Commutative Algebra Exam

Padova, 24/01/2015

Exercise 1. Let $L = \mathbb{Q}(\alpha)$, where $\alpha^3 = 12$. Let $A = \mathbb{Z}[\alpha]$ and write $\mathcal{O}_L$ for the integral closure of $\mathbb{Z}$ in L.

a) Compute $\Delta(1, \alpha, \alpha^2)$.

b) Describe the decompostion of 5, 7 and 13 in L and compute the inertia and ramification degree above these primes.

c) Let $S = \mathbb{Z} - 3\mathbb{Z}$. Show that $\alpha S^{-1}A$ is a maximal ideal. Show that $S^{-1}A = S^{-1}\mathcal{O}_L$ and compute the decomposition of 3 in L.

d) Is $\mathcal{O}_L$ a free $\mathbb{Z}$-module? If so, compute a $\mathbb{Z}$-basis. Is $A = \mathcal{O}_L$?

c) Compute the decomposition of 2 in L.

Exercise 2. Let R be a ring and M a finitely generated non-zero R-module. Let $\Sigma(M) = \{\mathrm{Ann}\,(m);\ m \in M\}$ the set of proper ideals in R which are annihilators of some element in M.

a) Show that any maximal element in $\Sigma(M)$ is a prime ideal.

b) From now on, assume that R is noetherian. Show that Σ contains maximal elements.

c) Denote $\Pi(M) \subseteq \Sigma(M)$ the subset of prime ideals. For $\mathfrak{p} \in \Pi(M)$, let $\varphi_\mathfrak{p} : M \to M_\mathfrak{p}$ be the canonical map $\varphi_\mathfrak{p}(m) = \frac{m}{1}$. Show that $\bigcap_{\mathfrak{p}\in\Pi(m)} \ker \varphi_\mathfrak{p} = 0$.

d) Show that there exists a finite sequence of submodules $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$, for $1 \leq i \leq n$.

e) Show that if $0 \neq m \in M$ has $\mathrm{Ann}\,(m) \in \Pi(M)$ and $0 \neq n \in Rm$, then $\mathrm{Ann}\,(n) = \mathrm{Ann}\,(m)$.

f) Show that if $0 \to M' \to M \to M'' \to 0$ is short exact, then $\Pi(M) \subseteq \Pi(M') \cup \Pi(M'')$. [Hint: show that if $m \in M$ has $\mathrm{Ann}\,(m) \in \Pi(M) - \Pi(M')$, then $Rm \cap M' = 0$]

g) Show that $\Pi(M)$ is a finite set.

h) Let $I \subset R$ be an ideal. Suppose that for any $x \in I$ there exists $0 \neq m_x \in M$ such that $xm_x = 0$. Show that there exists $0 \neq m \in M$ such that $xm = 0$ for all $x \in I$.

For exercise 3, recall the following facts. Let R be a local noetherian integrally closed domain, with maximal ideal $\mathfrak{m}$. Let $\mathfrak{m}' = \{x \in Frac(R)\,|\, xy \in R\ \forall y \in \mathfrak{m}\}$. Then $\mathfrak{m} \subseteq \mathfrak{m}\mathfrak{m}' \subseteq R$. Moreover, if $\mathfrak{m}\mathfrak{m}' = R$, then $\mathfrak{m}$ is invertible (*claim 1*); if $\mathfrak{m}\mathfrak{m}' = \mathfrak{m}$, then $\mathfrak{m}' = R$ (*claim 2*).

Exercise 3. Let R be an integrally closed noetherian domain. For $a, b \in R$, let $\overline{a} \in R/bR$ be the class of a mod b.

a) Show that if $b \neq 0$ then $\mathrm{Ann}\,(\overline{a}) \neq 0$ and that $a \in bR \iff \mathrm{Ann}\,(\overline{a}) = R$.

b) Let $b \in R$ and $\overline{x} \in R/bR$ such that $0 \neq \mathfrak{p} = \mathrm{Ann}\,(\overline{x})$ is a prime. Show that $R_\mathfrak{p}$ is a DVR.

c) Show that $R = \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} R_\mathfrak{p}$.

You may submit your answers in english, french or italian.

Exercise 1. The minimal polynomial of α if X^3-12, hence that of α^2 is Y^3-144. Therefore $\Delta(1,\alpha,\alpha^2) = -N_{L/\mathbb{Q}}(3\alpha^2) = -3^3 N_{L/\mathbb{Q}}(\alpha^2) = -2^4 3^5$. Only 2 and 3 may ramify, and 3 ramifies for sure, since it appears with odd exponent. Moreover, $A\left[\frac{1}{6}\right] = \mathcal{O}_L\left[\frac{1}{6}\right]$.

$X^3-12 \equiv X^3-2 \equiv (X-3)(X^2+3X+9) \bmod 5$, and the quadratic factor is irreducible. Therefore $5\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, with $e_1=e_2=f_1=1$ and $f_1=2$.

$X^3-12 \equiv X^3-5 \bmod 7$, which is irreducible. Therefore 7 is inert in L.

$X^3-12 \equiv X^3+1 \equiv (X+1)(X+3)(x-4) \bmod 13$. Therefore 13 splits completely in L.

X^3-12 is an Eisenstein polynomial with respect to 3. We have $\alpha^3 S^{-1}\mathcal{O}_L = 3S^{-1}\mathcal{O}_L$. Let's decompose $\alpha S^{-1}\mathcal{O}_L = \mathfrak{m}_1^{n_1}\cdots\mathfrak{m}_s^{n_s}$. Then $\alpha^3 S^{-1}\mathcal{O}_L = \mathfrak{m}_1^{3n_1}\cdots\mathfrak{m}_s^{3n_s} = 3S^{-1}\mathcal{O}_L$. We get an equation $3n_1+\ldots+3n_s=3$, whose only possible solutions in natural numbers force $s=1=n_1$. Thus $\alpha S^{-1}\mathcal{O}_L = \mathfrak{m}_1$ is maximal. Therefore $\alpha S^{-1}A$ is maximal ($\mathcal{O}_L$ is integral over A). Hence $S^{-1}A$ is a local noetherian domain whose maximal ideal is principal: it is a DVR. Therefore $S^{-1}A = S^{-1}\mathcal{O}_L$.

$\mathbb{Z}$ is a PID and $\mathcal{O}_L$ is a torsion-free module, so $\mathcal{O}_L$ is free. From what we have computed so far, we know $A\left[\frac{1}{2}\right] = \mathcal{O}_L\left[\frac{1}{2}\right]$. We look for elements in $\mathcal{O}_L$ of the form $\frac{1}{2}(a+b\alpha+c\alpha^2)$, with $a,b,c\in\{0,1\}$. Since $\mathrm{Tr}_{L/\mathbb{Q}}(\alpha) = \mathrm{Tr}_{L/\mathbb{Q}}(\alpha^2) = 0$, we get $\mathrm{Tr}_{L/\mathbb{Q}}\left(\frac{1}{2}(a+b\alpha+c\alpha^2)\right) = \frac{3a}{2}$, an integer only for $a=0$. We also have $N_{L/\mathbb{Q}}(\frac{\alpha}{2}) = \frac{12}{8} \notin \mathbb{Z}$, but $N_{L/\mathbb{Q}}(\frac{\alpha^2}{2}) = \frac{144}{8} = 18$. Indeed, $\frac{\alpha^2}{2} \in \mathcal{O}_L$, since it is a root of Y^3-18.

In order to conclude that $\{1,\alpha,\frac{\alpha^2}{2}\}$ is a basis for $\mathcal{O}_L$, we have two possibilities. The first is to check elements $\frac{1}{4}(a+b\alpha+c\alpha^2)$, with $a,b,c\in\{0,1,2,3\}$. Again the trace forces $a=0$ and $N_{L/\mathbb{Q}}(\frac{b\alpha}{4}) = \frac{3b^3}{16} \notin \mathbb{Z}$, so there are no solutions for $c=0$. Moreover $N_{L/\mathbb{Q}}(\frac{b\alpha+c\alpha^2}{4}) = \frac{3}{16}N_{L/\mathbb{Q}}(b+c\alpha)$. For $c\neq 0$, the minimal polynomial $P(T)$ of $\varepsilon = b+c\alpha$ over $\mathbb{Q}$ is computed substituting $\alpha = \frac{\varepsilon-b}{c}$:

$$P(Z) = c^3\left[\left(\frac{Z-b}{c}\right)^3 - 12\right] = Z^3 - 3bcZ^2 + 3c^2b^2Z - b^3c^3 - 12c^3.$$

Hence $N_{L/\mathbb{Q}}(b+c\alpha) = c^3(b^3+12)$. For $b,c\in\{0,1,2,3\}$, this is divisible by 16 only for $c=2$ and $b=0$ or $b=2$.

As an alternative, we may consider $L = \mathbb{Q}(\frac{\alpha^2}{2})$. We have already noticed that the minimal polynomial of $\beta = \frac{\alpha^2}{2}$ is Y^3-18, which is Eisenstein in 2. Taking $B=\mathbb{Z}[\beta]$ and $T=\mathbb{Z}-2\mathbb{Z}$, we may proceed as in c) and get $T^{-1}B = T^{-1}\mathcal{O}_L$. Therefore $\{1,\beta,\beta^2\} = \{1,\frac{\alpha^2}{2},3\alpha\}$ is a basis for $\mathcal{O}_L\left[\frac{1}{3}\right]$, thus $\{1,\alpha,\frac{\alpha^2}{2}\}$ is also a $\mathbb{Z}_{(2)}$-basis of $T^{-1}\mathcal{O}_L$. Therefore $M = \mathbb{Z}\oplus\mathbb{Z}\alpha\oplus\mathbb{Z}\frac{\alpha^2}{2} \subseteq \mathcal{O}_L$ and $M_{(p)} \cong (\mathcal{O}_L)_{(p)}$ for all primes, so they coincide.

Exercise 2. Let $\mathrm{Ann}\,(m)\in\Sigma(M)$ be a maximal element, $xy\in\mathrm{Ann}\,(m)$ but $y\notin\mathrm{Ann}\,(m)$. Then $xym=0$ but $ym\neq 0$. Thus $\mathrm{Ann}\,(m)\subseteq\mathrm{Ann}\,(ym)\subsetneq R$. By maximality, we conclude $x\in\mathrm{Ann}\,(ym)=\mathrm{Ann}\,(m)$.

Since R is noetherian, any chain $\mathrm{Ann}\,(m_1)\subseteq\mathrm{Ann}\,(m_2)\subseteq\ldots$ in $\Sigma(M)$ is stationary, hence by Zorn's lemma $\Sigma(M)$ contains maximal elements.

Suppose $m\neq 0$. Then $\mathrm{Ann}\,(m)\in\Sigma(M)$ is contained in some $\mathfrak{p}\in\Pi(M)$. Then $\varphi_{\mathfrak{p}}(m)\in M_{\mathfrak{p}}$ is not zero, otherwise there would be $s\in R-\mathfrak{p}\subseteq R-\mathrm{Ann}\,(m)$ such that $sm=0$, which is absurd.

By b), $\Sigma(M)\neq\varnothing$ and by a) there exists $0\neq m_1\in M$ such that $\mathfrak{p}_1=\mathrm{Ann}\,(m_1)$ is a prime ideal. Therefore $M_1 = Rm_1 \simeq R/\mathfrak{p}_1$. Consider $\pi_1 : M \twoheadrightarrow M/M_1$ and repeat the construction: get

$\overline{m}_2 \in M/M_1$ such that $\mathfrak{p}_2 = \mathrm{Ann}\,(\overline{m}_2)$ is prime. Put $\overline{M}_2 = R\overline{m}_2 \subsetneq M/M_1$ and $M_2 = \pi_1^{-1}(\overline{M}_2)$. Notice $M_1 \subseteq M_2$ and $M_2/M_1 \simeq \overline{M}_2 \simeq R/\mathfrak{p}_2$. We can iterate again and again, producing an ascending chain $M_1 \subsetneq M_2 \subsetneq \ldots$ of submodules of M, which must eventually stop, since R is noetherian.

Let $m \in M$ with $\mathrm{Ann}\,(m) \in \Pi(M)$. For $0 \neq n \in Rm$, since $n = am$, we have $\mathrm{Ann}\,(m) \subseteq \mathrm{Ann}\,(n)$. We have $a \notin \mathrm{Ann}\,(m)$ (since $n \neq 0$), therefore if $x \in \mathrm{Ann}\,(am)$, then $xam = 0$, so $ax \in \mathrm{Ann}\,(m)$, hence $x \in \mathrm{Ann}\,(m)$ because $\mathrm{Ann}\,(m)$ is prime. We conclude that $\mathrm{Ann}\,(n) = \mathrm{Ann}\,(m)$.

Assume $\mathrm{Ann}\,(m) \notin \Pi(M')$ and let $m' \in Rm \cap M'$. If $m' \neq 0$, by e), we have $\mathrm{Ann}\,(m') = \mathrm{Ann}\,(m) \notin \Pi(M')$, a contradiction. Hence $Rm \cap M' = 0$. Therefore Rm is isomorphic to its image in M'' and so $\mathrm{Ann}\,(m) \in \Pi(M'')$.

If $M = Rm$ for some m such that $\mathrm{Ann}\,(m) \in \Pi(M)$, then e) implies that $\Pi(M) = \{\mathrm{Ann}\,(m)\}$. From d), we get a sequence $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$, whence $\Pi(M_i/M_{i-1}) = \{\mathfrak{p}_i\}$. Applying f) recursively, we get

$$\begin{aligned}\Pi(M) &\subseteq \Pi(M_{n-1}) \cup \Pi\left(M_n/M_{n-1}\right) \\ &\subseteq \Pi(M_{n-2}) \cup \Pi\left(M_{n-1}/M_{n-2}\right) \cup \Pi\left(M_n/M_{n-1}\right) \\ &\subseteq \ldots \\ &\subseteq \bigcup_{i=1}^{n} \Pi\left(M_i/M_{i-1}\right) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}\,.\end{aligned}$$

By a) and b) we get $I \subseteq \bigcup_{\mathfrak{p}\in\Pi(M)} \mathfrak{p}$, and by g) this union is finite. By a result seen in class (*prime avoidance*), I must be contained in one of these primes, which by definition is the annihilator of some nonzero element.

Exercise 3. Since $b \in \mathrm{Ann}\,(\overline{a})$, the first claim is trivial. If $a \in bR$, then $\overline{a} = 0$ and $\mathrm{Ann}\,(0) = R$. If $\mathrm{Ann}\,(\overline{a}) = R$, then $\overline{a} = 1\overline{a} = 0$ so $a \in bR$.

Work in $R_\mathfrak{p}$, thus $\mathfrak{m} = \mathfrak{p}R_\mathfrak{p}$. We want to show that $\mathfrak{m}\mathfrak{m}' \neq \mathfrak{m}$. Suppose $\mathfrak{m}\mathfrak{m}' = \mathfrak{m}$ and thus $\mathfrak{m}' = R_\mathfrak{p}$. Let $x \in R$ be a lifting of $\overline{x} \in R/bR$. By definition, $\mathfrak{p} = \mathrm{Ann}\,(\overline{x})$, so for every $y \in \mathfrak{p}$ we have $xy \in bR$ i.e. $x\mathfrak{p} \subseteq bR$. Since $\mathfrak{m} = \mathfrak{p}R_\mathfrak{p} = \mathrm{Ann}\,(\overline{x})R_\mathfrak{p}$, we have $x\mathfrak{m} \subseteq bR_\mathfrak{p}$ and therefore $\frac{x}{b} \in \mathfrak{m}' = R_\mathfrak{p}$. Hence $x \in bR_\mathfrak{p}$. But this means $\overline{x} = 0$ in $(R/bR)_\mathfrak{p}$, thus $\mathfrak{p}R_\mathfrak{p} = \mathrm{Ann}\,(\overline{x})R_\mathfrak{p} = R_\mathfrak{p}$. By Nakayama we would get $\mathfrak{p} = 0$, and this contradicts the assumption $\mathfrak{p} \neq 0$.

Since R is a domain, the maps $R \to R_\mathfrak{p}$ are all injective, hence $R \subseteq \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} R_\mathfrak{p}$. Conversely, let $\frac{a}{b} \in \bigcap_{\mathrm{ht}\,\mathfrak{p}=1} R_\mathfrak{p}$: we need to show that $a \in bR$. If $\mathrm{Ann}\,(\overline{a}) \subsetneq R$ is a proper ideal (nonzero, since $0 \neq b \in \mathrm{Ann}\,(\overline{a})$). By exercise 2 a) and b), we get that $\mathrm{Ann}\,(\overline{a})$ is contained in a prime ideal of the form $\mathfrak{p} = \mathrm{Ann}\,(\overline{z})$, for some $\overline{z} \in R/bR$. We have just established that the localisation of R at such primes is a DVR, so $\mathrm{ht}\,\mathfrak{p} = 1$. Since $\mathrm{Ann}\,(\overline{a}) \subseteq \mathfrak{p}$, we have $s\overline{a} \neq 0$ for all $s \notin \mathfrak{p}$, hence $a \notin bR_\mathfrak{p}$ and therefore $\frac{a}{b} \notin R_\mathfrak{p}$, contrary to the assumption.