Commutative Algebra Exam

Padova, 26/01/2017

Exercise 1. Let $L = \mathbb{Q}(\alpha)$, where $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$. Let $A = \mathbb{Z}[\alpha]$ and write $\mathcal{O}_L$ for the integral closure of $\mathbb{Z}$ in L.

a) Compute $\Delta(1, \alpha, \alpha^2)$. Show that $A\left[\frac{1}{2}\right] = \mathcal{O}_L\left[\frac{1}{2}\right]$.

b) Describe the decompostion of 3, 5 and 503 in L and compute the inertia and ramification degree above these primes.

c) Manipulating the equation defining α, show that $\beta = \frac{4}{\alpha} \in \mathcal{O}_L$.

d) Prove that $\alpha^2 = -\alpha + 2 - 2\beta$ and that $\beta^2 = -2\alpha - 2 + \beta$. Conclude that $A \subsetneq \mathcal{O}_L$.

e) Show that $\{1, \alpha, \beta\}$ is a basis for $\mathcal{O}_L$ as a $\mathbb{Z}$-module.

f) Let $\mathfrak{p} \subset \mathcal{O}_L$ be a prime ideal, $2 \in \mathfrak{p}$. Let $\pi : \mathcal{O}_L \to \mathcal{O}_L/\mathfrak{p}$ be the projection map. Computing the value of π on α and β, show that $\mathcal{O}_L/\mathfrak{p} = \mathbb{F}_2$.

g) Describe all the possible ring homomorphisms $\pi : \mathcal{O}_L \to \mathbb{F}_2$.

h) Compute the decomposition of 2.

i) For every prime $\mathfrak{p} \ni 2$, find $\theta \in \mathcal{O}_L$ such that $\mathfrak{p} = (2, \theta)$.

Exercise 2. Let k be a field, A_1 and A_2 two k-algebras of finite type. Show that $\dim (A_1 \otimes_k A_2) = \dim A_1 + \dim A_2$. [Hint: Noether's Normalisation Lemma]

Exercise 3. Let k be a field, $A = k[X_1, \ldots, X_n]$. Let $\mathfrak{q}_1, \mathfrak{q}_2 \subset A$ be prime ideals, put $A_i = A/\mathfrak{q}_i$ and $\pi_i : A \to A_i$ the natural projections. Let $\mathfrak{p} \subset A$ be a prime ideal, minimal among those containing $\mathfrak{q}_1 + \mathfrak{q}_2$.

a) Let $\varphi = \pi_1 \otimes \pi_2 : A \otimes_k A \to A_1 \otimes_k A_2$. Show that $J = \ker \varphi = \mathfrak{q}_1 \otimes_k A + A \otimes_k \mathfrak{q}_2$.

b) Let $\mu : A \otimes_k A \to A$ be the multiplication map. Recall (remark 1.3.8) that $I = \ker \mu$ is the ideal generated by the elements $X_i \otimes 1 - 1 \otimes X_i$ for $i = 1, \ldots, n$. Show that $\mu(I + J) = \mathfrak{q}_1 + \mathfrak{q}_2$.

c) Let $\widetilde{\mathfrak{p}} = \mu^{-1}(\mathfrak{p})$. Show that $\widetilde{\mathfrak{p}}$ is minimal among the primes containing $I + J$.

d) Let $\overline{\mathfrak{p}} = \varphi(\widetilde{\mathfrak{p}})$. Show that $\overline{\mathfrak{p}} \subset A_1 \otimes_k A_2$ is prime and minimal among those containing $\varphi(I)$.

e) Show that $\operatorname{ht} \overline{\mathfrak{p}} \leq n$.

f) Using exercise 2 and proposition 6.1.35, show that $\operatorname{ht} \mathfrak{p} \leq \operatorname{ht} \mathfrak{q}_1 + \operatorname{ht} \mathfrak{q}_2$.

You may submit your answers in english, french or italian.

Exercise 1. The minimal polynomial of α if $F(X) = X^3 + X^2 - 2X + 8$, hence $\Delta(1, \alpha, \alpha^2) = -N_{L/\mathbb{Q}}(F'(\alpha)) = -N_{L/\mathbb{Q}}(3\alpha^2 + 2\alpha - 2)$. Writing the matrix of multiplication by $3\alpha^2 + 2\alpha - 2$ in the basis $\{1, \alpha, \alpha^2\}$ we get

$$\Delta(1, \alpha, \alpha^2) = -N_{L/\mathbb{Q}}(3\alpha^2 + 2\alpha - 2) = -\det \begin{pmatrix} -2 & -24 & 8 \\ 2 & 4 & -26 \\ 3 & -1 & 5 \end{pmatrix} = -2^2 503.$$

Only 2 and 503 may ramify, and 503 ramifies for sure, since it appears with odd exponent. Moreover, by exercise 5.8, we get $A\left[\frac{1}{2}\right] = \mathcal{O}_L\left[\frac{1}{2}\right]$.

We may compute the decompositions of the odd primes using Kummer's lemma. $F(X)$ has no roots mod 3, so it is irreducible. Therefore 3 is inert in L.

$F(X) \equiv (X+1)(X^2+3)$ mod 5, and the quadratic factor is irreducible. Therefore $5\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$, with $e_1 = e_2 = f_1 = 1$ and $f_1 = 2$.

We know that 503 ramifies. Writing $503\mathcal{O}_L = \prod_{i=1}^r \mathfrak{q}_i^{e_i}$ with $\sum_{i=1}^r e_i f_i = 3$ and $e_1 \geq 2$, we have only two possibilities: either $503\mathcal{O}_L = \mathfrak{q}_1^3$ or $503\mathcal{O}_L = \mathfrak{q}_1^2\mathfrak{q}_2$ and $f_i = 1$ in any case. From Kummer's lemma, we see that the first case occurs if F has a triple root mod 503, while the second if it has a double root. In the first case F' must have a double root, while in the second it has simple roots. $F'(X) = 3X^2 + 2X - 2$ has discriminant $28 \not\equiv 0$ mod 503. Hence F' has no double roots and F can't have a triple one, hence $503\mathcal{O}_L = \mathfrak{q}_1^2\mathfrak{q}_2$. If you really must know, $F(X) \equiv (X+299)(X+354)^2$, but it's advisable not to compute this by hand.

Dividing $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$ by α^3 we get $1 + \frac{1}{\alpha} - \frac{2}{\alpha^2} + \frac{8}{\alpha^3} = 0$. Multiplying by 8 we get $8 + 2\frac{4}{\alpha} - \left(\frac{4}{\alpha}\right)^2 + \left(\frac{4}{\alpha}\right)^3 = 0$, thus β is a root of $G(Y) = 8 + 2Y - Y^2 + Y^3 \in \mathbb{Z}[Y]$ hence integral.

Dividing $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$ by α we get $\alpha^2 = -\alpha + 2 - 2\beta$ and dividing $8 + 2\beta - \beta^2 + \beta^3 = 0$ by β we get that $\beta^2 = -2\alpha - 2 + \beta$. The first equation gives $\beta = 1 - \frac{1}{2}\alpha - \frac{1}{2}\alpha^2 \notin A$, while $\beta \in \mathcal{O}_L$.

From equation $\alpha^2 = -\alpha + 2 - 2\beta$ we see that $A \subset \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta \subseteq \mathcal{O}_L$ and we get that the matrix expressing the basis $\{1, \alpha, \alpha^2\}$ in terms of the basis $\{1, \alpha, \beta\}$ is

$$U = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since $\Delta(1, \alpha, \alpha^2) = \det U^2 \Delta(1, \alpha, \beta) = 4\Delta(1, \alpha, \beta)$, from a) we conclude that $\Delta(1, \alpha, \beta) = -503$. As this has no square factors, we conclude that $\{1, \alpha, \beta\}$ is a $\mathbb{Z}$-basis for $\mathcal{O}_L$. We also deduce that 2 is unramified, as it does not divide the discriminant anymore.

As $2 \in \mathfrak{p}$ we have $2 = 0$ in $\mathcal{O}_L/\mathfrak{p}$. Since the latter is a domain, from the equation $\alpha\beta = 4$ in $\mathcal{O}_L$ we get $\pi(\alpha\beta) = \pi(\alpha)\pi(\beta) = 0$, so either $\pi(\alpha) = 0$ or $\pi(\beta) = 0$. Put $\overline{\alpha} = \pi(\alpha)$ and $\overline{\beta} = \pi(\beta)$. If $\overline{\alpha} = 0$, from $\beta^2 = -2\alpha - 2 + \beta$ in $\mathcal{O}_L$ we get that $\overline{\beta}^2 = \overline{\beta}$, so either $\overline{\beta} = 1$ or $\overline{\beta} = 0$. Similarly, if $\overline{\beta} = 0$, from $\alpha^2 = -\alpha + 2 - 2\beta$ in $\mathcal{O}_L$ we get that $\overline{\alpha}$ is idempotent, so either $\overline{\alpha} = 1$ or $\overline{\alpha} = 0$. In all cases $\overline{\alpha}, \overline{\beta} \in \mathbb{F}_2 \subseteq \mathcal{O}_L/\mathfrak{p}$. Since $\{1, \alpha, \beta\}$ is a $\mathbb{Z}$-basis for $\mathcal{O}_L$ and π is $\mathbb{Z}$-linear, the classes $1, \overline{\alpha}, \overline{\beta}$ generate $\mathcal{O}_L/\mathfrak{p}$ as a $\mathbb{Z}$-module, hence as a $\mathbb{Z}/2\mathbb{Z}$-module. Thus $\mathcal{O}_L/\mathfrak{p} = \mathbb{F}_2$. Therefore all primes above 2 have inertia degree 1: we conclude that 2 splits completely.

The map π is determined by its values on the basis elements. From the computations in f), we get that there are only three $\mathbb{Z}$-linear maps $\pi_i : \mathcal{O}_L \to \mathbb{F}_2$, defined by

$$\pi_0(\alpha) = \pi_0(\beta) = 0; \qquad \pi_1(\alpha) = 1,\ \pi_1(\beta) = 0; \qquad \pi_2(\alpha) = 0,\ \pi_2(\beta) = 1.$$

Since 2 splits completely, there must be three different ring homomorphisms $\mathcal{O}_L \to \mathbb{F}_2$, so the π_i must be ring homomorphisms. We can check this directly: from the relations in d) and $\alpha\beta = 4$ we get the multiplication formula for $\xi = x + y\alpha + x\beta$ and $\eta = u + v\alpha + w\beta$ in $\mathcal{O}_L$:

$$\xi\eta = (xu + 4yw + 4zv + 2yv - 2zw) + (xv + yu - yv - 2zw)\alpha + (xw + zu - 2yv + zw)\beta.$$

Therefore we need to check the identity in $\mathbb{F}_2$

$$(x + y\overline{\alpha} + x\overline{\beta})(u + v\overline{\alpha} + w\overline{\beta}) \stackrel{?}{=} (xu) + (xv + yu + yv)\overline{\alpha} + (xw + zu + zw)\overline{\beta} \tag{1}$$

for the pairs $(\overline{\alpha}, \overline{\beta}) \in \{(0,0),(0,1),(1,0)\}$. It holds in all three cases, so the three maps π_i are ring homomorphism, hence $\mathfrak{p}_i = \ker \pi_i$ are distinct prime ideals containing 2.

We conclude from g) that $2\mathcal{O}_L = \mathfrak{p}_0\mathfrak{p}_1\mathfrak{p}_2$.

From g) we have $\mathfrak{p}_0 = (2, \alpha, \beta)$, $\mathfrak{p}_1 = (2, \alpha - 1, \beta)$ and $\mathfrak{p}_2 = (2, \alpha, \beta - 1)$. Using $\alpha\beta = 4$ it is easy to get rid of one generator in $\mathfrak{p}_1$ and $\mathfrak{p}_2$:

$$\beta(\alpha - 1) = 4 - \beta \quad \Longrightarrow \quad \beta = 4 - \beta(\alpha - 1) \in (2, \alpha - 1) = \mathfrak{p}_1;$$

and symetrically $\alpha = 4 - \alpha(\beta - 1) \in (2, \beta - 1) = \mathfrak{p}_2$. The case of $\mathfrak{p}_0$ is more complicated and we need to exploit the relations d) to get $\mathfrak{p}_0 = (2, \alpha - \beta)$ as

$$\alpha = 2 - \alpha^2 - 2\beta = -2 + \alpha\beta - \alpha^2 - 2\beta = -\alpha(\alpha - \beta) - 2(1 + \beta) \in (2, \alpha - \beta);$$
$$\beta = 2 + 2\alpha + \beta^2 = 6 + 2\alpha - \alpha\beta + \beta^2 = -\beta(\alpha - \beta) + 2(3 + \alpha) \in (2, \alpha - \beta).$$

Exercise 2. If $A_i = k[X_1, \ldots, X_{n_i}]$ then $A_1 \otimes_k A_2 \cong k[X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2}]$ and the result is clear. By Noether's Normalisation Lemma, there exists polynomial algebras $R_i \subset A_i$ such that A_i is a finitely generated R_i-module. Therefore $A_1 \otimes_k A_2$ is a finitely generated $R_1 \otimes_k R_2$-module, hence $A_1 \otimes_k A_2$ is integral over $R_1 \otimes_k R_2$. It now follows from proposition 6.1.32 that

$$\dim (A_1 \otimes_k A_2) = \dim (R_1 \otimes_k R_2) = \dim R_1 + \dim R_2 = \dim A_1 + \dim A_2.$$

Exercise 3. Identifying $A \otimes_k A \cong k[X_1 \ldots, X_n, Y_1, \ldots, Y_n]$, if $\mathfrak{q}_1 = (F_1, \ldots, F_r)$ and $\mathfrak{q}_2 = (G_1, \ldots, G_s)$ then $J = (F_1(\underline{X}), \ldots, F_r(\underline{X}), G_1(\underline{Y}), \ldots, G_s(\underline{Y})) \cong \mathfrak{q}_1 \otimes_k A + A \otimes_k \mathfrak{q}_2$. It is now trivial that $\mu(I + J) = \mu(J) = \mathfrak{q}_1 + \mathfrak{q}_2$.

Since μ is surjective, the minimality of $\widetilde{\mathfrak{p}} = \mu^{-1}(\mathfrak{p})$ follows from b) and the correspondence between primes in $A \cong (A \otimes_k A)/I$ and primes in $A \otimes_k A$ containing I.

Since φ is surjective and $\widetilde{\mathfrak{p}}$ contains its kernel, the statement follows from the correspondence between primes in $A \otimes_k A$ containing J and primes in $(A \otimes_k A)/J$.

Since I is generated by n elements, $\varphi(I)$ is generated by their images. We conclude by the generalised Principal Ideal theorem 6.1.19.

By proposition 6.1.35, we have

$$\dim [(A_1 \otimes_k A_2)] = \operatorname{ht} \overline{\mathfrak{p}} + \dim [(A_1 \otimes_k A_2)/\overline{\mathfrak{p}}] \tag{2}$$

By exercise 2, $\dim (A_1 \otimes_k A_2) = \dim A_1 + \dim A_2$. From the definitions of the ideals, we have $(A_1 \otimes_k A_2)/\overline{\mathfrak{p}} = (A \otimes_k A)/\widetilde{\mathfrak{p}} = A/\mathfrak{p}$. Substituting this in (2) we get

$$\dim A/\mathfrak{q}_1 + \dim A/\mathfrak{q}_1 = \operatorname{ht} \overline{\mathfrak{p}} + \dim A/\mathfrak{p} \tag{3}$$

Recalling that $A_i = A/\mathfrak{q}_i$ and $\dim A = n$, applying proposition 6.1.35 to both sides of (3) we get

$$(n - \operatorname{ht} \mathfrak{q}_1) + (n - \operatorname{ht} \mathfrak{q}_2) = \operatorname{ht} \overline{\mathfrak{p}} + n - \operatorname{ht} \mathfrak{p}.$$

Therefore $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{q}_1 + \operatorname{ht} \mathfrak{q}_2 + (\operatorname{ht} \overline{\mathfrak{p}} - n)$. The claim now follows from e): $\operatorname{ht} \overline{\mathfrak{p}} \leq n$.