Commutative Algebra Exam

Padova, 18/02/2016

Exercise 1. Let $\mathbb{Q} \subset K$ be a field extension, $[K : \mathbb{Q}] = n$. Let $\sigma_1, \dots, \sigma_n$ the distinct embeddings of K in $\overline{\mathbb{Q}}$. We assume the following standard result: for any $x \in K$, $\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{i=1}^{n} \sigma_i(x)$.

a) For $x_1, \dots, x_n \in K$, show that $\Delta_{K/\mathbb{Q}}(x_1, \dots, x_n) = \det(\sigma_i(x_j))^2$.

b) Expand $\det(\sigma_i(x_j))$ (in $\overline{\mathbb{Q}}$) as a sum of terms with plus and minus sign. Call P (resp. N) the sum of the terms with plus (resp. minus) sign, so that $\det(\sigma_i(x_j)) = P - N$. Show that $P + N$ and PN are rational numbers.

c) If $x_1, \dots, x_n \in \mathcal{O}_K$, show that $\Delta_{K/\mathbb{Q}}(x_1, \dots, x_n) \equiv 0$ or $1 \bmod 4$

Exercise 2. Let R be a noetherian ring, $I, J \subseteq R$ ideals.

a) Show that if $J \subseteq \sqrt{I}$ then there exists an $n \in \mathbb{N}$ such that $J^n \subseteq I$.

b) Show that every nonzero ideal in R contains a product of nonzero prime ideals.

Exercise 3. Let R be a Dedekind domain and $S \subseteq R$ a subring with $\mathrm{Frac}\, S = \mathrm{Frac}\, R = K$. We assume that S is noetherian and R is the integral closure of S in K. Fix a prime ideal $\mathfrak{p}$ in S.

a) Show that $\dim S = 1$.

b) Let $0 \neq b \in \mathfrak{p}$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ (not necessarily distinct) nonzero prime ideals of S such that $\prod_{i=1}^{n} \mathfrak{q}_i \subseteq bS$. Show that $\mathfrak{q}_i = \mathfrak{p}$ for some $1 \leq i \leq n$.

c) With notation as in b), assume that $\mathfrak{a} = \prod_{j \neq i} \mathfrak{q}_j \nsubseteq bS$. Show that there exists $a \in \mathfrak{a}$, $a \notin bS$, such that $a\mathfrak{p} \subseteq bS$.

d) Conclude that there exists $x \in K$, $x \notin S$, such that $x\mathfrak{p} \subseteq S$.

e) With notation as in d), show that $\mathfrak{p}$ is an invertible prime if and only if $x\mathfrak{p} \nsubseteq \mathfrak{p}$, and in this case compute the inverse $\mathfrak{p}^{-1}$. [Hint: consider the ideal $\mathfrak{p} + x\mathfrak{p}$]

f) Suppose $S = R$ and let $I \subseteq R$ be an ideal. Show that $I \subseteq \mathfrak{p}$ if and only if $xI \subseteq R$.

g) With notation as in f), show that $v_{\mathfrak{p}}(I) = \max\{\nu \in \mathbb{N} \mid x^\nu I \subseteq R\}$.

h) Let $S = \mathbb{Z}[\sqrt{-3}]$. Show that $(7, 2 + \sqrt{-3})$ is an invertible prime ideal in S.

i) Show that $(2, 1 + \sqrt{-3})$ is a non-invertible prime ideal in $S = \mathbb{Z}[\sqrt{-3}]$.

Exercise 4. Let $R \subseteq A$ be an integral extension of noetherian rings. Show that R is artinian if an only if A is artinian.

You may submit your answers in english, french or italian.

Solutions[1]

Exercise 1. $\Delta_{K/\mathbb{Q}}(x_1,\ldots,x_n) = \det\left(\mathrm{Tr}_{K/\mathbb{Q}}(x_ix_j)\right) = \det\left(\sum_{k=1}^n \sigma_k(x_ix_j)\right) = \det\left(\sum_{k=1}^n \sigma_k(x_i)\sigma_k(x_j)\right) = \det\left(\sigma_k(x_i)\right)\det\left(\sigma_k(x_j)\right) = \det\left(\sigma_i(x_j)\right)^2$.

We should prove that $\sigma_k(P+N) = P+N$ and $\sigma_k(PN) = PN$ for all k. This follows from the properties of determinants, since σ_k acts as a permutation on the rows of the matrix $(\sigma_i(x_j))$.

From a) and b) we have $\Delta_{K/\mathbb{Q}}(x_1,\ldots,x_n) = (P-N)^2 = (P+N)^2-4PN$. Since $x_1,\ldots,x_n \in \mathcal{O}_K$, the coefficients $\sigma_i(x_j)$ are elements in $\overline{\mathbb{Q}}$ that are integral over $\mathbb{Z}$, and so are their algebraic expressions P and N. Therefore $P+N$ and PN are integral over $\mathbb{Z}$ and belong to $\mathbb{Q}$. Hence $P+N$, $PN \in \mathbb{Z}$. We conclude, since the only squares mod 4 are 0 and 1.

Exercise 2. Since R is noetherian, J is generated by finitely many elements, say $x_1,\ldots,x_s$. By definition, there exists an integer $m \in \mathbb{N}$ such that $x_i^m \in I$ for $1 \le i \le s$. Then $n = ms+1$ will do: any element in J^n expands as a sum of terms of the form $ax_1^{\nu_1}\cdots x_s^{\nu_s}$ with $\nu_i \ge m$ for at least one i.

We know (eg exercise 4.2) that $\sqrt{I}$ is equal to the intersection of finitely many prime ideals. Say $\sqrt{I} = \bigcap_{i=1}^r \mathfrak{p}_i \supseteq \prod_{i=1}^r \mathfrak{p}_i$. By a) we conclude that $I \supseteq \prod_{i=1}^r \mathfrak{p}_i^n$.

Exercise 3. R is a Dedekind domain, so $\dim R = 1$; it is integral over S so, by proposition 6.1.32, $\dim S = \dim R$.

Since $\prod_{i=1}^n \mathfrak{q}_i \subseteq bS \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is prime, we have $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some $1 \le i \le n$. The inclusion must be an equality because nonzero primes in a ring of dimension 1 are maximal.

Taking any $a \in \mathfrak{a}$, $a \notin bS$ we have $a\mathfrak{p} \subseteq \mathfrak{a}\mathfrak{p} \subseteq bS$.

By exercise 2.b) the ideal bS contains a finite product of nonzero primes $\prod_{i=1}^n \mathfrak{q}_i$. If $\prod_{j\neq k}\mathfrak{q}_j \subseteq bS$ for some $1 \le k \le n$, remove $\mathfrak{q}_k$ from $\prod_{i=1}^n \mathfrak{q}_i$. Repeating the process, we may assume that $\prod_{j\neq h}\mathfrak{q}_j \not\subseteq bS$ for all $1 \le h \le n$. From b) and c) it follows now that there exists an $a \in S$ with $a \notin bS$ but $a\mathfrak{p} \subseteq bS$. Take $x = a/b$.

We have $\mathfrak{p} \subseteq \mathfrak{p} + x\mathfrak{p} \subseteq S$. By maximality of $\mathfrak{p}$ either $\mathfrak{p} + x\mathfrak{p} = S$ or $\mathfrak{p} = \mathfrak{p} + x\mathfrak{p}$. In the first case, $x\mathfrak{p}$ cannot be contained in $\mathfrak{p}$ (otherwise $S = \mathfrak{p}$) and $(S + xS)\mathfrak{p} = \mathfrak{p} + x\mathfrak{p} = S$, so $\mathfrak{p}$ is invertible with $\mathfrak{p}^{-1} = S + xS$. On the other hand, if $\mathfrak{p} = \mathfrak{p} + x\mathfrak{p}$ then $x\mathfrak{p} \subseteq \mathfrak{p}$ and $(S+xS)\mathfrak{p} = \mathfrak{p}$. If $\mathfrak{p}$ were invertible, multiplying this last equality by $\mathfrak{p}^{-1}$ we would get $S + xS = S$, hence $x \in S$: absurd.

If $S = R$ of course $\mathfrak{p}$ is invertible. If $I \subseteq \mathfrak{p}$ then $xI \subseteq x\mathfrak{p} \subseteq R$. If $xI \subseteq R$ then $x\mathfrak{p}I \subseteq \mathfrak{p}$. The prime $\mathfrak{p}$ contains the product of the integral ideals $x\mathfrak{p}$ and I, so must contain one of the two. Since it is invertible, by e) it doesn't contain $x\mathfrak{p}$, so it contains I.

Put $m = \max\{\nu \in \mathbb{N} \mid x^\nu I \subseteq R\}$. We have that $I = \mathfrak{p}^{v_\mathfrak{p}(I)}J$ with $v_\mathfrak{p}(J) = 0$. Therefore $x^{v_\mathfrak{p}(I)}I = (x\mathfrak{p})^{v_\mathfrak{p}(I)}J \subseteq J \subseteq R$ so $v_\mathfrak{p}(I) \le m$. For the converse, notice that from $x\mathfrak{p} \subseteq R$ and $x\mathfrak{p} \not\subseteq \mathfrak{p}$ we conclude that $x\mathfrak{p}R_\mathfrak{p} = R_\mathfrak{p}$, hence $v_\mathfrak{p}(x) = -1$. So if $x^mI \subseteq R$ we get $v_\mathfrak{p}(x^mI) \ge 0$. But $v_\mathfrak{p}(x^mI) = m \cdot v_\mathfrak{p}(x) + v_\mathfrak{p}(I) = -m + v_\mathfrak{p}(I)$. Hence $m \le v_\mathfrak{p}(I)$.

Set $\alpha = \sqrt{-3}$. We have that $S/(7, 2+\alpha) \simeq \mathbb{Z}[X]/(7, X^2+3, X+2) \simeq (\mathbb{Z}/7\mathbb{Z})[X]/(X^2-4, X+2) \simeq (\mathbb{Z}/7\mathbb{Z})[X]/(X+2) \simeq \mathbb{Z}/7\mathbb{Z}$, so $(7, 2+\alpha)$ is prime. Moreover $(7, 2+\alpha) = (2+\alpha)$ because $7 = (2+\alpha)(2-\alpha) = N_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}(2+\alpha)$. Every principal ideal is invertible. We can also apply e) with $x = \frac{1}{2+\alpha}$.

$S/(2, 1+\alpha) \simeq \mathbb{Z}[X]/(2, X^2+3, X+1) \simeq (\mathbb{Z}/2\mathbb{Z})[X]/\left((X+1)^2, X+1\right) \simeq (\mathbb{Z}/2\mathbb{Z})[X]/(X+1) \simeq \mathbb{Z}/2\mathbb{Z}$, so $\mathfrak{p} = (2, 1+\alpha)$ is prime. Recalling that $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, for $x = \frac{1-\alpha}{2} \notin S$ we get $x \cdot 2 = 1-\alpha = 2-(1+\alpha) \in \mathfrak{p}$ and $x \cdot (1+\alpha) = 2 \in \mathfrak{p}$. By e), the prime $\mathfrak{p}$ is not invertible.

Exercise 4. By proposition 6.1.32, $\dim R = \dim A$. The claim now follows from corollary 6.1.7.

[1]References to the lecture notes are numbered as in the version of 21/01/2016.