Richiami di analisi 1 e 2:

(1,1): $f: I \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}$, $f'(x) = \frac{df}{dx}(x) = \partial_x f(x) = f_x(x)$ — derivata di f in x

$df: I \xrightarrow{\subseteq \mathbb{R}} \mathcal{L}(\mathbb{R},\mathbb{R})$, $x \longmapsto df(x): \mathbb{R} \to \mathbb{R}$ lineare (mappa lineare), $v \mapsto df(x)\cdot v := f'(x)v$

$x \longmapsto$ migliore approssimazione lineare di f intorno a x.

$df(x) = f'(x)\,dx$ — $dx(p) = id$ per ogni p (è poco chiaro!)

differenziale di f in x: migliore approssimazione lineare di f intorno a x.

(n,m): $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$

derivate direzionali in direzione $v \in \mathbb{R}^n$

$\partial_v f(x) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}$

$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} f(x)$ $\quad \frac{\partial}{\partial x_i} = \partial_{e_i}$

$\frac{\partial f}{\partial x_i} = \partial_{e_i} f = \partial_i f = f_{x_i}$

$df: U \subseteq \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $x \mapsto df(x): \mathbb{R}^n \to \mathbb{R}^m$ lineare continua, $v \mapsto df(x)\cdot v = \partial_v f(x) = \sum_i v_i \partial_i f(x)$

$df(x) = \sum_i \partial_i f(x)\, dx_i$ $\quad (dx_i)(x) = \pi_i$, $x \mapsto x_i$

$df(x)$ in base canonica di $\mathbb{R}^n, \mathbb{R}^m$ ha matrice Jacobiana

$\alpha_E(df(x)) = \left(\frac{\partial}{\partial x_i} f_j(x)\right)_{j,i}$

composizione:

$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^k$, $g\circ f$

$d(g\circ f)(x) = dg(fx)\circ df(x)$

$\mathbb{R}^n \to \mathbb{R}^k$, $df(x): \mathbb{R}^n \to \mathbb{R}^m$, $dg(fx): \mathbb{R}^m \to \mathbb{R}^k$

prodotto di matrici jacobiane

$d(g\circ f)(x) = dg(fx)\cdot df(x)$

$\big(\!\big(d(g\circ f) = dg \circ f) \cdot df \big)\!\big)$

(n,1)

(1,m)

Derivate superiori:

(1,1): $f''(x) = \frac{d^2}{dx^2} f(x) = \partial_{xx}^2 f(x) = f_{xx}(x)$ $\Rightarrow$ Taylor $f(x+\varepsilon) = f(x) + f'(x)\varepsilon + f''(x)\frac{\varepsilon^2}{2!} + \dots$

$f^{(2)}(x)$

(n,1): $\partial_{xy} f(x) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x) = f_{xy}(x) = f_{yx}(x)$

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$\Rightarrow$ Taylor $f\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = f(x) + \sum_i^n \partial_i f(x)\varepsilon_i + \frac{1}{2!}\sum \partial_{ij} f(x)\varepsilon_i\varepsilon_j + \dots$

$(Jf(x))\varepsilon$ — matrice Jacobiana $\quad \varepsilon^t (H_f(x))\varepsilon$ — matrice hessiana (forma quadratica).

$\nabla f(x)\cdot\varepsilon$

Proiezione di f: $(\partial_1 f, \dots \partial_n f)(x): \mathbb{R}^n \to \mathbb{R}$ lineare, lineare vera!

$u\cdot(v\times w) = |u\ v\ w|$ — se vettori $u,v,w \in \mathbb{R}^3$ e lunghezze

$(u_1\times u_2)\cdot(v_1\times v_2) = \begin{vmatrix} u_1\cdot v_1 & u_1\cdot v_2 \\ u_2\cdot v_1 & u_2\cdot v_2 \end{vmatrix}$

$\|v\times w\|^2 = \begin{vmatrix} \|v\|^2 & v\cdot w \\ w\cdot v & \|w\|^2 \end{vmatrix} = \|v\|^2\|w\|^2 - (v\cdot w)^2$ — (Lagrange)

$u\times(v\times w) = (u\cdot w)v - (u\cdot v)w$ $\quad v\times(v\times w) = (v\cdot w)v - (v\cdot v)w$ $\quad \perp v$

$w\times(v\times w) = (w\cdot w)v - (w\cdot v)w$ $\quad \perp w$

$(u_1\times u_2)\times(v_1\times v_2) = ?\ v_1 + ?\ v_2 = ?\ u_1 + ?\ u_2$

Usiamo funzioni definite su $I \subseteq \mathbb{R}$

$v: I \to \mathbb{R}^3$ "punto mobile" = curva "vettore mobile" = campo vettoriale

$A: I \to GL_n(\mathbb{R})$ "riferimenti mobili"

$U: I \to SO_n(\mathbb{R})$ "riferimento mobile ortonormale"

det sempre $\neq 0$ (perché det $U(t)$ è sempre >0 o sempre <0)

(perché abbiamo una curva in $GL_n(\mathbb{R})$, det $\neq 0$ $\forall t$)

Regole del calcolo differenziale:

derivata: $(\alpha(t))' = \alpha'(t) + \alpha(t)$ $\quad \frac{d}{dt}(\alpha(t)v(t))$

$(v\cdot w)' = v'\cdot w + v\cdot w'$ $\quad (v\times w)' = v'\times w + v\times w'$

$(\|v\|^2)' = 2\,v\cdot v'$

$v\cdot v'$

$\|v\|$ costante $\Leftrightarrow v\cdot v' = 0$ $\Leftrightarrow v \perp v'$

modulo: $(\|v\|^2)' = 2v\cdot v'$ $\quad = 2\|v\|\cdot(\|v\|)'$ $\Rightarrow (\|v\|)' = \frac{v\cdot v'}{\|v\|}$

(attenti $\|v\| = (v\cdot v)^{\frac{1}{2}} \Rightarrow \|v\|' = \big((v\cdot v)^{\frac{1}{2}}\big)' = \frac{1}{2}(v\cdot v)^{\frac{1}{2}-1}(v\cdot v)' = \dots$)

$\left(\frac{1}{\|v\|}\right)' = -\frac{\|v\|'}{\|v\|^2} = -\frac{v\cdot v'/\|v\|}{\|v\|^2} = -\frac{v\cdot v'}{\|v\|^3}$

versore: $\left(\frac{v}{\|v\|}\right)' = \frac{v'}{\|v\|} + v\left(\frac{1}{\|v\|}\right)' = \frac{v'}{\|v\|} - \frac{v\cdot v'}{\|v\|^3}v = \frac{\|v\|^2v' - (v\cdot v')v}{\|v\|^3}$ $\quad$ su $\mathbb{R}^3$ ci serve $\times$

$= \frac{1}{\|v\|}\left(v' - \frac{v\cdot v'}{\|v\|^2}v\right) = \frac{v\times(v'\times v)}{\|v\|^3}$ $\quad \left(\frac{v}{\|v\|}\right)'$

proiezione ortogonale, facciamo G-S.

versore della velocità modulo 1, costante

$\left(\frac{v\times w}{\|v\times w\|}\right)' = \dots$

matrici: $(AB)' = A'B + AB'$

$(A^2)' = \sum_{ij=n} A^iA'A^j$ (ci serve A mezzo commuta con A')

$(A^{-1})' = -A^{-1}A'A^{-1}$

perché $A^{-1}A = \mathbb{1}$

$(A^{-1}A)' = 0$

$(A^{-1})'A + A^{-1}A' = 0$

$(A^{-1})'A = -A^{-1}A'$

$(\mathrm{tr}(A))' = \mathrm{tr}(A')$

$(\det(A))' = \sum_{i=n}^n \det(A_i')$ dove A_i' = matrice A in cui abbiamo derivato la i-esima colonna

$= \det(A)\,\mathrm{tr}(A^{-1}A')$ $\quad = \mathrm{tr}(A^cA')$

formula di Jacobi: problema: INTEGRARE

$A^cA = \det(A)\,\mathbb{1}$

$\mathrm{tr}(A^cA) = n\det(A)$

$\mathrm{tr}(A^cA)' = n\det(A)'$

$\mathrm{tr}((A^cA)') = \mathrm{tr}\left((A^c)'A + A^cA'\right)$

$\mathrm{tr}\left((A^c)'A + AA^cA'\right)$

$\mathrm{tr}\left((A^c)'A\right) + \mathrm{tr}(AA^cA')$

?

se $U \in O_n(\mathbb{R})$ (cioè $U^tU = \mathbb{1}$) allora: $U' = UK$ (K matrice antisimmetrica: la colonne sono basi di $\mathbb{R}^n$)

con K antisimmetrica

perché da $U^tU = \mathbb{1}$ otteniamo $(U^tU)' = 0$

$(U^t)'U + U^tU' = 0$

$(U')^tU + U^t(UK) = 0$

$(UK)^tU + U^t(UK) = 0$

$K^t + K = 0$ //

Composizioni di derivazioni:

cambiamento di variabile:

$\gamma(t) \in \mathbb{R}^n$, $t \mapsto \gamma(t)$, $I \xrightarrow{\subseteq \mathbb{R}} \mathbb{R}^n$

funzione $J \xrightarrow{\varphi} I \xrightarrow{\gamma} \mathbb{R}^n$, $s \mapsto t(s)$, differenziabile e ad inversa di $\mathbb{R}$

$\gamma(t(s)) =: \tilde{\gamma}(s)$

vogliamo esprimere la derivata di $\tilde{\gamma}$ (rispetto ad s) con derivate di γ (rispetto a t).

(1) $\tilde{\gamma}'(s) = \frac{d}{ds}\tilde{\gamma}(s) = \frac{d}{ds}\gamma(t(s)) = \frac{d}{dt}\gamma(t(s))\cdot\frac{dt}{ds}(s)$

$\tilde{\gamma}' = \gamma'\cdot\frac{dt}{ds}$

(2) $\tilde{\gamma}(s)'' = \frac{d^2}{ds^2}\tilde{\gamma}(s) = \frac{d}{ds}\gamma(t(s))' = \frac{d}{ds}\left(\gamma'(t(s))\cdot\frac{dt}{ds}(s)\right)$

$= \frac{d}{ds}\left(\gamma'(t(s))\cdot\frac{dt}{ds}(s)\right) + \gamma'(t(s))\cdot\frac{d}{ds}\left(\frac{dt}{ds}(s)\right)$

$= \frac{d}{dt}\left(\gamma'(t(s))\cdot\frac{dt}{ds}(s)\right)\frac{dt}{ds}(s) + \gamma'(t(s))\cdot\frac{d^2t}{ds^2}(s)$

$= \gamma''(t(s))\left(\frac{dt}{ds}(s)\right)^2 + \gamma'(t(s))\cdot\frac{d^2t}{ds^2}(s)$

$\tilde{\gamma}'' = \gamma''\left(\frac{dt}{ds}\right)^2 + \gamma'\frac{d^2t}{ds^2}$

(3) $\tilde{\gamma}(s)''' = \frac{d^3}{ds^3}\tilde{\gamma}(s)$

$\tilde{\gamma}''' = \frac{d}{ds}\left(\tilde{\gamma}''\right) = \frac{d}{ds}\left(\gamma''\left(\frac{dt}{ds}\right)^2 + \gamma'\left(\frac{d^2t}{ds^2}\right)\right)$

$= \frac{d}{ds}\left(\gamma^k\right)\cdot\left(\frac{dt}{ds}\right)^2 + \gamma''\frac{d}{ds}\left(\frac{dt}{ds}\right)^2 + \frac{d}{ds}\left(\gamma'\right)\cdot\frac{d^2t}{ds^2} + \gamma'\frac{d}{ds}\left(\frac{d^2t}{ds^2}\right) =$

$= \gamma^k\frac{dt}{ds}\cdot\left(\frac{dt}{ds}\right)^2 + \gamma''\cdot 2\left(\frac{dt}{ds}\right)\frac{d}{dt}\left(\frac{dt}{ds}\right) + \gamma''\cdot\frac{dt}{ds}\cdot\frac{d^2t}{ds^2} + \gamma'\frac{d^3t}{ds^3}$ $\quad \left(\frac{d^2t}{ds^2}\right)$

$\tilde{\gamma}''' = \gamma'''\left(\frac{dt}{ds}\right)^3 + 3\gamma''\frac{dt}{ds}\frac{d^2t}{ds^2} + \gamma'\frac{d^3t}{ds^3}$

(4) problema: $\tilde{\gamma}^{(4)} = \frac{d^4}{ds^4}\tilde{\gamma}(s) = ?$

(n) $\tilde{\gamma}^{(n)} = \gamma^{(n)}\left(\frac{dt}{ds}\right)^n + \dots\dots\dots\dots\dots + \gamma'\frac{d^nt}{ds^n}$

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