Commutative Algebra Exam

Padova, 18/02/2015

Exercise 1. Let $L = \mathbb{Q}(\alpha)$, where $\alpha^3 = 11$. Let $A = \mathbb{Z}[\alpha]$ and write $\mathcal{O}_L$ for the integral closure of $\mathbb{Z}$ in L.

a) Compute $\Delta(1, \alpha, \alpha^2)$.

b) Describe the decompostion of 2, 5, 7 and 13 in L and compute the inertia and ramification degree above these primes.

c) Let $S = \mathbb{Z} - 11\mathbb{Z}$. Show that $\alpha S^{-1}A$ is a maximal ideal. Show that $S^{-1}A = S^{-1}\mathcal{O}_L$ and compute the decomposition of 11 in L.

d) For any $h \in \mathbb{Z}$, compute the minimal polynomial of $\alpha + h$.

e) Compute the decomposition of 3 in L and show that $A = \mathcal{O}_L$.

f) Show that if $\beta \in \mathcal{O}_L$ is such that $N_{L/\mathbb{Q}}(\beta) = p$ is a prime number, then $\beta\mathcal{O}_L$ is a prime ideal.

g) Show that all the primes dividing 3,7,11, 13 are principal.

h) Show that $u = 1 + 4\alpha - 2\alpha^2 \in \mathcal{O}_L^\times$.

We now borrow some classical results from Number Theory. For this particular field, the Hasse-Minkowsky bound implies that $\mathrm{Pic}(\mathcal{O}_L)$ is generated by the primes of norm strictly less than 17. Dirichlet's Unit theorem says that $\mathcal{O}_L^\times = \{\pm u^n, \ \forall n \in \mathbb{Z}\}$.

i) Show that $\mathrm{Pic}(\mathcal{O}_L)$ is a cyclic group generated by the class of a prime dividing 2. [Hint: find $x \in \mathcal{O}_L$ such that $N_{L/\mathbb{Q}}(x) = 10$.]

j) Compute $N_{L/\mathbb{Q}}(\alpha^2 - 5)$ and show that $\frac{2}{\alpha^2-5} \notin \mathcal{O}_L$. Conclude that $|\mathrm{Pic}(\mathcal{O}_L)| \leq 2$.

k) Show that if $\mathrm{Pic}(\mathcal{O}_L)$ is trivial, then there exist $\varepsilon, \delta \in \{\pm 1\}$ such that $y = \varepsilon u^\delta(\alpha^2 - 5)$ is a square in $\mathcal{O}_L$.

l) Show that $\mathfrak{p} = (\alpha - 2)$ is a prime ideal, compute $\mathcal{O}_L/\mathfrak{p}$. Reduce y mod $\mathfrak{p}$ and determine ε.

m) Find a prime factor $\mathfrak{q}$ of $(\alpha + 3)\mathcal{O}_L$ such that y mod $\mathfrak{q}$ is not a square. Conclude that $\mathrm{Pic}(\mathcal{O}_L) = \{\pm 1\}$.

n) Find a fractional ideal $I \subset L$ whose class $[I] \in K_0(\mathcal{O}_L)$ satisfies $\chi([I]) = (1, -1) \in \mathbb{Z} \oplus \mathrm{Pic}(\mathcal{O}_L)$. Is it possible to find an integral fractional ideal $I \subseteq R$ with this property?

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Exercise 2. Let k be a field, $\mathfrak{p}_1, \mathfrak{p}_2 \subset k[X_1, \dots, X_n]$ two prime ideals with $\operatorname{ht} \mathfrak{p}_i = h_i$.

a) Let $\sigma : k[X_1, \dots, X_n] \to k[Y_1, \dots, Y_n]$ be given by $\sigma(X_j) = Y_j$. Put $I = \mathfrak{p}_1 + \sigma(\mathfrak{p}_2)$. Show that $k[X_1, \dots, X_n, Y_1, \dots, Y_n]/I \simeq k[X_1, \dots, X_n]/\mathfrak{p}_1 \otimes_k k[Y_1, \dots, Y_n]/\sigma(\mathfrak{p}_2)$.

b) Show that $\dim k[X_1, \dots, X_n, Y_1, \dots, Y_n]/I = \dim k[X_1, \dots, X_n]/\mathfrak{p}_1 + \dim k[Y_1, \dots, Y_n]/\sigma(\mathfrak{p}_2)$. [Hint: Normalisation lemma].

c) Let $D = (X_1 - Y_1, \dots, X_n - Y_n)$. Show that $k[X_1, \dots, X_n, Y_1, \dots, Y_n]/(I + D) \simeq k[X_1, \dots, X_n]/\left(\mathfrak{p}_1 + \mathfrak{p}_2\right)$.

d) Let $\widetilde{\mathfrak{q}} \subset k[X_1, \dots, X_n, Y_1, \dots, Y_n]$ be a minimal prime ideal containing $I + D$. Show that $\operatorname{ht} \widetilde{\mathfrak{q}}/I \leq n$.

e) Let $\mathfrak{q} \subset k[X_1, \dots, X_n]$ be a minimal prime ideal containing $\mathfrak{p}_1 + \mathfrak{p}_2$. Show that $\operatorname{ht} \mathfrak{q} \leq h_1 + h_2$.

f) Let $V_1, V_2 \subseteq \mathbb{A}^n(k)$ be irreducible closed subsets. Show that $\dim W \geq \dim V_1 + \dim V_2 - n$ for any irreducible component W of $V_1 \cap V_2$.

You can submit your answers in english, french or italian.

Exercise 1. a) The minimal polynomial of α if X^3-11, hence that of α^2 is Y^3-121. Therefore $\Delta(1,\alpha,\alpha^2) = -N_{L/\mathbb{Q}}(3\alpha^2) = -3^3 N_{L/\mathbb{Q}}(\alpha^2) = -3^3 11^2$. Only 3 and 11 may ramify, and 3 ramifies for sure, since it appears with odd exponent. Moreover, $A\left[\frac{1}{33}\right] = \mathcal{O}_L\left[\frac{1}{33}\right]$.

b) $X^3-11 \equiv X^3-1 \equiv (X-1)(X^2+X+1) \bmod 2$, and the quadratic factor is irreducible. Therefore $2\mathcal{O}_L = \mathfrak{p}_{2,1}\mathfrak{p}_{2,2}$, with $e_{2,1}=e_{2,2}=f_{2,1}=1$ and $f_{2,2}=2$.

$X^3-11 \equiv X^3-1 \equiv (X-1)(X^2+X+1) \bmod 5$, and the quadratic factor is irreducible. Therefore $2\mathcal{O}_L = \mathfrak{p}_{5,1}\mathfrak{p}_{5,2}$, with $e_{5,1}=e_{5,2}=f_{5,1}=1$ and $f_{5,2}=2$.

$X^3-11 \equiv X^3-4 \bmod 7$, which is irreducible. Therefore 7 is inert in L.

X^3-11 is irreducible mod 13. Therefore 13 is inert in L.

c) X^3-11 is an Eisenstein polynomial with respect to 11. We have $\alpha^3 S^{-1}\mathcal{O}_L = 11S^{-1}\mathcal{O}_L$. Let's decompose $\alpha S^{-1}\mathcal{O}_L = \mathfrak{p}_{11,1}^{n_1}\cdots\mathfrak{p}_{11,s}^{n_s}$. Then $\alpha^3 S^{-1}\mathcal{O}_L = \mathfrak{p}_{11,1}^{3n_1}\cdots\mathfrak{p}_{11,s}^{3n_s} = 11S^{-1}\mathcal{O}_L$. We get an equation $3n_1+\ldots+3n_s=3$, whose only possible solutions in natural numbers force $s=1=n_1$. Thus $\alpha S^{-1}\mathcal{O}_L = \mathfrak{p}_{11}$ is maximal. Therefore $\alpha S^{-1}A$ is maximal ($\mathcal{O}_L$ is integral over A). Hence $S^{-1}A$ is a local noetherian domain whose maximal ideal is principal: it is a DVR. Therefore $S^{-1}A = S^{-1}\mathcal{O}_L$. As a consequence $A[\frac{1}{3}] = \mathcal{O}_L[\frac{1}{3}]$.

d) Expanding, we get $p(Z) = (Z-h)^3-11 = Z^3-3hZ^2+3h^2Z-(h^3+11)$.

e) Clearly $L=\mathbb{Q}(\alpha+h)$ and $A=\mathbb{Z}[\alpha+h]$ for all $h\in\mathbb{Z}$. Taking $h=-2$ we get that the minimal polynomial of $\alpha-2$ is Eisenstein with respect to 3. Putting $S'=\mathbb{Z}-3\mathbb{Z}$ we may repeat the argument in c): $(\alpha-2)^3 S'^{-1}\mathcal{O}_L = 3S'^{-1}\mathcal{O}_L$. Decomposing $(\alpha-2)S'^{-1}\mathcal{O}_L = \mathfrak{p}_{3,1}^{m_1}\cdots\mathfrak{p}_{3,r}^{m_r}$ we get $(\alpha-2)^3 S'^{-1}\mathcal{O}_L = \mathfrak{p}_{3,1}^{3m_1}\cdots\mathfrak{p}_{3,r}^{3m_r} = 3S'^{-1}\mathcal{O}_L$. Whence equation $3m_1+\ldots+3m_r=3$, whose only possible solutions in natural numbers force $r=1=m_1$. Thus $(\alpha-2)S'^{-1}\mathcal{O}_L = \mathfrak{p}_3$ is maximal. Therefore $(\alpha-2)S'^{-1}A$ is maximal ($\mathcal{O}_L$ is integral over A). Hence $S'^{-1}A$ is a local noetherian domain whose maximal ideal is principal: it is a DVR. Therefore $S'^{-1}A = S'^{-1}\mathcal{O}_L$ and thus $A=\mathcal{O}_L$.

f) Factoring $\beta\mathcal{O}_L = \mathfrak{m}_1^{\nu_1}\ldots\mathfrak{m}_t^{\nu_t}$ we get $N_{L/\mathbb{Q}}(\beta) = N_{L/\mathbb{Q}}(\beta\mathcal{O}_L) = N_{L/\mathbb{Q}}(\mathfrak{m}_1)^{\nu_1}\cdots N_{L/\mathbb{Q}}(\mathfrak{m}_t)^{\nu_t} = p$, which forces $t=1$ and $\nu_1=1$. Notice moreover that the inertia degree of the prime $\beta\mathcal{O}_L$ is $f_{\beta/p}=1$.

g) We have already seen in b) that $7\mathcal{O}_L$ and $13\mathcal{O}_L$ are primes. Since $N_{L/\mathbb{Q}}(\alpha)=11$, we get $\mathfrak{p}_{11,1}=\alpha\mathcal{O}_L$ and from e) we know that $N_{L/\mathbb{Q}}(\alpha-2)=3$, so $\mathfrak{p}_{3,1}=(\alpha-2)\mathcal{O}_L$.

h) is a byproduct of e): $(\alpha-2)^3 = \alpha^3-6\alpha^2+12\alpha-8 = 3(1+4\alpha-2\alpha^2)$ and $3\mathcal{O}_L = (\alpha-2)^3\mathcal{O}_L$.

i) Recall that the Picard group of a Dedekind domain is generated by the classes of prime ideals. If p is a prime number and $\mathfrak{p}$ is a prime dividing $p\mathcal{O}_L$ then $N_{L/\mathbb{Q}} = p^{f(\mathfrak{p}/p)}$. Since primes dividing 3, 7, 11 and 13 are principal, we only need to check the other primes of norm less than 17, namely $\mathfrak{p}_{2,1}$, $\mathfrak{p}_{2,2}$ and $\mathfrak{p}_{5,1}$. Since $2\mathcal{O}_L = \mathfrak{p}_{2,1}\mathfrak{p}_{2,2}$ we have the relation $\mathfrak{p}_{2,2} = \mathfrak{p}_{2,1}^{-1}$ in $\mathrm{Pic}(\mathcal{O}_L)$. From d) we get $N_{L/\mathbb{Q}}(\alpha-1)=10$. The argument used in f) shows that the only possible factorisation of $(\alpha-1)\mathcal{O}_L$ is as the product of two primes of norm 2 and 5 respectively. Therefore $(\alpha-1)\mathcal{O}_L = \mathfrak{p}_{2,1}\mathfrak{p}_{5,1}$, hence $\mathfrak{p}_{5,1} = \mathfrak{p}_{2,1}^{-1}$ in $\mathrm{Pic}(\mathcal{O}_L)$. We conclude that $\mathrm{Pic}(\mathcal{O}_L)$ is cyclic, generated by $\mathfrak{p}_{2,1}$.

j) In a) we have computed the minimal polynomial Y^3-121 of α^2, hence that of α^2-5 is $T^3+15T^2-45T+4=0$, thus $N_{L/\mathbb{Q}}(\alpha^2-5)=-4$. The only possible factorisations are $(\alpha^2-5)\mathcal{O}_L = \mathfrak{p}_{2,1}^2$ or $(\alpha^2-5)\mathcal{O}_L = \mathfrak{p}_{2,2}$. In the second case, since $2\mathcal{O}_L = \mathfrak{p}_{2,1}\mathfrak{p}_{2,2}$, we would get $\mathfrak{p}_{2,1} = \mathfrak{p}_{2,2}^{-1}2\mathcal{O}_L = \frac{2}{\alpha^2-5}\mathcal{O}_L$. Computing the coordinates of $\frac{2}{\alpha^2-5}$ in the basis $\{1,\alpha,\alpha^2\}$ we get

$$\frac{2}{\alpha^2-5} = -\frac{25}{2} - \frac{11}{2}\alpha - \frac{5}{2}\alpha^2 \notin \mathbb{Z}[\alpha] = \mathcal{O}_L.$$

Hence $\mathfrak{p}_{2,1}^2 = (\alpha^2 - 5)\mathcal{O}_L$, so $\mathrm{Pic}(\mathcal{O}_L)$ has order at most 2.

k) If $\mathrm{Pic}(\mathcal{O}_L) = 1$ then $\mathfrak{p}_{2,1}$ is principal, say $\mathfrak{p}_{2,1} = x\mathcal{O}_L$. Since $\mathfrak{p}_{2,1}^2 = (\alpha^2 - 5)\mathcal{O}_L$, we would have $(\alpha^2 - 5) = vx^2$, for some unit $v \in \mathcal{O}_L^\times$. The claim now follows from Dirichlet's theorem.

l) We have established in g) that $\mathfrak{p} = (\alpha - 2)\mathcal{O}_L = \mathfrak{p}_{3,1}$ is the unique prime dividing 3, which is tolally ramified, hence $\mathcal{O}_L/\mathfrak{p} \simeq \mathbb{Z}/3\mathbb{Z}$. Reducing $y = \varepsilon u^\delta(\alpha^2 - 5)$ mod $\mathfrak{p}$ we substitute $\alpha = 2$ in y and in u to get $y \equiv -\varepsilon$. Since the only squares in $\mathbb{Z}/3\mathbb{Z}$ are 0 and 1, we conclude $\varepsilon = -1$.

m) From d) we find $N_{L/\mathbb{Q}}(\alpha + 3) = 38$, so $(\alpha + 3)\mathcal{O}_L = \mathfrak{p}_{2,1} \cdot \mathfrak{q}$ for some prime $\mathfrak{q}$ dividing 19 and such that $f(\mathfrak{q}/19) = 1$. Substituting $\alpha = -3$ in y and in u we get that $y \equiv -(1-12-18)^\delta(9-5) \equiv -4 \cdot 9^\delta$ mod $\mathfrak{q}$. But -1 is not a square mod 19, so y is not a square mod $\mathfrak{q}$ and so it is not a square in $\mathcal{O}_L$. So $\mathrm{Pic}(\mathcal{O}_L)$ is non-trivial, hence cyclic of order 2.

n) We have $(1, -1) = (1, 0) + (0, -1) = \chi(\mathcal{O}_L) + \chi(\mathcal{O}_L/\mathfrak{p}_{2,1})$. From the exact sequence

$$0 \longrightarrow \mathfrak{p}_{2,1} \longrightarrow \mathcal{O}_L \longrightarrow \mathcal{O}_L/\mathfrak{p}_{2,1} \longrightarrow 0.$$

we get that $[\mathfrak{p}_{2,1}] = [\mathcal{O}_L] - [\mathcal{O}_L/\mathfrak{p}_{2,1}]$ in $K_0(\mathcal{O}_L)$, so $\chi([\mathfrak{p}_{2,1}]) = (1, 0) - (0, -1) = (1, -1)$ (don't be fooled by the change between additive and multiplicative notation, -1 is the opposite of -1 in $\{\pm 1\}$).

Exercise 2. a) Define $\varphi : k[X_1, \ldots, X_n, Y_1, \ldots, Y_n] \to k[X_1, \ldots, X_n]/\mathfrak{p}_1 \otimes_k k[Y_1, \ldots, Y_n]/\sigma(\mathfrak{p}_2)$ by $\varphi(X_i) = x_i \otimes 1$ and $\varphi(Y_i) = 1 \otimes y_i$. The kernel of φ clearly contains I, so φ factors through $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/I$. There is also a k-bilinear map $k[X_1, \ldots, X_n] \times k[Y_1, \ldots, Y_n] \to k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/I$ sending $(X_i, 1)$ to x_i and $(1, Y_i)$ to y_i. Its kernel contains $\mathfrak{p}_1 \times \sigma(\mathfrak{p}_2)$, whence a bilinear map $(k[X_1, \ldots, X_n]/\mathfrak{p}_1) \times (k[Y_1, \ldots, Y_n]/\sigma(\mathfrak{p}_2)) \to k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/I$, which is clearly an inverse to φ.

b) By the normalisation lemma, $k[X_1, \ldots, X_n]/\mathfrak{p}_1$ is finite over $k[T_1, \ldots, T_{n-h_1}]$ and $k[Y_1, \ldots, Y_n]/\sigma(\mathfrak{p}_2)$ is finite over $k[S_1, \ldots, S_{n-h_2}]$. Through φ, we deduce that $k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/I$ is finite over $k[T_1, \ldots, T_{n-h_1}, S_1, \ldots, S_{n-h_2}]$.

c)

$$\begin{aligned} k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(I + D) &\simeq (k[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/I) \,/\, (I + D/I) \\ &\simeq k[X_1, \ldots, X_n]/\, (\mathfrak{p}_1 + \mathfrak{p}_2) \end{aligned}$$

d) $\widetilde{\mathfrak{q}}/I$ is a minimal prime ideal containing n elements, the classes of $X_i - Y_i$ mod I for $i = 1, \ldots, n$. Hence $\mathrm{ht}\, \widetilde{\mathfrak{q}}/I \leq n$.

e) By the isomorphism in c), we have

$$\begin{aligned} \dim k[X_1, \ldots, X_n]/\mathfrak{q} &= \dim k[X_1, \ldots, X_n, Y_1 \ldots, Y_n]/\widetilde{\mathfrak{q}} \\ &= \dim\left((k[X_1, \ldots, X_n, Y_1 \ldots, Y_n]/I) \,/\, (\widetilde{\mathfrak{q}}/I)\right) \\ &= \dim\left(k[X_1, \ldots, X_n, Y_1 \ldots, Y_n]/I\right) - \mathrm{ht}\, \widetilde{\mathfrak{q}}/I \\ &= \dim k[X_1, \ldots, X_n]/\mathfrak{p}_1 + \dim k[Y_1, \ldots, Y_n]/\mathfrak{p}_2 - \mathrm{ht}\, \widetilde{\mathfrak{q}}/I \\ &= n - \mathrm{ht}\, \mathfrak{p}_1 + n - \mathrm{ht}\, \mathfrak{p}_2 - \mathrm{ht}\, \widetilde{\mathfrak{q}}/I \\ &\geq 2n - \mathrm{ht}\, \mathfrak{p}_1 - \mathrm{ht}\, \mathfrak{p}_2 - n \\ &= n - \mathrm{ht}\, \mathfrak{p}_1 - \mathrm{ht}\, \mathfrak{p}_2. \end{aligned}$$

Hence $\mathrm{ht}\, \mathfrak{q} = \dim k[X_1, \ldots, X_n] - \dim k[X_1, \ldots, X_n]/\mathfrak{q} \leq n - (n - \mathrm{ht}\, \mathfrak{p}_1 - \mathrm{ht}\, \mathfrak{p}_2) = \mathrm{ht}\, \mathfrak{p}_1 + \mathrm{ht}\, \mathfrak{p}_2$.

f) Let $\mathfrak{p}_i = I(V_i)$. then $I(W)$ is a minimal prime containing $\mathfrak{p}_1 + \mathfrak{p}_2$. We conclude by e).