COVERING DIRECT PRODUCTS WITH PROPER SUBGROUPS

A. Lucchini, M. Garonzi

University of Padova

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1. History, definitions and easy results

2. Sigma-elementary groups

3. Covering direct products
Exercise

No group can be written as set-theoretical union of two proper subgroups.
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**Theorem (Scorza 1926)**

*A group $G$ is union of three proper subgroups if and only if it admits an epimorphic image isomorphic to $C_2 \times C_2$.***
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These elementary considerations led Cohn in 1994 to define for every group $G$:

$$\sigma(G) \quad \text{Sum of } G: \text{ the least cardinality of a cover of } G \text{ consisting of proper subgroups.}$$
**Example**

If $G$ is a cyclic group then it is not a union of proper subgroups, because the generators of $G$ do not lie in proper subgroups. In this case we make the convention $\sigma(G) = \infty$, with $n < \infty$ for every integer $n$. 
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Exercise

If $p$ is a prime number then $\sigma(C_p \times C_p) = p + 1$. Indeed, $C_p \times C_p$ has exactly $p + 1$ maximal subgroups, all of them isomorphic to $C_p$ and pairwise intersecting in the identity subgroup, so they cover $1 + (p - 1)(p + 1) = p^2$ elements.
The following result is due to Tomkinson (1997):

**Theorem (Tomkinson)**

Let $G$ be a solvable non-cyclic group. Then $\sigma(G) = |S/K| + 1$ where $|S/K|$ is the least order of a chief factor of $G$ with more than one complement.
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**Example**

If $q$ is a prime power then $\sigma(\mathbb{F}_q \rtimes \mathbb{F}_{q^*}) = q + 1$. 
**Remark**

If $N$ is a normal subgroup of $G$ then $\sigma(G) \leq \sigma(G/N)$, because every cover of $G/N$ corresponds to a cover of $G$.

This suggests to study the quotients $G/N$ such that $\sigma(G) = \sigma(G/N)$, and leads to the following:

**Definition ($\sigma$-elementary groups)**

A group $G$ is said to be “$\sigma$-elementary” if $\sigma(G) < \sigma(G/N)$ for every $1 \neq N \trianglelefteq G$. We say that $G$ is “$n$-elementary” if $G$ is $\sigma$-elementary and $\sigma(G) = n$. 
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**Example**

There exist $\sigma$-elementary groups with non-cyclic proper quotients. For example if $p$ is a large enough prime number and $G := \text{Alt}(5) \wr \text{Alt}(p)$ then $\sigma(G) \leq |\text{Alt}(5)|^p = 60^p$ and $\sigma(\text{Alt}(p)) \geq (p-2)! \geq 60^p$. 
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**Example**

If the $\sigma$-elementary group $G$ is abelian then $G \cong C_p \times C_p$ for some prime $p$ (cf. [2], Theorem 3).
A natural question arises:

**DIRECT PRODUCTS OF SIMPLE GROUPS**

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The answer is no: if $T_1, \ldots, T_k$ are non-abelian simple groups then

$$\sigma(T_1 \times \ldots \times T_k) = \min\{\sigma(T_1), \ldots, \sigma(T_k)\}.$$

It is not difficult to prove.
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But why not asking the general question:

**Direct Products at All**

Can a direct product of groups be $\sigma$-elementary?

This is the question we answered in [1], and the answer is again no, with the exception of $C_p \times C_p$. 
For example, let us prove that if $S$ is a finite simple non-abelian group then $\sigma(S \times S) = \sigma(S)$. 

$\blacksquare$
For example, let us prove that if $S$ is a finite simple non-abelian group then $\sigma(S \times S) = \sigma(S)$.

- We know that the maximal subgroups of $S \times S$ are of the following three types:

  1. $K \times S$
  2. $S \times K$
  3. $\Delta_\varphi := \{(x, \varphi(x)) \mid x \in S\}$,

where $K$ is a maximal subgroup of $S$ and $\varphi \in \text{Aut}(S)$.
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- Let $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ be a minimal cover of $S \times S$, where $\mathcal{M}_i$ consists of subgroups of type $(i)$. 

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- Let \( \Omega := S \times S - \bigcup_{M \in M_1 \cup M_2} M = \Omega_1 \times \Omega_2 \), where \( \Omega_1 = S - \bigcup_{K \times S \in M_1} K \) and \( \Omega_2 = S - \bigcup_{S \times K \in M_2} K \).
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- We prove that $\Omega = \emptyset$. Suppose $\Omega \neq \emptyset$. Let $\omega \in \Omega_1$. Notice that $\{K < S \mid S \times K \in \mathcal{M}_2\} \cup \{\langle \varphi(\omega) \rangle \mid \Delta_\varphi \in \mathcal{M}_3\}$ covers $S$. 
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- It follows that

  $$|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = |\mathcal{M}| = \sigma(S \times S) \leq \sigma(S) \leq |\mathcal{M}_2| + |\mathcal{M}_3|.$$ 

  This implies that $\mathcal{M}_1 = \emptyset$. Analogously $\mathcal{M}_2 = \emptyset$. So $\mathcal{M} = \mathcal{M}_3$. 


For example, let us prove that if $S$ is a finite simple non-abelian group then $\sigma(S \times S) = \sigma(S)$.

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- Let $M = M_1 \cup M_2 \cup M_3$ be a minimal cover of $S \times S$, where $M_i$ consists of subgroups of type $(i)$.

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- It follows that

  \begin{align*}
  |M_1| + |M_2| + |M_3| = |M| = \sigma(S \times S) \leq \sigma(S) \leq |M_2| + |M_3|.
  \end{align*}

  This implies that $M_1 = \emptyset$. Analogously $M_2 = \emptyset$. So $M = M_3$.

- Since $S = \bigcup_{S \ni s \neq 1} \langle s \rangle$ and $|M| = |S|$ for every $M \in M_3$,

  \begin{align*}
  |S| - 1 \geq \sigma(S) \geq \sigma(S \times S) = |M| = |M_3| \geq |S|,
  \end{align*}

  contradiction.
In fact there is a much more general conjecture ([3]):

**Conjecture (Lucchini, Detomi)**

Every non-abelian $\sigma$-elementary group is monolithic.

There are partial results supporting this conjecture. Let $G$ be a non-abelian $\sigma$-elementary group. Then:

- ([3], Corollary 14) $G$ has at most one abelian minimal normal subgroup. In particular if it is solvable, it is monolithic.
- ([3], Corollary 14) $G$ is a subdirect product of monolithic primitive groups.
- ([3], Proposition 21) Any solvable proper quotient of $G$ is cyclic.
- ([3], Theorem 24) Suppose $G$ has no abelian minimal normal subgroups. Then either $G$ is a primitive monolithic group and $G/\text{soc}(G)$ is cyclic, or $G/\text{soc}(G)$ is non-solvable and all the non-abelian composition factors of $G/\text{soc}(G)$ are alternating groups of odd degree.
- ([5], Lemma 3) If $\sigma(G) \leq 33$ then $G$ is monolithic.
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**Theorem**

Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ where $\mathcal{X}$ is a minimal cover of $H_1$. In this case $\sigma(G) = \sigma(H_1)$.

2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ where $\mathcal{X}$ is a minimal cover of $H_2$. In this case $\sigma(G) = \sigma(H_2)$.

3. There exist $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ with $H_1/N_1 \cong H_2/N_2 \cong C_p$ and $\mathcal{M}$ consists of the maximal subgroups of $H_1 \times H_2$ containing $N_1 \times N_2$. In this case $\sigma(G) = p + 1$. 
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3. There exist $N_1 \triangleleft H_1$, $N_2 \triangleleft H_2$ with $H_1/N_1 \cong H_2/N_2 \cong C_p$ and $\mathcal{M}$ consists of the maximal subgroups of $H_1 \times H_2$ containing $N_1 \times N_2$. In this case $\sigma(G) = p + 1$.

**Remark**

This theorem can be re-stated in the general case $G = H_1 \times \ldots \times H_n$. 
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**Proposition (Maximal Subgroups of a Direct Product)**

Let $G = H_1 \times H_2$ be the direct product of two finite groups. A maximal subgroup of $G$ is called “(of) standard (type)” if it is of the form $M \times H_2$ with $M$ a maximal subgroup of $H_1$ or $H_1 \times M$ with $M$ a maximal subgroup of $H_2$, it is called “(of) diagonal (type)” if it is of the form $\{(x, y) \in G \mid \varphi(xN_1) = yN_2\}$ where $N_i$ is a maximal normal subgroup of $H_i$ for $i = 1, 2$ and $\varphi : H_1/N_1 \to H_2/N_2$ is an isomorphism. It is well known that every maximal subgroup of $G$ is either of standard type or of diagonal type.
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For a proof see [7], Chap. 2, (4,19).
Step 1. Let $\mathcal{M}$ be a minimal cover of $G = H_1 \times H_2$ consisting of maximal subgroups. Assume that $\mathcal{M}$ contains no subgroup of diagonal type whose index is a prime number. We want to show that in this case either $H_1 \times 1$ or $1 \times H_2$ is contained in $\bigcap_{M \in \mathcal{M}} M$. 
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\[ \mathcal{M}_1 := \{ M \in \mathcal{M} | M \supseteq 1 \times H_2 \}, \quad \mathcal{M}_2 := \{ M \in \mathcal{M} | M \supseteq H_1 \times 1 \}, \]

\[ \mathcal{M}_3 := \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_2), \]

\[ \Omega_1 := H_1 - \left( \bigcup_{L \times H_2 \in \mathcal{M}_1} L \right), \quad \Omega_2 := H_2 - \left( \bigcup_{H_1 \times L \in \mathcal{M}_2} L \right). \]
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Suppose by contradiction that $\Omega := G - \bigcup_{M \in \mathcal{M}_1 \cup \mathcal{M}_2} M = \Omega_1 \times \Omega_2 \neq \emptyset$. 
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- $\mathcal{M}_1 := \{M \in \mathcal{M} \mid M \supseteq 1 \times H_2\}$, $\mathcal{M}_2 := \{M \in \mathcal{M} \mid M \supseteq H_1 \times 1\}$, $\mathcal{M}_3 := \mathcal{M} - (\mathcal{M}_1 \cup \mathcal{M}_2)$,

- $\Omega_1 := H_1 - (\bigcup_{L \times H_2 \in \mathcal{M}_1} L)$, $\Omega_2 := H_2 - (\bigcup_{H_1 \times L \in \mathcal{M}_2} L)$.

- Suppose by contradiction that $\Omega := G - \bigcup_{M \in \mathcal{M}_1 \cup \mathcal{M}_2} M = \Omega_1 \times \Omega_2 \neq \emptyset$.

- Let $K_i$ be intersection of the maximal normal subgroups of $H_i$, for $i = 1, 2$. There are simple groups $S_1, \ldots, S_\alpha, T_1, \ldots, T_\beta$ such that

\[
H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a, \quad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b.
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\]

\[
\rho_i : H_i \to H_i/K_i, \quad \pi_1, a : H_1 \to S_a, \quad \Delta_i := \rho_i(\Omega_i), \quad \pi_2, b : H_2 \to T_b.
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To any \( M \in \mathcal{M}_3 \) we may associate a triple \((a, b, \phi)\) with \(1 \leq a \leq \alpha, 1 \leq b \leq \beta\) and \( \phi : S_a \to T_b \) a group isomorphism such that \( M = M(a, b, \varphi) \) equals

\[ \{(h_1, h_2) \in H_1 \times H_2 \mid \phi(\pi_1,a(h_1)) = \pi_2,b(h_2)\}. \]

By the hypothesis, if \( M(a, b, \phi) \in \mathcal{M}_3 \) then \( S_a \cong T_b \) is non-abelian.
\[ \frac{H_1}{K_1} = \prod_{1 \leq a \leq \alpha} S_a, \quad \frac{H_2}{K_2} = \prod_{1 \leq b \leq \beta} T_b. \]

\[ \rho_i : H_i \to H_i/K_i, \quad \Delta_i := \rho_i(\Omega_i), \]
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Fix \( \omega \in \Omega_1. \ \rho_1(\omega) =: (s_1, \ldots, s_\alpha) \in \Delta_1. \) For \( M(a, b, \phi) \in \mathcal{M}_3 \) let

\[ U(a, b, \phi) := \{ h \in H_2 \mid \pi_{2,b}(h) \in \langle \phi(s_a) \rangle \}. \]

\( T_b \) is non-abelian \( \Rightarrow \langle \phi(s_a) \rangle \neq T_b \Rightarrow U(a, b, \phi) < H_2. \)
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H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a, \quad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b.
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\[\rho_i : H_i \rightarrow H_i/K_i, \quad \Delta_i := \rho_i(\Omega_i),\]

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The following family of proper subgroups of \( H_2 \) covers \( H_2 \):

\[
\{K < H_2 \mid H_1 \times K \in \mathcal{M}_2\} \cup \{U(a, b, \phi) \mid M(a, b, \phi) \in \mathcal{M}_3\}.
\]
\[ H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a, \quad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b. \]

\[ \rho_i : H_i \rightarrow H_i/K_i, \quad \Delta_i := \rho_i(\Omega_i), \]

\[ \pi_{1,a} : H_1 \rightarrow S_a, \quad \pi_{2,b} : H_2 \rightarrow T_b. \]

To any \( M \in \mathcal{M}_3 \) we may associate a triple \((a, b, \phi)\) with \(1 \leq a \leq \alpha, 1 \leq b \leq \beta\) and \(\phi : S_a \rightarrow T_b\) a group isomorphism such that \(M = M(a, b, \phi)\) equals

\[ \{(h_1, h_2) \in H_1 \times H_2 \mid \phi(\pi_{1,a}(h_1)) = \pi_{2,b}(h_2)\}. \]

By the hypothesis, if \( M(a, b, \phi) \in \mathcal{M}_3 \) then \(S_a \cong T_b\) is non-abelian.

Fix \( \omega \in \Omega_1. \rho_1(\omega) =: (s_1, \ldots, s_\alpha) \in \Delta_1. \) For \( M(a, b, \phi) \in \mathcal{M}_3 \) let

\[ U(a, b, \phi) := \{ h \in H_2 \mid \pi_{2,b}(h) \in \langle \phi(s_a) \rangle \}. \]

\( T_b \) is non-abelian \(\Rightarrow \langle \phi(s_a) \rangle \neq T_b \Rightarrow U(a, b, \phi) < H_2.\)

The following family of proper subgroups of \( H_2 \) covers \( H_2: \)

\[ \{K < H_2 \mid H_1 \times K \in \mathcal{M}_2\} \cup \{U(a, b, \phi) \mid M(a, b, \phi) \in \mathcal{M}_3\}. \]

It follows that

\[ |\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = |\mathcal{M}| = \sigma(H_1 \times H_2) \leq \sigma(H_2) \leq |\mathcal{M}_2| + |\mathcal{M}_3|. \]

This implies that \( \mathcal{M}_1 = \emptyset. \) Analogously \( \mathcal{M}_2 = \emptyset. \) So \( \mathcal{M} = \mathcal{M}_3. \)
Now the conclusion of Step 1 follows easily. For sake of exposition, we will state and use the following (quite useful) technical lemma.

**Lemma**

Let $G$ be a finite group, let $N$ be a proper normal subgroup of $G$, and let $U_1, \ldots, U_h, V_1, \ldots, V_k$ be proper subgroups of $G$ such that $U_1, \ldots, U_h$ contain $N$, $V_1, \ldots, V_k$ supplement $N$, and $\beta_1 \leq \ldots \leq \beta_k$, where $\beta_i = |G : V_i|$ for $i = 1, \ldots, k$.

If $U_1 \cup \ldots \cup U_h \cup V_1 \cup \ldots \cup V_k = G$ and $U_1 \cup \ldots \cup U_h \neq G$ then $\beta_1 \leq k$.

Moreover, if $\beta_1 = k$ then $\beta_1 = \ldots = \beta_k = k$ and $V_i \cap V_j \leq U_1, \ldots, U_h$ for every $i \neq j$ in $\{1, \ldots, k\}$.

Apply this lemma with $N = H_1 \times \{1\} = U_1$, $h = 1$, $\{V_1, \ldots, V_k\} = \mathcal{M}_3$. The index of $V_1 \in \mathcal{M}_3$ is the order of a simple non-abelian group $S$ which is an epimorphic image of $G$.

$$|S| = |G : V_1| = \beta_1 \leq k = |\mathcal{M}_3| = |\mathcal{M}| = \sigma(G) \leq \sigma(S) \leq |S| - 1,$$

contradiction.
Step 2. Assume now that there exists $M \in \mathcal{M}$ of diagonal type and index $p$, say $M = \{(x, y) \in G \mid \varphi(xN_1) = yN_2\}$ for some $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ with $H_i/N_i \cong C_p$ for $i = 1, 2$. We prove that then $\mathcal{M}$ consists of normal subgroups of index $p$. 
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Note that $\sigma(G) \leq \sigma(H_1 \times H_2/N_1 \times N_2) = \sigma(C_p \times C_p) = p + 1$.

Induction: assume that no non-trivial normal subgroup $N$ of $G$ is contained in $H_i$ and in every element of $\mathcal{M}$, for $i = 1, 2$. 
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- WLOG, let $g \in Z(G) \cap H_1$ of order $p$, and let $N = \langle g \rangle \trianglelefteq G$. 


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- Assume that \( p \) divides \( |Z(G)| \).
- WLOG, let \( g \in Z(G) \cap H_1 \) of order \( p \), and let \( N = \langle g \rangle \trianglelefteq G \).
- \( \mathcal{U} := \{M \in \mathcal{M} \mid M \nsubseteq N\} \) is non-empty by the above assumption.
Step 2. Assume now that there exists $M \in \mathcal{M}$ of diagonal type and index $p$, say $M = \{(x, y) \in G \mid \varphi(xN_1) = yN_2\}$ for some $N_1 \leq H_1$, $N_2 \leq H_2$ with $H_i/N_i \cong C_p$ for $i = 1, 2$. We prove that then $\mathcal{M}$ consists of normal subgroups of index $p$.

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- $\mathcal{U} := \{M \in \mathcal{M} \mid M \not\supseteq N\}$ is non-empty by the above assumption.

- Let $M \in \mathcal{U}$. $G = MN \cong M \times N$, so $M$ is normal of index $|N| = p$. 
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- Let $N = M_1 \cap M_2$ and $\{U_1, \ldots, U_h, V_1, \ldots, V_k\} = \mathcal{M}$. Applying the lemma we get $h = 1$, $M_1 \cap \ldots \cap M_p \cap K = M_1 \cap M_2 = N$. Therefore $\mathcal{M}$ corresponds to the unique cover of $G/N \cong C_p \times C_p$. 
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It is possible to find a minimal normal subgroup $N$ of $G$ contained either in $H_1$ or $H_2$ with the property that $A = G/C_G(N)$ has a chief factor of order $p$. The set $\mathcal{U}$ of the subgroups in $\mathcal{M}$ not containing $N$ is non-empty.
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Let $\beta := \min_{K \in \mathcal{U}} |G : K|$, and let $M \in \mathcal{M}$ be such that $\beta = |G : M|$. By the lemma, $p + 1 \geq \sigma(G) \geq |\mathcal{U}| \geq \beta$. 
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We only discuss the case in which $N$ is a non-abelian simple group. In this case $C_p$ is isomorphic to a chief factor of a subgroup of $\text{Out}(N)$ hence $p \leq |\text{Out}(N)|$. 
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$\beta = |G : M| = |N : M \cap N|$ is the index of a proper subgroup of $N$, therefore $\beta > 2p$ (this follows from [6], Lemma 2.7, which relies on the CFSG). Then $p + 1 \geq \beta > 2p$, contradiction.
M. Garonzi, A. Lucchini, Direct products of groups as unions of proper subgroups.


M. Garonzi, Finite Groups that are Union of at most 25 Proper Subgroups, Journal of Algebra and its Applications, ISSN: 0219-4988.
