TEOREMI SPETTRALI

Introduciamo un po' di terminologia:
(generalizzazione delle funzioni bilineari su $\mathbb{C}=\mathbb{R}(\mathbb{C})$)

V spazio vettoriale su $\mathbb{C}$

una **forma** è $\varphi : V\times V \longrightarrow \mathbb{C}$ di cui richiediamo che:
- ① sesquilineare in ciascuna delle sue variabili:
  $\varphi(\alpha v+\beta' w, u) = \alpha\,\varphi(v,u) + \beta'\varphi(w,u)$
- ② diciamo anche Hermitiana:
  $\varphi(v,\beta u+\gamma' w) = \bar\beta\,\varphi(v,u) + \bar\gamma'\varphi(v,w)$
- ③ semidefinita positiva (o anche):
  $\varphi(v,v)\geq 0 \quad \forall v\in V$
  (implicitamente richiediamo che $\varphi(v,v)\in\mathbb{R}$)

Nota: ③ $\Rightarrow$ ② (e anche ①)

da forme quadr. forme hermitiane e viceversa

Esempio tipico: $V=\mathbb{C}^n$, $\varphi(v,w) = v^t\bar w$ (prodotto hermitiano standard)

(1) definite, cioè altrimenti: $\varphi(v,v)=0 \Rightarrow v=0$
(equivalente: $\varphi(v,w)=0\ \forall w \Rightarrow v=0$)

nucleo di φ: $\ker(\varphi) = \{v : \varphi(v,w)=0 \ \forall w\in V\}$

(2) φ def. positiva se $\varphi(v,v)>0 \quad \forall v\neq 0$
(vedi $\varphi(v,v)\in\mathbb{R}$)
def. negativa se $\varphi(v,v)<0 \quad \forall v\neq 0$

(3) ortogonali tra $U,W\subseteq V$ se $\varphi(u,w)=0$
basi ortogonali: $\mathcal{B}=\{v_1,\dots,v_n\}$: $\varphi(v_i,v_j)=0$ se $i\neq j$
basi ortonormali: $\varphi(v_i,v_j)=\delta_{ij}$

(4) vettori isotropi, sottospazi isotropi

(5) classificazione con Teo Sylvester:

la forma φ determina φ (come forma sesquilineare, per cambiamento di variabile...)

(6) scelta basi nei V, φ diventa tutto un numero di indici:

$\Gamma_{\mathcal{B}}(\varphi) = \big(\varphi(v_i,v_j)\big)_{ij} = \big(\overline{\varphi(v_j,v_i)}\big)_{ij} = \overline{\Gamma_{\mathcal{B}}(\varphi)}^{\,t}$

sono indici hermitiani:

$A^k = \bar A \ , \text{ cioè } \bar A^t = A$

considerando basi diverse con P^tAP con $P\in GL_n(\mathbb{C})$

(7) isometrie $\varphi : V\longrightarrow V$ lineare
$\varphi(\varphi v,\varphi w) = \varphi(v,w) \quad \forall v,w$

indici di isometrie: φ di base $\mathcal{B}$
$H = \Gamma_{\mathcal{B}}(\varphi)$
$A = \alpha_{\mathcal{B}}(\varphi)$

φ isometria se $\bar A^t H A = H$

basta $\varphi(\varphi v,\varphi w) = \varphi(v,w) \quad \forall v,w$
$(\bar A\bar X)^t H(AY) = \bar X^t H Y \quad \forall X,Y$
$\bar Y^t \bar A^t H A Y$

$|\bar A^t H A| = |H|$
$|A|^2|H| = |H|$
$|A| = 1$

la isometria $\mathcal{H} = \Gamma_{\mathcal{B}}^{-1}(H)$ dimostriamo completamente $\mathbb{C}$:

se v_i, w basi ortn. nei cons X, Y hanno lor φ allora $\varphi(x,y) = \varphi\big(\sum x_i v_i, \sum y_j v_j\big) = \sum_{i,j}\bar x_i y_j \varphi(v_i,v_j) = \bar X^t H Y$

$\Rightarrow$ φ non dipende da $\mathcal{B}$ ortn. (Gram-Schmidt) $\Rightarrow$ $\det(H)\in\mathbb{R}^\times$

$\mathrm{Gr}(\varphi) \longrightarrow \mathbb{C}^\times$
$\det : \mathrm{Isom}(\varphi) \longrightarrow \mathbb{R}^\times$
$\mathrm{Isom}(\varphi,\mathrm{id}) \longrightarrow \mathrm{SU}_n$

se $H=\mathbb{1}$ ($\mathbb{C}$ ne ha anche un'altra) allora $\mathrm{Isom}(\varphi,\mathrm{id}) = \{A : \bar A^t A = \mathbb{1}\}$
e $\mathrm{Isom}(U_n(\mathbb{C}) \to \mathbb{C}^\times) = SU_n(\mathbb{C}) \subseteq U_n(\mathbb{C})$

e $U_n(\mathbb{C}) \supseteq S^1$.
$\mathrm{Isom}(\varphi,\mathrm{id})$
Funzionale di paragone(simplettico).

Appendice e ricordo veloce a forme (o bilineari o hermitiane complesse)

$\varphi : V\longrightarrow W$ lineare tra spazi (con forma hermitiana/$\mathbb{C}$ con forma quadr/$\mathbb{C}$)

(V,φ) (W,ψ) (nota: $\varphi : V\times V\to\mathbb{C}$, $\psi : W\times W\to\mathbb{C}$)
uno degenere;

allora definiscono $\Phi^\varphi : W\longrightarrow V$ (FUNZIONE ASSOCIATA AL DUALE DI φ, $\mathbb{C}$)
un punto unico:

in quel modo $w\in W$ definiscono $\Phi^\varphi(w)$
funzionale:
$\varphi(v,\Phi^\varphi(w)) := \psi(\varphi v, w) \quad \forall v\in V$

Nota: siccome φ è non degenere, questo identifica univocamente w dove $\Phi^\varphi(w)$

Osservazioni varie:

(5) $\varphi(w,w)\neq 0$ $\quad \varphi(v,v)\neq 0$
$\varphi(\varphi+\psi)^2 = \varphi^2 + \psi^2 + \varphi\psi + \psi\varphi$
$(\varphi^2)^2 = \varphi^4$
$(\lambda\varphi_{id})^2 = \lambda^2\varphi_{id}$

(4) $\varphi(v,w)=0$ $\quad \varphi(w,v)=0$
$\varphi(v,\varphi^2(w)) = \varphi(\varphi v,\varphi^3(w)) = \varphi(v,\varphi(w)) = 0$
$\varphi(w,\varphi^2(v)) = \varphi(\varphi w,\varphi^3(v)) = \varphi(w,\varphi(v)) = 0$

$\varphi(w,w)=0$

$\varphi_1 : V\to\mathbb{C}$, $\varphi_1(v) = \varphi(v,w)$
$\varphi_2 : V\to\mathbb{C}$, $\varphi_2(v) = \varphi(\varphi v,w) = \varphi(v,\varphi w)$
$\varphi_1(\varphi^2(v)) = \varphi(-,\varphi^2 w)$
$(\varphi_1\circ\varphi^2)(v) : V\to\mathbb{C}$, $(\varphi_1(\varphi^2(v)))(-) = \varphi(-,\varphi^2 w)$
$(\varphi^2\circ\varphi_1)(v) : V\to\mathbb{C}$, $(\varphi^2\circ\varphi_1(v))(-) = \varphi(\varphi v,\varphi w)$
$\varphi^2(\varphi_1(v))$
$\varphi(-,w)\circ\varphi = \varphi(\varphi(-),w)$

Vediamo matricole:

Se $\varphi : V\longrightarrow W$ di base V di base W

$\Gamma_{\mathcal{B}}^{\mathcal{V}}(\varphi) = G$
$\Gamma_{\mathcal{W}}^{\mathcal{W}}(\psi) = H$
$\alpha_{\mathcal{V},\mathcal{W}}(\Phi^\varphi) = A$

dimostrare:
che Φ^φ manda basi $\mathcal{W},\mathcal{V}$
$\Phi^\varphi : W\to V$
$\alpha_{\mathcal{W},\mathcal{V}}(\Phi^\varphi) = A^t$?

$\varphi(v,\Phi^\varphi(w)) = \psi(\varphi v,w) \quad \forall v,w$
$X^t G(\bar A^t Y) = (\bar A\bar X)^t H Y \quad \forall X,Y$
$GA^t = \bar A^t H$
$A^t = G^{-1}\bar A^t H$

se $H=\mathbb{1}=G$ (basi ortonormali) allora $A^t = \bar A^t$

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Ricordo, $(V,\varphi)\xrightarrow{T}(W,\psi)$ lineare,
φ,ψ forme sesqui/quad. non degenere

allora definiamo $\Phi^\varphi : W\to V$
$\forall v\in V : \Phi^\varphi(w)$ unico vettore $\in V$ t.c.
$\varphi(v,\Phi^\varphi(w)) = \psi(\varphi v, w) \quad \forall v\in V$
(definizione vettore associato $\Phi^\varphi(w)$ grazie a non degenere)

In matrici moltiplicando $\Gamma_{\mathcal{B}}(\varphi) = G$, $\Gamma_{\mathcal{W}}(\psi) = H$ $\Rightarrow$ $\alpha_{\mathcal{W},\mathcal{V}}(\Phi^\varphi) = \bar G^t\bar A^t H$

$\bar Y^t G(\bar A^t X) = (\bar A\bar Y)^t H X$
$\bar Y^t \bar G^t A^t X = \bar Y^t \bar A^t H X \quad \forall X,Y$
$GA^t = \bar A^t H$

se $\varphi=\psi$ $G=H$: $A^t = \bar A^t$

Per particolari quando $V=W$:

① ISOMETRIE : φ isometria se $\Phi^\varphi = \Phi^\varphi$
se $\Phi^\varphi\circ\Phi^\varphi = \Phi^{\varphi^2} = \mathrm{id}$

Anche: isometria se $\varphi(\Phi^\varphi(v),\Phi^\varphi(w)) = \varphi(v,w) \quad \forall v,w$
se $\Phi^\varphi\circ\Phi^\varphi = \mathrm{id}$ (usa matrice inverse ?)

② AUTOAGGIUNTE (o φ-HERMITIANE) : $\varphi : V\to V$ t.c. $\Phi^\varphi = \varphi$
se $\varphi(\Phi^\varphi(v),w) = \varphi(v,\Phi^\varphi(w))$ (usa matrici coniugate):
$A^t = A$ se $\bar A^t G = GA$

③ NORMALI : $\varphi : V\to V$ t.c. $\Phi^\varphi\circ\Phi^\varphi = \Phi^\varphi\circ\Phi^\varphi$
(φ e Φ^φ commutano tra loro)

nota: $\varphi(\varphi,\Phi^\varphi(w)) = \varphi(\varphi,\Phi^\varphi(\Phi^\varphi(w)))$ $\quad \forall v,w$
$\varphi(\Phi^\varphi v, w)$
$\varphi(\Phi^\varphi(\Phi^\varphi(v)),\Phi^\varphi(w))$
$\varphi(\Phi^\varphi(v),\Phi^\varphi(w)) \quad \forall v,w$

OSSERVAZIONE Coniugato o ortogonale $\Rightarrow$ normale
se

Se esempio antisimmetrica (Φ t.c. $\Phi^2=-\mathrm{id}$)
Senza autovalori.

TEOREMI SPETTRALI in $\mathbb{C}$ e $\mathbb{R}$:

caso reale: (V,φ) spazio hermitiano, $\varphi\geq 0$

$\Phi : V\to V$ **simmetrica rispetto** a φ,

$\Rightarrow$ allora Φ è ortogonalmente diagonalizzabile cioè $\exists$ una base di V ortonormale per φ di autovettori per Φ.

nuova: (V,φ) $\mathbb{R}$-simmetrica, $\varphi\geq 0$

$\Phi : V\longrightarrow V$ φ-ortogonale

allora Φ è φ-ortogonalmente diagonalizzabile cioè $\exists$ base di V φ-ortonormale di autovettori per Φ.

dimostrazione:

(1) $\ker\Phi = \ker\Phi^2$
dunque
$V_0(\Phi) = V_0(\Phi^2)$
$\ker(\Phi-\lambda\mathrm{id}) \quad \ker(\Phi^2-\bar\lambda\lambda\mathrm{id})$
$\ker((\Phi-\lambda\mathrm{id})(\Phi-\bar\lambda\mathrm{id}))$

$\varphi\geq 0$ $\quad \varphi(\Phi v,\Phi v)=0$
sia $\varphi(v,\Phi\Phi v)=0$
$\varphi(v,\Phi^2 v)=0$
$\varphi(\Phi^2 v,\Phi^2 v)=0$
$\varphi(\Phi v,\Phi^3 v)=0$
$\varphi(\Phi^3 v,\Phi^3 v)=0$
$\Phi^3(v)=0$

(2) se $\lambda\neq\mu$ allora $V_\lambda(\Phi)\perp_\varphi V_\mu(\Phi)$
se v,w autovettori di certi autovalori λ,μ (*)
allora $v\perp_\varphi w$

$\Phi v = \lambda v$: $\varphi(\Phi v,w) = \varphi(v,\Phi^* w)$
$\Phi w = \mu w$ $\quad \varphi(\lambda v,w) \quad \varphi(v,\bar\mu w)$
$\bar\lambda\,\varphi(v,w) \quad \bar\mu\,\varphi(v,w)$
$(\bar\lambda-\bar\mu)\,\varphi(v,w) = 0$
$\neq 0 \Rightarrow \varphi(v,w)=0$
$v\perp_\varphi w$

(3) se $W\leq V$, φ $\Phi(W)\leq W$ allora $\Phi^*(W^\perp)\leq W^\perp$
(φ $\Phi^*(W^\perp)\leq W^\perp$ allora $\Phi(W^\perp)\leq W^\perp$)

se $\Phi(w)\in W$ (cioè $\forall w\in W$ c'è un $\Phi w\in W$)
vediamo cosa succede in $w\in W^\perp$ $\quad (\Phi(v,w)=0 \ \forall v\in W)$
dimostr. $\Phi w\in W^\perp$?
$\varphi(v,\Phi w) = \varphi(\Phi v,w) = 0 \quad \forall v\in W$
$\Phi w\in W^\perp$

(4) conclusione:
consideriamo una matrice HERMITIANA (in $\mathbb{R}$ è simmetrica) per Φ
(possiamo supporre che diano 2×2 autoquadri: se hanno autovalori $\neq$ siamo a pt ②,
se hanno stesso autovalore siano una base ortonormale degli autospazi)
e ricordiamo che ci sono due stessi spazi:
ortonormali presi qualsiasi per $w\in V$,
che rendono diagonale la matrice
e diagonale in $\mathbb{R}$ e per Φ^t,
dunque
$W^\perp$ è stabile per Φ
se $W\neq V$ allora, $W^\perp\neq 0$ e $W^\perp$ è stabile
ordi autovettori / autospazi (iterato da finite facendo ortogonali).

OSSERVAZIONE : i teoremi sono dei SSE ?

OSSERVAZIONE : i teoremi spettrali valgono in forma definita (>0 o <0) in una unica semidefinita (es. matrice di forma $0,1,0,1$) ovviamente forma indefinita non vanno.

Esempi particolari:

(a) Φ è $\mathbb{R}$-normale $\Leftrightarrow$ Φ normale con spettro $\subseteq S^1$
(Caso unitario)

(b) Φ è φ-autoaggiunta $\Leftrightarrow$ Φ normale con spettro $\subseteq\mathbb{R}$

(c) Φ è φ-antiaggiunta $\Leftrightarrow$ Φ normale con spettro $\subseteq i\mathbb{R}$
(Caso hamiltoniano)

(d) Φ è $\mathbb{R}$-normale e $\mathbb{R}$-ortogonale
$\Leftrightarrow$ Φ normale e spettro $\subseteq\{\pm 1\}$
$\Leftrightarrow$ simmetrica rispetto a $\mathbb{R}$.

Osservazioni uniformanti:

(1) Il teorema spettrale ci dice veramente cosa: un risultato di diagonalizzabilità di endomorfismi per coniugazione di basi (basi/matrici di cambiamento) in una di coordinate tra.

Vediamo poi vedo vedere classificazione esattamente quelle questi dei:

$S=S^t>0$ $\quad A=\bar A^t$
$\Rightarrow \exists H\in GL_n : \begin{cases} H^tSH=\mathbb{1} \\ H^*AH \text{ diagonale} \end{cases}$ (stesso H per entrambi)
Stesso autospazi

$\exists K\in GL_n : \begin{cases} K^tIK=\mathbb{1} \\ K^*AK=\Lambda \end{cases}$
Cioè $K\in O_n$, $K^t=K^{-1}$

Così $\begin{cases} K^t(H^tSH)K = \mathbb{1} \\ K^t(H^tAH)K = \text{diagonale} \end{cases}$

Così $\begin{cases} (HK)^tS(HK) = \mathbb{1} \\ (HK)^tA(HK) = \text{diagonale} \end{cases}$

(diagonalizzare una matrice euclidea vuol dire cercare una base ortonormale per cui la matrice diventa diagonale, ovvero cercare una matrice ortogonale P tale che P^tAP sia diagonale)

(2) Unità un gran aiuto sup (Gram) : (V,φ)

Data una forma hermitiana/simmetrica φ su V (un particolare se $\dim V>0$)
c'è una bilineare non:

$\{\text{forme hermitiane rispetto a } \varphi\} \longleftrightarrow \{\text{forme bilineari su } V\}$
$\Phi : V\to V$ $\quad \varphi^2 = \varphi$

$\Phi(\varphi\Phi) \longmapsto \psi_\Phi : V\times V\to\mathbb{R}$
$\psi_\Phi(v,w) = \varphi(\Phi v,w) = \varphi(v,\Phi w)$
$= \overline{\varphi(v,\Phi w)}$
$\psi_\Phi(w,v) = \varphi(\Phi w,v)$

$\Phi : V\to V$ $\longleftarrow$ ψ
$\Phi_\psi(v) = \psi(\varphi(v,-))$ $\quad \forall v\in V$ (condizioni viene implicite su $\Phi_\psi(v)$)

Le invarianti dei modelli: se $\mathcal{B}$ base V, $G=\Gamma_{\mathcal{B}}^{\mathcal{B}}(\varphi)$ invertibile
$\Phi : V\to V$ antisimmetrica per φ $\quad A=\alpha_{\mathcal{B}}(\Phi)$
(antisimmetrica : $\bar A^t G = A$, $\bar A^t G = GA$)
allora in forma $\mathcal{B}$ le matrice $\Gamma_{\mathcal{B}}^{\mathcal{B}}(\psi_\Phi)$:

$\psi_\Phi(v,w) = \varphi(\Phi v,w) = \varphi(\Phi Y,X)$
$\bar X^t(\Gamma_{\mathcal{B}}^{\mathcal{B}}(\psi_\Phi))Y \quad (\bar A\bar X)^t GY \quad \bar X^t\bar G(AY)$

$\boxed{\Gamma_{\mathcal{B}}^{\mathcal{B}}(\psi_\Phi) = \bar A^t G = GA}$

$A \longmapsto GA = \bar A\bar G$
$\bar G^t(GA) \longleftarrow GA$
$\bar G^t \longleftarrow \bar G$ (simmetrica)
e passato ai coniugati,
legame di isomorfismo di strutture

Caso particolare $\varphi>0$, $G=\mathbb{1}$
allora $A \longmapsto \bar A = A$
↑ base ortonormale per φ ↑ base ortonormale per ψ_Φ
(Autospazi per ψ_Φ)

Problema (Teorico): (V,φ) ψ forme non degenere

$\Phi \longmapsto \psi_\Phi(v,w) := \varphi(\Phi v,w) = \psi(v,\Phi w)$
$\psi_1(v,w) := \varphi(\Phi^2 v,w) = \psi(v,\Phi^2 w)$
$\psi_2(v,w) := \varphi(\Phi^3 v,w) = \psi(\Phi v,\Phi w)$
$\psi_3(v,w) := \varphi(\Phi^4 v,w) = \psi(\Phi^2 v,\Phi^2 w)$
$\psi_4(v,w) := \varphi(\Phi^5 v,w) = \psi(\Phi^3 v,\Phi^3 w)$
$\psi_5(v,w) := \varphi(\Phi^6 v,w) = \psi(\Phi^4 v,\Phi^4 w)$
$\psi_6(v,w) := \varphi(\Phi^7 v,w) = \psi(\Phi^5 v,\Phi^5 w)$
$\psi_7(v,w) := \varphi(\Phi^8 v,w) = \psi(\Phi^6 v,\Phi^6 w)$
$\psi_8(v,w) := \varphi(\Phi^9 v,w) = \psi(\Phi^7 v,\Phi^7 w)$
$\psi_9(v,w) := \varphi(\Phi^{10} v,w) = \psi(\Phi^8 v,\Phi^8 w)$
$\psi_{10}(v,w) := \varphi(\Phi^{11} v,w) = \psi(\Phi^9 v,\Phi^9 w)$
$\psi_{11}(v,w) := \varphi(\Phi^{12} v,w) = \psi(\Phi^{10} v,\Phi^{10} w)$
$\psi_{12}(v,w) := \varphi(\Phi^{13} v,w) = \psi(\Phi^{11} v,\Phi^{11} w)$
$\psi_{13}(v,w) := \varphi(\Phi^{14} v,w) = \psi(\Phi^{12} v,\Phi^{12} w)$
$\psi_{14}(v,w) := \varphi(\Phi^{15} v,w) = \psi(\Phi^{13} v,\Phi^{13} w)$
$\psi_{15}(v,w) := \varphi(\Phi^{16} v,w) = \psi(\Phi^{14} v,\Phi^{14} w)$
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$\psi_{69}(v,w) := \varphi(\Phi^{70} v,w) = \psi(\Phi^{68} v,\Phi^{68} w)$
$\psi_{70}(v,w) := \varphi(\Phi^{71} v,w) = \psi(\Phi^{69} v,\Phi^{69} w)$
$\psi_{71}(v,w) := \varphi(\Phi^{72} v,w) = \psi(\Phi^{70} v,\Phi^{70} w)$
$\psi_{72}(v,w) := \varphi(\Phi^{73} v,w) = \psi(\Phi^{71} v,\Phi^{71} w)$
$\psi_{73}(v,w) := \varphi(\Phi^{74} v,w) = \psi(\Phi^{72} v,\Phi^{72} w)$
$\psi_{74}(v,w) := \varphi(\Phi^{75} v,w) = \psi(\Phi^{73} v,\Phi^{73} w)$
$\psi_{75}(v,w) := \varphi(\Phi^{76} v,w) = \psi(\Phi^{74} v,\Phi^{74} w)$
$\psi_{76}(v,w) := \varphi(\Phi^{77} v,w) = \psi(\Phi^{75} v,\Phi^{75} w)$
$\psi_{77}(v,w) := \varphi(\Phi^{78} v,w) = \psi(\Phi^{76} v,\Phi^{76} w)$
$\psi_{78}(v,w) := \varphi(\Phi^{79} v,w) = \psi(\Phi^{77} v,\Phi^{77} w)$
$\psi_{79}(v,w) := \varphi(\Phi^{80} v,w) = \psi(\Phi^{78} v,\Phi^{78} w)$
$\psi_{80}(v,w) := \varphi(\Phi^{81} v,w) = \psi(\Phi^{79} v,\Phi^{79} w)$
$\psi_{81}(v,w) := \varphi(\Phi^{82} v,w) = \psi(\Phi^{80} v,\Phi^{80} w)$
$\psi_{82}(v,w) := \varphi(\Phi^{83} v,w) = \psi(\Phi^{81} v,\Phi^{81} w)$
$\psi_{83}(v,w) := \varphi(\Phi^{84} v,w) = \psi(\Phi^{82} v,\Phi^{82} w)$
$\psi_{84}(v,w) := \varphi(\Phi^{85} v,w) = \psi(\Phi^{83} v,\Phi^{83} w)$
$\psi_{85}(v,w) := \varphi(\Phi^{86} v,w) = \psi(\Phi^{84} v,\Phi^{84} w)$
$\psi_{86}(v,w) := \varphi(\Phi^{87} v,w) = \psi(\Phi^{85} v,\Phi^{85} w)$
$\psi_{87}(v,w) := \varphi(\Phi^{88} v,w) = \psi(\Phi^{86} v,\Phi^{86} w)$
$\psi_{88}(v,w) := \varphi(\Phi^{89} v,w) = \psi(\Phi^{87} v,\Phi^{87} w)$
$\psi_{89}(v,w) := \varphi(\Phi^{90} v,w) = \psi(\Phi^{88} v,\Phi^{88} w)$
$\psi_{90}(v,w) := \varphi(\Phi^{91} v,w) = \psi(\Phi^{89} v,\Phi^{89} w)$
$\psi_{91}(v,w) := \varphi(\Phi^{92} v,w) = \psi(\Phi^{90} v,\Phi^{90} w)$
$\psi_{92}(v,w) := \varphi(\Phi^{93} v,w) = \psi(\Phi^{91} v,\Phi^{91} w)$
$\psi_{93}(v,w) := \varphi(\Phi^{94} v,w) = \psi(\Phi^{92} v,\Phi^{92} w)$
$\psi_{94}(v,w) := \varphi(\Phi^{95} v,w) = \psi(\Phi^{93} v,\Phi^{93} w)$
$\psi_{95}(v,w) := \varphi(\Phi^{96} v,w) = \psi(\Phi^{94} v,\Phi^{94} w)$
$\psi_{96}(v,w) := \varphi(\Phi^{97} v,w) = \psi(\Phi^{95} v,\Phi^{95} w)$
$\psi_{97}(v,w) := \varphi(\Phi^{98} v,w) = \psi(\Phi^{96} v,\Phi^{96} w)$
$\psi_{98}(v,w) := \varphi(\Phi^{99} v,w) = \psi(\Phi^{97} v,\Phi^{97} w)$
$\psi_{99}(v,w) := \varphi(\Phi^{100} v,w) = \psi(\Phi^{98} v,\Phi^{98} w)$
$\psi_{100}(v,w) := \varphi(\Phi^{101} v,w) = \psi(\Phi^{99} v,\Phi^{99} w)$
$\psi_{101}(v,w) := \varphi(\Phi^{102} v,w) = \psi(\Phi^{100} v,\Phi^{100} w)$
$\psi_{102}(v,w) := \varphi(\Phi^{103} v,w) = \psi(\Phi^{101} v,\Phi^{101} w)$
$\psi_{103}(v,w) := \varphi(\Phi^{104} v,w) = \psi(\Phi^{102} v,\Phi^{102} w)$
$\psi_{104}(v,w) := \varphi(\Phi^{105} v,w) = \psi(\Phi^{103} v,\Phi^{103} w)$
$\psi_{105}(v,w) := \varphi(\Phi^{106} v,w) = \psi(\Phi^{104} v,\Phi^{104} w)$
$\psi_{106}(v,w) := \varphi(\Phi^{107} v,w) = \psi(\Phi^{105} v,\Phi^{105} w)$
$\psi_{107}(v,w) := \varphi(\Phi^{108} v,w) = \psi(\Phi^{106} v,\Phi^{106} w)$
$\psi_{108}(v,w) := \varphi(\Phi^{109} v,w) = \psi(\Phi^{107} v,\Phi^{107} w)$
$\psi_{109}(v,w) := \varphi(\Phi^{110} v,w) = \psi(\Phi^{108} v,\Phi^{108} w)$
$\psi_{110}(v,w) := \varphi(\Phi^{111} v,w) = \psi(\Phi^{109} v,\Phi^{109} w)$
$\psi_{111}(v,w) := \varphi(\Phi^{112} v,w) = \psi(\Phi^{110} v,\Phi^{110} w)$
$\psi_{112}(v,w) := \varphi(\Phi^{113} v,w) = \psi(\Phi^{111} v,\Phi^{111} w)$
$\psi_{113}(v,w) := \varphi(\Phi^{114} v,w) = \psi(\Phi^{112} v,\Phi^{112} w)$
$\psi_{114}(v,w) := \varphi(\Phi^{115} v,w) = \psi(\Phi^{113} v,\Phi^{113} w)$
$\psi_{115}(v,w) := \varphi(\Phi^{116} v,w) = \psi(\Phi^{114} v,\Phi^{114} w)$
$\psi_{116}(v,w) := \varphi(\Phi^{117} v,w) = \psi(\Phi^{115} v,\Phi^{115} w)$
$\psi_{117}(v,w) := \varphi(\Phi^{118} v,w) = \psi(\Phi^{116} v,\Phi^{116} w)$
$\psi_{118}(v,w) := \varphi(\Phi^{119} v,w) = \psi(\Phi^{117} v,\Phi^{117} w)$
$\psi_{119}(v,w) := \varphi(\Phi^{120} v,w) = \psi(\Phi^{118} v,\Phi^{118} w)$
$\psi_{120}(v,w) := \varphi(\Phi^{121} v,w) = \psi(\Phi^{119} v,\Phi^{119} w)$
$\psi_{121}(v,w) := \varphi(\Phi^{122} v,w) = \psi(\Phi^{120} v,\Phi^{120} w)$
$\psi_{122}(v,w) := \varphi(\Phi^{123} v,w) = \psi(\Phi^{121} v,\Phi^{121} w)$
$\psi_{123}(v,w) := \varphi(\Phi^{124} v,w) = \psi(\Phi^{122} v,\Phi^{122} w)$
$\psi_{124}(v,w) := \varphi(\Phi^{125} v,w) = \psi(\Phi^{123} v,\Phi^{123} w)$
$\psi_{125}(v,w) := \varphi(\Phi^{126} v,w) = \psi(\Phi^{124} v,\Phi^{124} w)$
$\psi_{126}(v,w) := \varphi(\Phi^{127} v,w) = \psi(\Phi^{125} v,\Phi^{125} w)$
$\psi_{127}(v,w) := \varphi(\Phi^{128} v,w) = \psi(\Phi^{126} v,\Phi^{126} w)$
$\psi_{128}(v,w) := \varphi(\Phi^{129} v,w) = \psi(\Phi^{127} v,\Phi^{127} w)$
$\psi_{129}(v,w) := \varphi(\Phi^{130} v,w) = \psi(\Phi^{128} v,\Phi^{128} w)$
$\psi_{130}(v,w) := \varphi(\Phi^{131} v,w) = \psi(\