Commutative Algebra Exam
Padova, 23/07/2015

Exercise 1. Let R be a ring, $\mathfrak{a}, \mathfrak{b} \subset R$ ideals and assume that the rings $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are noetherian. Let $\mathfrak{c} \subset R$ be an ideal such that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{c}$.

a) Show that $\mathfrak{c}/(\mathfrak{a} \cap \mathfrak{c})$ and $(\mathfrak{a} \cap \mathfrak{c})/(\mathfrak{a} \cap \mathfrak{b})$ are finitely generated R-modules.

b) Show that $R/(\mathfrak{a} \cap \mathfrak{b})$ is a noetherian ring.

c) Show that if $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are artinian rings, then $R/(\mathfrak{a} \cap \mathfrak{b})$ is an artinian ring.

Let moreover M be a finitely generated R-module.

d) Show that if $R/\mathrm{Ann}\,(M)$ is a noetherian (resp. artinian) ring, then M is a noetherian (resp. artinian) R-module.

e) Let $m_1, \ldots, m_r$ be generators of M. Show that $\mathrm{Ann}\,(M) = \bigcap_{i=1}^{r} \mathrm{Ann}\,(m_i)$.

f) Show that if M is a noetherian (resp. artinian) R-module then $R/\mathrm{Ann}\,(M)$ is a noetherian (resp. artinian) ring.

Exercise 2. Let $L = \mathbb{Q}(\alpha)$, where $\alpha^3 = 20$. Let $A = \mathbb{Z}[\alpha]$ and write $\mathcal{O}_L$ for the integral closure of $\mathbb{Z}$ in L.

a) Compute $\Delta(1, \alpha, \alpha^2)$.

b) Describe the decompostion of 7, 11 and 13 in L and compute the inertia and ramification degree above these primes.

c) Let $A_3 = (\mathbb{Z} - 3\mathbb{Z})^{-1}A$ and $\mathcal{O}_{L,3} = (\mathbb{Z} - 3\mathbb{Z})^{-1}\mathcal{O}_L$. Show that $(\alpha - 2)A_3$ is a maximal ideal. Show that $A_3 = \mathcal{O}_{L,3}$ and compute the decomposition of 3 in L.

d) Let $A_5 = (\mathbb{Z} - 5\mathbb{Z})^{-1}A$ and $\mathcal{O}_{L,5} = (\mathbb{Z} - 5\mathbb{Z})^{-1}\mathcal{O}_L$. Show that αA_5 is a maximal ideal. Show that $A_5 = \mathcal{O}_{L,5}$ and compute the decomposition of 5 in L.

e) Is $\mathcal{O}_L$ a free $\mathbb{Z}$-module? If so, compute a $\mathbb{Z}$-basis. Is $A = \mathcal{O}_L$?

f) Compute the decomposition of 2 in L.

Exercise 3. Let R be an integrally closed noetherian domain, K its fraction field. For any prime ideal $\mathfrak{p} \subset R$ of height 1, let $v_{\mathfrak{p}}$ be the discrete valuation of $R_{\mathfrak{p}}$. Fix $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ distinct prime ideals of height 1 in R.

a) Let $S = \bigcap_{i=1}^{r}(R - \mathfrak{p}_i)$ and $R' = S^{-1}R$. Put $\mathfrak{p}'_i = \mathfrak{p}_i R'$. Show that R' is a semi-local ring and that $\mathfrak{p}'_1, \ldots, \mathfrak{p}'_r$ are its maximal ideals.

b) Show that R' is a PID.

c) For any $n_1, \ldots, n_r \in \mathbb{N}$, show that the following system has a nonzero solution $x \in R$:

$$v_{\mathfrak{p}_i}(x) \geq n_i \quad i = 1, \ldots, r; \qquad v_{\mathfrak{q}}(x) \geq 0 \quad \forall \mathfrak{q} \notin \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}.$$

You can submit your answers in english, french or italian.

Solutions

Exercise 1. We have $\mathfrak{c}/(\mathfrak{a}\cap\mathfrak{c}) \cong (\mathfrak{a}+\mathfrak{c})/\mathfrak{a}$ and the latter, as an ideal of the noetherian ring $R/\mathfrak{a}$, is finitely generated, as an $R/\mathfrak{a}$-module and thus as an R-module. On the other hand, $(\mathfrak{a}\cap\mathfrak{c})/(\mathfrak{a}\cap\mathfrak{b})$ is an R-submodule of $\mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b}) \cong (\mathfrak{a}+\mathfrak{b})/\mathfrak{b}$, an ideal in the noetherian ring $R/\mathfrak{b}$. Moreover, every R-submodule of $R/\mathfrak{b}$ is an $R/\mathfrak{b}$-module. We conclude that $(\mathfrak{a}\cap\mathfrak{c})/(\mathfrak{a}\cap\mathfrak{b})$ is finitely generated as $R/\mathfrak{b}$-module and thus as R-module.

From a) we conclude that $\mathfrak{c}/(\mathfrak{a}\cap\mathfrak{b}) \cong (\mathfrak{c}/(\mathfrak{a}\cap\mathfrak{c}))\,/\,((\mathfrak{a}\cap\mathfrak{c})/(\mathfrak{a}\cap\mathfrak{b}))$ is finitely generated. Since every ideal of $R/(\mathfrak{a}\cap\mathfrak{b})$ is of the form $\mathfrak{c}/(\mathfrak{a}\cap\mathfrak{b})$ for a suitable $\mathfrak{c}\subseteq R$, we conclude that R is noetherian.

We have an exact sequence $0 \longrightarrow \mathfrak{a}/(\mathfrak{a}\cap\mathfrak{b}) \longrightarrow R/(\mathfrak{a}\cap\mathfrak{b}) \longrightarrow R/\mathfrak{a} \longrightarrow 0$. Being artinian, $R/\mathfrak{a}$ has a composition series (as an $R/\mathfrak{a}$-module and thus as an $R/(\mathfrak{a}\cap\mathfrak{b})$-module. Also $\mathfrak{a}/(\mathfrak{a}\cap\mathfrak{c}) \cong (\mathfrak{a}+\mathfrak{a})/\mathfrak{b}$ has a composition series, as an $R/\mathfrak{b}$-module and thus as an $R/(\mathfrak{a}\cap\mathfrak{b})$-module. We conclude that $R/(\mathfrak{a}\cap\mathfrak{b})$ has a composition series, and is therefore artinian.

If $x\in R$, $xm_i = 0$ for $i = 1,\dots,r$ then for every $m = a_1m_1+\ldots+a_rm_r \in M$ we get $xm = x(a_1m_1+\ldots+a_rm_r) = a_1 0+\ldots+a_r 0 = 0$, so $\bigcap_{i=1}^r \operatorname{Ann}(m_i) \subseteq \operatorname{Ann}(M)$. The reverse inclusion is trivial.

If $R/\operatorname{Ann}(M)$ is a noetherian (resp. artinian) ring and M is a finitely generated R-module, it is finitely generated as an $R/\operatorname{Ann}(M)$-module and thus noetherian (resp. artinian), as an $R/\operatorname{Ann}(M)$-module and therefore as an R-module. For the converse, notice that $R/\operatorname{Ann}(m_i) \cong Rm_i$ is a noetherian (resp. artinian) ring. We now conclude by b) (resp. c)).

Exercise 2. The minimal polynomial of α if X^3-20, hence that of α^2 is Y^3-400. Therefore $\Delta(1,\alpha,\alpha^2) = -N_{L/\mathbb{Q}}(3\alpha^2) = -3^3N_{L/\mathbb{Q}}(\alpha^2) = -2^4 3^3 5^2$. Only 2, 3 and 5 may ramify, and 3 ramifies for sure, since it appears with odd exponent. Moreover, $A\left[\frac{1}{30}\right] = \mathcal{O}_L\left[\frac{1}{30}\right]$.

$X^3-20 \equiv X^3+1 \equiv (X-3)(X-5)(X+1) \bmod 7$, splits completely in L.

$X^3-20 \equiv X^3-64 = (X-4)(X^2+4X+16) \bmod 11$, and the quadratic factor is irreducible. Therefore $11\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, with $e_1 = e_2 = f_1 = 1$ and $f_1 = 2$.

$X^3-20 \equiv X^3-7 \bmod 13$, which is irreducible. Therefore 13 is inert in L.

The minimal polynomial of $\alpha-2$ is $(Y+2)^3-20 = Y^3+6Y^2+12Y-12$, Eisenstein with respect to 3. The reduction mod 3 of this polynomial is Y^3, so by exercise 5, sheet 5 we have that A_3 is local with maximal ideal $\mathfrak{m} = (\alpha-2, 3)$, and in fact $\mathfrak{m} = (\alpha-2)A_3$ because $3 = \frac{1}{4}\left((\alpha-2)^3+6(\alpha-2)^2+12(\alpha-2)\right) \in (\alpha-2)A_3$. So A_3 is a DVR, because it is local noetherian with maximal ideal generated by a non-nilpotent element. Hence $A_3 = \mathcal{O}_{L,3}$. We know that 3 ramifies and there is only one prime above it, so it is totally ramified.

X^3-20 is an Eisenstein polynomial with respect to 5. We have $\alpha^3\mathcal{O}_{L,5} = 5\mathcal{O}_{L,5}$. Let's decompose $\alpha\mathcal{O}_{L,5} = \mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_s^{n_s}$. Then $\alpha^3\mathcal{O}_{L,5} = \mathfrak{m}_1^{3n_1}\cdots\mathfrak{m}_s^{3n_s} = 5\mathcal{O}_{L,5}$. We get an equation $3n_1+\ldots+3n_s = 3$, whose only possible solutions in natural numbers force $s = 1 = n_1$. Thus $\alpha\mathcal{O}_{L,5} = \mathfrak{m}_1$ is maximal. Therefore αA_5 is maximal ($\mathcal{O}_L$ is integral over A). Hence A_5 is a local noetherian domain whose maximal ideal is principal: it is a DVR. Therefore $A_5 = \mathcal{O}_{L,5}$.

$\mathbb{Z}$ is a PID and $\mathcal{O}_L$ is a torsion-free module, so $\mathcal{O}_L$ is free. From what we have computed so far, we know $A\left[\frac{1}{2}\right] = \mathcal{O}_L\left[\frac{1}{2}\right]$. We look for elements in $\mathcal{O}_L$ of the form $\frac{1}{2}(a+b\alpha+c\alpha^2)$, with $a,b,c \in \{0,1\}$. Since $\operatorname{Tr}_{L/\mathbb{Q}}(\alpha) = \operatorname{Tr}_{L/\mathbb{Q}}(\alpha^2) = 0$, we get $\operatorname{Tr}_{L/\mathbb{Q}}\left(\frac{1}{2}(a+b\alpha+c\alpha^2)\right) = \frac{3a}{2}$, an integer only for $a = 0$. We also have $N_{L/\mathbb{Q}}(\frac{\alpha}{2}) = \frac{20}{8} \notin \mathbb{Z}$, but $N_{L/\mathbb{Q}}(\frac{\alpha^2}{2}) = \frac{400}{8} = 50$. Indeed, $\frac{\alpha^2}{2} \in \mathcal{O}_L$, since it is a root of Y^3-50.

In order to conclude that $\{1, \alpha, \frac{\alpha^2}{2}\}$ is a basis for $\mathcal{O}_L$, we have two possibilities. The first is to check elements $\frac{1}{4}(a + b\alpha + c\alpha^2)$, with $a, b, c \in \{0, 1, 2, 3\}$. Again the trace forces $a = 0$ and $N_{L/\mathbb{Q}}(\frac{b\alpha}{4}) = \frac{5b^3}{16} \notin \mathbb{Z}$, so there are no solutions for $c = 0$. Moreover $N_{L/\mathbb{Q}}(\frac{b\alpha + c\alpha^2}{4}) = \frac{5}{16} N_{L/\mathbb{Q}}(b + c\alpha)$. For $c \neq 0$, the minimal polynomial $P(Z)$ of $\varepsilon = b + c\alpha$ over $\mathbb{Q}$ is computed substituting $\alpha = \frac{\varepsilon - b}{c}$:

$$P(Z) = c^3 \left[\left(\frac{Z - b}{c} \right)^3 - 20 \right] = Z^3 - 3bcZ^2 + 3c^2b^2Z - b^3c^3 - 20c^3.$$

Hence $N_{L/\mathbb{Q}}(b + c\alpha) = c^3(b^3 + 20)$. For $b, c \in \{0, 1, 2, 3\}$, this is divisible by 16 only for $c = 2$ and $b = 0$ or $b = 2$.

As an alternative, we may consider $L = \mathbb{Q}(\frac{\alpha^2}{2})$. We have already noticed that the minimal polynomial of $\beta = \frac{\alpha^2}{2}$ is $Y^3 - 50$, which is Eisenstein in 2. Taking $B = \mathbb{Z}[\beta]$ and $T = \mathbb{Z} - 2\mathbb{Z}$, we may proceed as in d) and get $B_2 = \mathcal{O}_{L,2}$. Therefore $\{1, \beta, \beta^2\} = \{1, \frac{\alpha^2}{2}, 5\alpha\}$ is a basis for $\mathcal{O}_{L,2}$, thus $\{1, \alpha, \frac{\alpha^2}{2}\}$ is also a $\mathbb{Z}_{(2)}$-basis of $\mathcal{O}_{L,2}$. Therefore $M = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\frac{\alpha^2}{2} \subseteq \mathcal{O}_L$ and $M_{(p)} \cong (\mathcal{O}_L)_{(p)}$ for all primes, so they coincide.

Exercise 3. We know that the prime ideals in R' are in bijection with the primes $\mathfrak{q} \subset R$ such that $\mathfrak{q} \cap S = \varnothing$. Since $S = R - \bigcup_{i=1}^r \mathfrak{p}_i$, this is the same as saying that $\mathfrak{q} \subseteq \bigcup_{i=1}^r \mathfrak{p}_i$. By the prime avoidance lemma (6.1.16), we conclude that any such prime is contained in one of the $\mathfrak{p}_i$.

Since $\dim R'_{\mathfrak{p}'_i} = \dim R_{\mathfrak{p}_i} = \operatorname{ht} \mathfrak{p}_i = 1$ and the $\mathfrak{p}'_i$ are its maximal ideals, the ring R' is of dimension 1. As a localisation of R, it is noetherian, integral and integrally closed. Therefore R' is a Dedekind domain. It is also semi-local: by exercise 4, sheet 7, it is a PID.

Let $\frac{x}{s}$ be a generator of $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r} R'$. Clearly x is a solution to the system.