\mathcal{I}}$. If F is a sheaf, then $\mathcal{C}^q(\mathbf{U},F)$ is. We have $\Gamma(X,\mathcal{C}^q(\mathbf{U},F))=C^q(\mathbf{U},F)$.

Remark that it is equivalent to define $\mathcal{C}^q(\mathbf{U},F)$ by means of:

$\Gamma(U,\mathcal{C}^q(\mathbf{U},F))=C^q(\mathbf{U}|_U,F|_U)$

or $\Gamma(U,\mathcal{C}^q(\mathbf{U},F))=C^q(\mathbf{U},i_*i^{-1}F)$ where i is the inclusion of U in X,

or also $\mathcal{C}^q(\mathbf{U},F)=C^q(\mathbf{U},pf(F))$ where $pf(F)$ is the presheaf on X with values in the category of sheaves on X sending U to $i_*i^{-1}F$; moreover $pf(F)$ is a sheaf if F is.

If I is ordered $\mathcal{C}^q(\mathbf{U},F)=\prod_{I_0<\cdots<I_q}i_{U_{(I_0,\dots,I_q)}*}F|_{U_{(I_0,\dots,I_q)}}$.

(1) We define the cohomology of the covering $\mathbf{U}$ w.r.t. F as the cohomology of the complex of q-cochains:

$$H^q(\mathbf{U},F)=H^q\left(C^{\cdot}(\mathbf{U},F)\right)$$

If F is a sheaf, then $\mathcal{C}^{\cdot}(\mathbf{U},F)$ is a resolution of F, i.e.

$$0\longrightarrow F\longrightarrow\mathcal{C}^0(\mathbf{U},F)\longrightarrow\mathcal{C}^1(\mathbf{U},F)\longrightarrow\mathcal{C}^2(\mathbf{U},F)\longrightarrow\cdots$$

is an exact sequence of sheaves.

(2) The hypercohomology of a covering $\mathbf{U}$ w.r.t. a complex $K^{\cdot}$ of sheaves is the total cohomology of the bicomplex we obtain developing for every K^n the cochain complex:

$$\mathbf{H}^{\cdot}(\mathbf{U},K^{\cdot})=H^{\cdot}_{tot}\left(C^{\cdot}(\mathbf{U},K^{\cdot})\right)$$

and it is the limit of two spectral sequences:

$$\begin{aligned}{}^IE_1^{p,q}&=H^q(\mathbf{U},K^p) & {}^{II}E_1^{p,q}&=C^p\left(\mathbf{U},h^q(K^{\cdot})\right)\\{}^IE_2^{p,q}&=H^p\left(H^q(\mathbf{U},K^{\cdot})\right) & {}^{II}E_2^{p,q}&=H^p\left(\mathbf{U},h^q(K^{\cdot})\right)\ .\end{aligned}$$

(3) *Cartan Leray sp.seq.:* take $K^\cdot = \mathcal{C}^\cdot(X,F)$, for example the canonical flabby resolution; then we have $\mathbf{H}^\cdot(\mathbf{U},K^\cdot) \simeq H^\cdot(X,F)$ and moreover

$$H^p(\mathbf{U},h^q(X,F)) \Longrightarrow H^{p+q}(X,F)$$

($C^p(\mathbf{U},h^q(X,F))$ at level one) where $h^q(X,F) = H^q(-,F)$ is the presheaf sending U in $H^q(U,F|_U)$. It is the Leray sp.seq. for the the composed functor

$$\mathbf{Ab}(X) \xrightarrow{C^\cdot(\mathbf{U},-)} \mathbf{C}^+(\mathbf{Ab}) \xrightarrow{H^0(-)} \mathbf{Ab} \ .$$

If $K^\cdot$ is a resolution for F, we have $h^0(K^\cdot) = F$, so $h^0(X,F) = F$ and we have the edge morphisms $H^p(\mathbf{U},F) \longrightarrow H^p(X,F)$ which are isomorphisms if the Cartan Leray sp.seq. degenerate; for this is sufficient that $H^p(U_I,F|_{U_I}) = 0$ for every $p > 0$, $I \in \mathcal{I}^{q+1}$ (for any q), i.e. by definition if the covering $\mathbf{U}$ is acyclic w.r.t. F.

(4) by using inductive limits on the coverings ordered by fineness, we can write the Čech spectral sequence:

$$E_2^{p,q} = \check{H}^p(\mathbf{U},h^q(X,F)) \Longrightarrow H^{p+q}(X,F).$$

(5) Let $K^\cdot$ be a complex of sheaves; then we have functorially two sp.seq. starting with

$$E_1^{p,q} = \bigoplus_{i+j=p} C^k(\mathbf{U},h^q(X,K^i)) \qquad E_1^{p,q} = C^p\left(\mathbf{U},\mathbf{h}^q(X,K^\cdot)\right)$$
$$E_2^{p,q} = \mathbf{H}^p(\mathbf{U},h^q(X,K^\cdot)) \qquad E_2^{p,q} = H^p\left(\mathbf{U},\mathbf{h}^q(X,K^\cdot)\right)$$

where $h^q(X,K^\cdot)$ is the complex of presheaves whose value in U is $H^q(U,K^\cdot)$, $\mathbf{h}^q(X,K^\cdot)$ by definition is the q-th presheaf of hypercohmology of $K^\cdot$, i.e. $U \mapsto \mathbf{H}^q(U,K^\cdot)$, and having limit the hypercohomology $\mathbf{H}^n(X,K^\cdot)$. The key point is: take a Cartan Eilenberg resolution of $L^{\cdot\cdot}$ of $K^\cdot$, then make the tricomplex $C^\cdot(\mathbf{U},L^{\cdot\cdot}) = C^k(\mathbf{U},L^{i,j})$ of cochains. By the reduction to a bicomplex by $p = i+j$ and $q = k$, the first sp.seq. degenerates:

$${}^I E_1^{p,q} = H^q\left(C^\cdot(\mathbf{U},L^{i,j})\right) = \begin{cases} 0 & \text{if } q > 0 \\ \Gamma(X,L^{i,j}) & \text{i } q = 0 \end{cases}$$

so that ${}^I E_2^{p,0} = H^p_{tot}\left(\Gamma(X,L^{\cdot\cdot})\right) = \mathbf{H}^p(X,K^\cdot)$ and by regularity $H^p_{tot}\left(C^\cdot(\mathbf{U},L^{\cdot\cdot})\right) \simeq \mathbf{H}^p(X,K^\cdot)$. Taking the second sp.seq. we obtain also ${}^{II}E_2^{p,q} = H^p\left(\mathbf{U},\mathbf{h}^q(X,K^\cdot)\right) \Longrightarrow \mathbf{H}^{p+q}(X,K^\cdot)$. By the reduction to a bicomplex by $p = i+k$ and $q = j$, the first sp.seq. starts with ${}^I E_1^{p,q} = H^q\left(C^k(\mathbf{U},L^{i,\cdot})\right) = C^k\left(\mathbf{U},h^q(X,K^i)\right)$ so that ${}^I E_2^{p,q} = H^p_{tot}\left(C^\cdot\left(\mathbf{U},h^q(X,K^\cdot)\right)\right) = \mathbf{H}^p\left(\mathbf{U},h^q(X,K^\cdot)\right)$.

(6) If $\mathbf{U}$ is a covering acyclic for the sheaves of $K^\cdot$ then we have a canonical isomorphism $\mathbf{H}^n(\mathbf{U},K^\cdot) \simeq \mathbf{H}^n(X,K^\cdot)$. In fact in this case $h^q(X,K^\cdot) = \begin{cases} K^\cdot & \text{if } q = 0 \\ 0 & \text{i } q \neq 0 \end{cases}$ on the open sets involved in $\mathbf{U}$, and the degeneration of the previous sp.seq.

Cohomology and Hypercohomology of Hom functors:

(0) Let I be an injective sheaf; then $\mathcal{H}om_X(F,I)$ is a flabby sheaf for any F, so acyclic for the global sections $\Gamma(X,-)$.

(1) Let F and G be sheaves on X; then we have a regular sp.seq. with

$$E_2^{p,q} = H^p\left(X,\mathcal{E}xt^q_X(F,G)\right) \Longrightarrow \mathrm{Ext}^n_X(F,G).$$

It is the Leray sp.seq. for the composition $\Gamma(X,\mathcal{H}om_X(F,G)) = \mathrm{Hom}_X(F,G)$, i.e. $\Gamma(X,-) \circ \mathcal{H}om_X(F,-) = \mathrm{Hom}_X(F,-)$. Moreover we have the edge morphisms

$$\begin{array}{ccc} H^p(X,\mathcal{H}om_X(F,G)) & \longrightarrow & \mathrm{Ext}^p_X(F,G) \\ \mathrm{Ext}^p_X(F,G) & \longrightarrow & H^0\left(X,\mathcal{E}xt^p_X(F,G)\right) \end{array}$$

(2) Another computation of this sp.seq. in the case of a ringed space can be done by a left resolution $L.$ of F with locally free Modules; for the complex $\mathcal{H}om_{\mathcal{O}_X}(L.,G)$ of $\mathcal{O}_X$ Modules we have $\mathcal{E}xt^p_{\mathcal{O}_X}(F,G) = H^p\left(\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right)$. The second sp.seq. of hypercohomology of $\Gamma(X,-)$ gives

$${}^{II}E_2^{p,q} = H^p\left(X,H^q(\mathcal{H}om_{\mathcal{O}_X}(L.,G))\right) = H^p\left(X,\mathcal{E}xt^q_{\mathcal{O}_X}(F,G)\right);$$

by biregularity we have $\mathrm{Ext}^n_{\mathcal{O}_X}(F,G) \simeq \mathbf{H}^n\left(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right)$ as limit, and the first sp.seq. starts with ${}^I E_2^{p,q} = H^p\left(H^q(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G))\right)$.

(3) With the same notations $\mathrm{Ext}^n_{\mathcal{O}_X}(F,G) \simeq H^n_{tot}\left(\mathrm{Hom}_{\mathcal{O}_X}(L.,\mathcal{C}^\cdot(X,G))\right)$ where $\mathcal{C}^\cdot(X,-)$ is the canonical flabby resolution. This can be seen using the bicomplex of $\mathcal{O}_X$ Modules $\mathcal{C}^\cdot\left(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right) \simeq \mathcal{H}om_{\mathcal{O}_X}\left(L.,\mathcal{C}^\cdot(X,G)\right)$ whose global sections give a bicomplex of modules $C^\cdot\left(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right) \simeq \mathrm{Hom}_{\mathcal{O}_X}\left(L.,\mathcal{C}^\cdot(X,G)\right)$ and we can use the second sp.seq. starting with

$$\begin{aligned} {}^{II}E_2^{p,q} &= H^p_{II}H^q_I\mathrm{Hom}_{\mathcal{O}_X}\left(L.,\mathcal{C}^\cdot(X,G)\right) \\ &= H^p_{II}\mathrm{Ext}^q_{\mathcal{O}_X}\left(F,\mathcal{C}^\cdot(X,G)\right) \\ &= H^p_{II}\Gamma\left(X,\mathcal{E}xt^q_{\mathcal{O}_X}\left(F,\mathcal{C}^\cdot(X,G)\right)\right) \\ &= H^p_{II}\Gamma\left(X,\mathcal{C}^\cdot\left(X,\mathcal{E}xt^q_{\mathcal{O}_X}(F,G)\right)\right) \\ &= H^p\left(X,\mathcal{E}xt^q_{\mathcal{O}_X}(F,G)\right) \ . \end{aligned}$$

The other sp.seq. gives

$$\begin{aligned} {}^I E_2^{p,q} &= H^p_I H^q_{II} C^\cdot\left(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right) \\ &= H^p\left(H^q\left(X,\mathcal{H}om_{\mathcal{O}_X}(L.,G)\right)\right) \ . \end{aligned}$$

Cohomology and Hypercohomology of direct images:

(0) Given a left exact functor T from an abelian category $\mathcal{A}$, with sufficiently many injectives, to $\mathbf{Ab}(X)$, the functor R^nT on A is obtained as the sheaf associated to the presheaf sending U to $R^n(\Gamma(U,-) \circ T)(A)$ where $\Gamma(U,-) \circ T$ is a functor $\mathcal{A} \to \mathbf{Ab}$.

In particular, given a continuous map $X \xrightarrow{f} Y$ and a sheaf on X, $R^qf_*(F)$ is the sheaf associated to the presheaf sending an open set V of Y to $H^n(f^{\leftarrow}V,F)$. So flabby sheaves are acyclic for the direct image functors.

(1) For $X \xrightarrow{f} Y$ and F sheaf on X, the Leray sp.seq. for the composite $\Gamma(Y,-) \circ f_* = \Gamma(X,-)$ gives

$$H^p(Y,R^qf_*(F)) \Longrightarrow H^{p+q}(X,F)$$

(Leray sp.seq. for a continuous map).

(2) For $X \xrightarrow{f} Y \xrightarrow{g} Z$, the Leray sp.seq. for $g_* \circ f_* = (g \circ f)_*$ gives

$$(R^pg_* \circ R^qf_*)(F) \Longrightarrow R^{p+q}(g \circ f)_*(F).$$

The edge morphisms are given by

$$\begin{array}{ccc} R^pg_*(f_*F) & \longrightarrow & R^p(g \circ f)_*(F) \\ R^p(g \circ f)_*(F) & \longrightarrow & g_*(R^pf_*(F)) \ . \end{array}$$

(3) The hypercohomology of the functor f_* w.r.t. a complex of sheaves $K^\cdot$ on X is $\mathbf{R}^nf_*(K^\cdot) = H^n_{tot}\left(f_*\left(CE(K^\cdot)\right)\right)$, and the sp.seq. converging to it are:

$${}^I\mathcal{E}_2^{p,q} = H^p\left(R^qf_*(K^\cdot)\right) \qquad {}^{II}\mathcal{E}_2^{p,q} = R^pf_*\left(H^q(K^\cdot)\right) \ .$$

(4) Given an open covering $\mathbf{U}$ of X we can use the bicomplex of sheaves $\mathcal{C}^\cdot(\mathbf{U},K^\cdot)$ giving two regular spectral sequences with limit $H^n_{tot}\left(\mathcal{C}^\cdot(\mathbf{U},K^\cdot)\right)$. They start with

$$\begin{aligned} {}^I E_1^{p,q} &= H^q\mathcal{C}^\cdot(\mathbf{U},K^p) & {}^{II}E_1^{p,q} &= H^q\mathcal{C}^p(\mathbf{U},K^\cdot) \\ &= \mathcal{H}^q(\mathbf{U},K^p) & &= \mathcal{C}^p\left(\mathbf{U},h^q(K^\cdot)\right) \\ {}^I E_2^{p,q} &= H^p\left(\mathcal{H}^q(\mathbf{U},K^\cdot)\right) & {}^{II}E_2^{p,q} &= H^p\left(\mathcal{C}^\cdot\left(\mathbf{U},h^q(K^\cdot)\right)\right) \\ & & &= \mathcal{H}^p\left(\mathbf{U},h^q(K^\cdot)\right) \end{aligned}$$

where $h^q(-)$ is the presheaf sending U to $H^q(\Gamma(U,K^\cdot))$. If the covering is acyclic for the sheaves in $K^\cdot$ we can write

$$ {}^{I}E_2^{p,q} = H^p\left(\mathcal{H}^q(X,K^\cdot)\right) \qquad {}^{II}E_2^{p,q} = \mathcal{H}^p\left(X,H^q(K^\cdot)\right) $$

where $\mathcal{H}^q(X,-)$ is the sheaf associated to the presheaf $h^q(X,-)$. Comparing the first two sp.seq. we see that

$$ H^n_{tot}\left(f_*\left(\mathcal{C}^\cdot(\mathbf{U},K^\cdot)\right)\right) \simeq \mathbf{R}^n f_*(K^\cdot) $$

and moreover we can write the terms ${}^{II}E_1^{p,q}$ as $C^p\left(\mathbf{U},pf\left(h^q(K^\cdot)\right)\right)$.

(5) Let $K^\cdot$ be a complex of sheaves; there exist functorially two sp.seq. starting at level two with $\mathbf{H}^p(\mathbf{U},h^q(f,K^\cdot))$, where $h^q(f,K^\cdot)$ is the complex of presheaves whose value in U is $R^q f_{|U*}K^\cdot_{|U})$ (sheaves on Y), and $H^p\left(\mathbf{U},\mathbf{h}^q(f,K^\cdot)\right)$ where $\mathbf{h}^q(f,K^\cdot)$ by definition is the presheaf $U \mapsto \mathbf{R}^q f_{|U*}K^\cdot_{|U})$, and having limit $\mathbf{R}^q f_*(K^\cdot)$.

The key point is: take a Cartan Eilenberg resolution of $L^{\cdot\cdot}$ of $K^\cdot$, then make the tricomplex $\mathcal{C}^\cdot(\mathbf{U},L^{\cdot\cdot}) = C^k(\mathbf{U},L^{i,j})$ of sheaves of cochains. By the reduction to a bicomplex with $p=i+j$ and $q=k$, we see the limits and the second sp.seq.; by the reduction to a bicomplex by $p=i+k$ and $q=j$, we see the first sp.seq.

References.

[CE] Cartan and Eilemberg, Homological Algebra.

[T] Grothendieck, *Sur quelques points d'algèbre homologique.*

[EgaIII] Grothendieck, Eléments de Géométrie Algébrique, III.

[G] Godement, Topologie Algébrique et Théorie des faisceaux.

[B] Bourbaki, Algèbre, X: Algèbre Homologique.