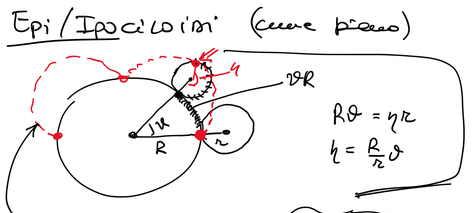
G2B-256-08

giovedì 8 gennaio 2026 15:23

Epi/Ipocicloidi (curve piane)



$R\vartheta = \eta r$

$\eta = \frac{R}{r}\vartheta$

$$\gamma(\vartheta) = \underbrace{(R+r)\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix}}_{\text{centro del cerchio piccolo ruotato}} + r\overbrace{\left(\begin{pmatrix}-\cos\vartheta\\ -\sin\vartheta\end{pmatrix}\text{ ruotato di }\eta\right)}$$

$$= (R+r)\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} + r\begin{pmatrix}\cos(\pi+\vartheta+\eta)\\ \sin(\pi+\vartheta+\eta)\end{pmatrix}$$

$$= (R+r)\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} + r\begin{pmatrix}\cos(\pi+\vartheta+\frac{R}{r}\vartheta)\\ \sin(\pi+\vartheta+\frac{R}{r}\vartheta)\end{pmatrix}$$

$$= r\left[\frac{R+r}{r}\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} - \begin{pmatrix}\cos(1+\frac{R}{r})\vartheta\\ \sin(1+\frac{R}{r})\vartheta\end{pmatrix}\right]$$

$$= r\left(\alpha\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} - \begin{pmatrix}\cos(\alpha\vartheta)\\ \sin(\alpha\vartheta)\end{pmatrix}\right)$$

$$\gamma(\vartheta) = r\begin{pmatrix}\alpha\cos\vartheta - \cos(\alpha\vartheta)\\ \alpha\sin\vartheta - \sin(\alpha\vartheta)\end{pmatrix} \qquad \alpha = \frac{R+r}{r} = \frac{R}{r}+1$$

$$\gamma(\vartheta) = r\begin{pmatrix}\alpha\cos\vartheta - \cos(\alpha\vartheta)\\ \alpha\sin\vartheta - \sin(\alpha\vartheta)\end{pmatrix}$$

$$\gamma' = r\begin{pmatrix}-\alpha\sin\vartheta + \alpha\sin(\alpha\vartheta)\\ \alpha\cos\vartheta - \alpha\cos(\alpha\vartheta)\end{pmatrix} = \alpha r\begin{pmatrix}-\sin\vartheta + \sin(\alpha\vartheta)\\ \cos\vartheta - \cos(\alpha\vartheta)\end{pmatrix}$$

$$\gamma'' = \alpha r\begin{pmatrix}-\cos\vartheta + \alpha\cos(\alpha\vartheta)\\ -\sin\vartheta + \alpha\sin(\alpha\vartheta)\end{pmatrix}$$

$$\|\gamma'\| = \alpha r\sqrt{2 - 2\sin\vartheta\sin(\alpha\vartheta) - 2\cos\vartheta\cos(\alpha\vartheta)}$$

$$= \alpha r\sqrt{2}\sqrt{1 - (\cos(\vartheta)\cos(\alpha\vartheta) + \sin(\vartheta)\sin(\alpha\vartheta))}$$

$$= \alpha r\sqrt{2}\sqrt{\frac{1-\cos(\alpha\vartheta-\vartheta)}{2}}\sqrt{2}$$

$$= \alpha r\, 2\cdot\sqrt{\sin^2\frac{(\alpha-1)\vartheta}{2}}$$

$$= 2\alpha r\cdot\sin\frac{(\alpha-1)\vartheta}{2}$$

$$\frac{1-\cos\varphi}{2}$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = 1 - 2\sin^2\varphi$$

$$\sin^2\varphi = \frac{1-\cos 2\varphi}{2}$$

$$\frac{1-\cos\varphi}{2} = \sin^2\frac{\varphi}{2}$$

$$\boxed{s(\vartheta) = \int\|\gamma'\|d\vartheta = -\frac{4\alpha r}{\alpha-1}\cos\frac{(\alpha-1)\vartheta}{2}}$$

$$\alpha - 1 = \frac{R+r}{r} - 1 = \frac{R}{r}$$

$$K = \frac{|\gamma'\gamma''|}{\|\gamma'\|^3} = \frac{\alpha^2 r^2\begin{vmatrix}-s\vartheta + s(\alpha\vartheta) & -c\vartheta + \alpha c(\alpha\vartheta)\\ c\vartheta - c(\alpha\vartheta) & -s\vartheta + \alpha s(\alpha\vartheta)\end{vmatrix}}{''}$$

$$= \frac{\alpha^2 r^2\begin{pmatrix}s^2\vartheta - \alpha s\vartheta\, s(\alpha\vartheta) - s\vartheta\, s(\alpha\vartheta) + \alpha s^2(\alpha\vartheta)\\ +c^2\vartheta - \alpha c\vartheta\, c(\alpha\vartheta) - c\vartheta\, c(\alpha\vartheta) + \alpha c^2(\alpha\vartheta)\end{pmatrix}}{''}$$

$$= \frac{\alpha^2 r^2\left(1+\alpha - \alpha(s\vartheta\, s(\alpha\vartheta) + c\vartheta\, c(\alpha\vartheta))\right)}{''}$$

$$= \frac{\alpha^2 r^2\,(1+\alpha)\left[1 - s\vartheta\, s(\alpha\vartheta) - c\vartheta\, c(\alpha\vartheta)\right]}{\alpha^3 r^3\sqrt{2}^3\sqrt{1 - s\vartheta\, s(\alpha\vartheta) - c\vartheta\, c(\alpha\vartheta)}^3}$$

$$= \frac{1+\alpha}{\alpha r\sqrt{2}^3}\,\frac{1}{\sqrt{1 - s\vartheta\, s(\alpha\vartheta) - c\vartheta\, c(\alpha\vartheta)}}$$

$$= \frac{1+\alpha}{\alpha r\, 4}\,\frac{1}{\sin\frac{(\alpha-1)\vartheta}{2}}$$

$$s(\vartheta) = -\frac{4\alpha r}{\alpha-1}\cos\frac{(\alpha-1)\vartheta}{2} \qquad K(\vartheta) = \frac{1+\alpha}{4r\alpha}\,\frac{1}{\sin\frac{(\alpha-1)\vartheta}{2}}$$

$$(1+\alpha)^2\frac{1}{K^2} + (\alpha-1)^2 s^2 = (4\alpha r)^2$$

$$\boxed{(1+\alpha)^2\frac{1}{K^2} + (\alpha-1)^2 s^2 = 16\alpha^2 r^2}$$

$$\boxed{\frac{1}{K^2} + \underset{<1}{*}^2 s^2 = \bullet^2}$$

viceversa, se abbiamo curve tali che K e s sono legate da

$$\frac{1}{K^2} + \omega^2 s^2 = A^2 \qquad \omega < 1$$

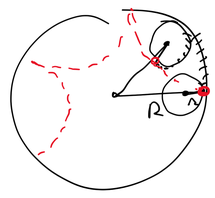
allora sono a meno di isometrie del piano, degli epicicloidi?

Calcolare la curva dei centri di γ

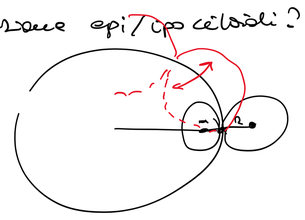
$$C_\gamma = \gamma + \frac{1}{K}n \qquad n = -\mathrm{cross}(\gamma')$$

e vedere se viene un epicicloide?

ipocicloidi:



relazione epi/ipocicloidi?



NOTA: non è l'inversione rispetto alla circonferenza grande (funzione che manda X in $\frac{R^2}{\|X\|^2}X$ per $X \neq 0$)!

ma è una specie di simmetria rispetto alla circonferenza grande:
se prendiamo i cerchi centrati nei punti della circonferenza grande e TANGENTI all'EPICICL., allora sono tangenti all'ipocicl (sic)

ESERCIZIO DA UN COMPITO:

$$\gamma(t) = \begin{pmatrix}a\cos t\\ a\sin t\\ b\cos(\beta t)\\ b\sin(\beta t)\end{pmatrix}$$

curva dentro $\mathbb{R}^4$, $a, b, \beta \in \mathbb{R}_{>0}$

$$\gamma' = \begin{pmatrix}-a\sin t\\ a\cos t\\ -b\beta\sin(\beta t)\\ b\beta\cos(\beta t)\end{pmatrix} \qquad \gamma'' = \begin{pmatrix}-a\cos t\\ -a\sin t\\ -b\beta^2\cos(\beta t)\\ -b\beta^2\sin(\beta t)\end{pmatrix} \qquad \gamma''' = \begin{pmatrix}a\sin t\\ -a\cos t\\ b\beta^3\sin(\beta t)\\ -b\beta^3\cos(\beta t)\end{pmatrix}$$

è $\perp\gamma'$

è $\perp\gamma''$, $\not\perp\gamma'$

Frenet di γ: $(e_1 e_2 e_3 e_4)' = (e_1 e_2 e_3 e_4)\begin{pmatrix}0 & -k_1 & 0 & 0\\ k_1 & 0 & -k_2 & 0\\ 0 & k_2 & 0 & -k_3\\ 0 & 0 & k_3 & 0\end{pmatrix}$

$$\begin{cases} e_1' = k_1 e_2 & \Rightarrow k_1 = e_1'\cdot e_2\\ e_2' = -k_1 e_1 + k_2 e_3 & \Rightarrow k_2 = e_2'\cdot e_3\\ e_3' = -k_2 e_2 + k_3 e_4 & \Rightarrow k_3 = e_3'\cdot e_4 = -e_4'\cdot e_3\\ e_4' = -k_3 e_3 \end{cases}$$

se siamo nel piano d'orbita, allora avremo matrice con $\omega_1, \omega_2, \omega_3$, e $k_i = \frac{1}{\|\gamma'\|}\omega_i$.

$$\|\gamma'\| = \sqrt{a^2+b^2\beta^2}$$

costante

$\gamma'' \perp \gamma'$

$$e_1 = \frac{\gamma'}{\|\gamma'\|}$$

$$\|\gamma''\| = \sqrt{a^2 + b^2\beta^4}$$

costante

$\gamma''' \perp \gamma''$ (ma $\not\perp\gamma'$)

$$e_2 = \frac{\gamma''}{\|\gamma''\|}$$

$$\omega_1 = e_1'\cdot e_2 = \left(\frac{\gamma'}{\|\gamma'\|}\right)'\cdot\frac{\gamma''}{\|\gamma''\|} = \frac{\gamma''\cdot\gamma''}{\|\gamma'\|\cdot\|\gamma''\|} = \frac{\|\gamma''\|}{\|\gamma'\|}$$

costante!

$$k_1 = \frac{1}{\|\gamma'\|}\omega_1$$

quindi:

$$\boxed{e_3 = \frac{\gamma''' - \frac{\gamma'''\cdot\gamma'}{\gamma'\cdot\gamma'}\cdot\gamma' - \cancel{\frac{\gamma'''\cdot\gamma''}{\gamma''\cdot\gamma''}\cdot\gamma''}}{\|\text{numeratore}\|}}$$

$$= \begin{pmatrix}a\,st\\ -a\,ct\\ b\beta^3 s(\beta t)\\ -b\beta^3 c(\beta t)\end{pmatrix} + \frac{+a^2+b^2\beta^4}{a^2+b^2\beta^2}\begin{pmatrix}-a\,st\\ a\,ct\\ -b\beta\, s(\beta t)\\ b\beta\, c(\beta t)\end{pmatrix} =$$

$$= \begin{pmatrix}a b^2\beta^2(1-\beta^2)\,st\\ -a b^2\beta^2(1-\beta^2)\,ct\\ a^2 b\beta(1-\beta^2)\,s(\beta t)\\ a^2 b\beta(1-\beta^2)\,c(\beta t)\end{pmatrix}\frac{1}{a^2+b^2\beta^2}$$

$$a(\cancel{a^2}+b^2\beta^2) + (\cancel{a^2}+b^2\beta^4)(-a)$$

$$-a(\quad) + (\quad)a$$

$$-b\beta^3(a^2+\cancel{b^2\beta^2}) + b\beta(a^2+\cancel{b^2\beta^4})$$

$$= \frac{a b\beta(1-\beta^2)}{a^2+b^2\beta^2}\begin{pmatrix}b\beta\, st\\ -b\beta\, ct\\ -a\, s(\beta t)\\ a\, c(\beta t)\end{pmatrix}$$

controllo che sia $\perp\gamma'$, $\perp\gamma''$

$$e_3 = \frac{1}{\sqrt{a^2+b^2\beta^2}}\begin{pmatrix}b\beta\, st\\ -b\beta\, ct\\ -a\, s(\beta t)\\ a\, c(\beta t)\end{pmatrix}$$

$\omega_2 = e_2'\cdot e_3 = \ldots$ costante dip a, b, β

Dati e_1, e_2, e_3 abbiamo

$$e_4 = -\mathrm{cross}(e_1, e_2, e_3) = -\text{minori sequenti}(e_1, e_2, e_3)$$

$$= -\frac{\mathrm{cross}\left(\gamma'\ \gamma''\ \left(\gamma''' - \frac{\gamma'\cdot\gamma'''}{\gamma'\cdot\gamma'}\gamma'\right)\right)}{\|\text{numeratore}\|}$$

$$= -\frac{\mathrm{cross}\begin{pmatrix}-a\,st & -a\,ct & b\beta\,st\\ a\,ct & -a\,st & -b\beta\,ct\\ -b\beta\, s(\beta t) & -b\beta\, c(\beta t) & -a\, s(\beta t)\\ b\beta\, c(\beta t) & -b\beta\, s(\beta t) & a\, c(\beta t)\end{pmatrix}}{\|\text{numeratore}\|}$$

combinazione $b\beta\,\mathrm{I} + a\,\mathrm{III}$

$$\mathrm{cross}\begin{pmatrix}-st & -a\,ct & 0\\ ct & -a\,st & 0\\ 0 & -b\beta^2 c(\beta t) & -s(\beta t)\\ 0 & -b\beta^2 s(\beta t) & c(\beta t)\end{pmatrix}$$

$a\,\mathrm{I} - b\beta\,\mathrm{III}$

$$= \frac{\quad}{\|\text{numeratore}\|}$$

$$= -\frac{\begin{pmatrix}-b\beta^2\, ct\\ -b\beta^2\, st\\ a\, c(\beta t)\\ a\, s(\beta t)\end{pmatrix}}{\sqrt{a^2+b^2\beta^4}}$$

Morale: le curvature vengono costanti, e tutte le curve in $\mathbb{R}^4$ a curvature costanti sono di questo tipo (elicoide TORICHE: sono elicoide sul toro $aS^1 \times bS^1 \subseteq \mathbb{R}^2\times\mathbb{R}^2 = \mathbb{R}^4$).

---

8 gennaio 2026 lunedì