Integral closure and completions.

Exercise 1. Let $R = \mathbb{C}[X,Y,Z]/(X^3 - Y^2 + XY)$ as in Zariski's example and $\mathfrak{m} = (X,Y,Z)$. Find two discrete valuation rings $A_1, A_2 \subseteq \mathrm{Frac}(R)$ such that $R_{\mathfrak{m}} \subseteq A_1 \cap A_2$.

Exercise 2. Show that $\mathbb{C}[X,Y,Z]/(Z^2 - XY)$ is an integrally closed domain.

Exercise 3. Let $R = \mathbb{C}[X,Y]/(X^3 - Y^2 + XY)$ and $\mathfrak{m} = (X,Y)$. Compute the integral closure of $R_{\mathfrak{m}}$ as the intersection of two discrete valuation rings in $\mathrm{Frac}(R)$.

Exercise 4. Let K be a field, complete with respect to a non-archimedean absolute value.

a) Show that a series $\sum_{n=0}^{\infty} a_n$ converges in K if and only if $\lim_{n\to\infty} a_n = 0$.

b) Let $f(X) \in K[X]$. Show that $f : K \to K$ is a continuous function i.e. $\lim_{x\to x_0} f(x) = f(x_0)$ for every $x_0 \in K$.

c) Recall that for $\alpha \in \mathbb{Q}$ and $n \in \mathbb{N}$ one defines $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ for $\alpha \neq 0$ and $\binom{0}{n} = 1$. Show that if $|\alpha|_p \leq 1$ then $|\binom{\alpha}{n}|_p \leq 1$ (where $|\ |_p$ denotes the p-adic absolute value).

b) Show that the polynomial $X^{p^2-1} - p - 1$ has a root in $\mathbb{Z}_p$. Can you deduce this from Hensel's Lemma?

Solutions

Exercise 1. We have computed the integral closure of R as $A = \mathbb{C}[T,Z]$, with T satisfying $T^2 - T = X$ and $T^3 - T^2 = Y$. Therefore $\mathfrak{m}A = (T^2 - T, T^3 - T^2, Z) = (T(T-1), Z)$. It is contained in only two prime (in fact, maximal) ideals $\mathfrak{m}_0 = (T,Z)$ and $\mathfrak{m}_1 = (T-1, Z)$, so $A_{\mathfrak{m}}$ is a semi-local ring. In order to find two DVRs containing $A_{\mathfrak{m}}$ we should consider two irreducible curves in the plane having in common precisely the two points $(0,0)$ and $(1,0)$. We may take $\mathcal{Z}(Z)$ and $\mathcal{Z}(T^2 - T - Z)$. Put $\mathfrak{p} = (Z)$ and $\mathfrak{q} = (T^2 - T - Z)$. The generators are irreducible polynomials, so every $f \in A$ factors uniquely as

$$f(T,Z) = Z^{v_{\mathfrak{p}}(f)} g(T,Z) = (T^2 - T - Z)^{v_{\mathfrak{q}}(f)} h(T,Z)$$

and we extend $v_{\mathfrak{p}}$ and $v_{\mathfrak{q}}$ to valuations $\mathbb{C}(T,Z)^{\times} \to \mathbb{Z}$ in the usual way. Since $\mathfrak{p} + \mathfrak{q} = \mathfrak{m}A$ (recall $\mathcal{Z}(I) \cap \mathcal{Z}(J) = \mathcal{Z}(I+J)$), we have $R_{\mathfrak{m}} \subseteq A_{\mathfrak{p}} \cap A_{\mathfrak{q}}$.

Exercise 2. $A = \mathbb{C}[X,Y,Z]/(Z^2 - XY)$ is a domain because $Z^2 - XY$ is irreducible: it is an Eisenstein polynomial with respect to X. We regard $L = \mathrm{Frac}\, A = \mathbb{C}(X,Y)[Z]/(Z^2 - XY)$ as a quadratic extension of the fraction field $K = \mathbb{C}(X,Y)$ of $R = \mathbb{C}[X,Y]$. We have $\mathrm{Tr}_{L/K}(Z) = 0$

and $N_{L/K}(Z) = XY$. Since A is integral over R, the integral closure of A is integral over R and so is equal to the integral closure of R in L. Every element in K can be written uniquely as $f + Zg$ with $f, g \in \mathbb{C}(X,Y)$. If it is integral over R then

$$\mathrm{Tr}_{L/K}(f + Zg) = 2f \in R; \qquad N_{L/K}(f + Zg) = f^2 - XYg^2 \in R.$$

The first condition gives $f \in R$. Substituting in the second we get $XYg^2 \in R = \mathbb{C}[X,Y]$. Writing g as a reduced fraction, we see that the denominator of g^2 divides XY, which has no square factors, hence $g \in \mathbb{C}[X,Y]$ and therefore $f + Zg \in A$.

Exercise 3. Arguing as in exercise 1 (dropping the variable Z) we see that the integral closure of R is $A = \mathbb{C}[T]$. Since integral closure commutes with fractions, the integral closure of $R_{\mathfrak{m}}$ is $A_{\mathfrak{m}} = (R - \mathfrak{m})^{-1}A$. Since $\mathfrak{m}A = (T^2 - T, T^3 - T^2) = (T(T-1))$ is the product of the two maximal ideals $\mathfrak{m}_0 = (T)$ and $\mathfrak{m}_1 = (T-1)$, we get two DVRs $A_0 = \mathbb{C}[T]_{(T)}$ and $A_1 = \mathbb{C}[T]_{(T-1)}$. Clearly $A_{\mathfrak{m}} \subseteq A_0 \cap A_1$. The conclusion follows if we show that A_0 and A_1 are the only valuation rings in $\mathbb{C}(T)$ containing $R_{\mathfrak{m}}$. Let $(B, \mathfrak{m}_B)$ be such a valuation ring. Since B is integrally closed, it contains $A_{\mathfrak{m}}$ and therefore contains A. Since $\mathfrak{m}_B \cap A$ is a prime ideal in A, either $\mathfrak{m}_B \cap A = (T)$ or $\mathfrak{m}_B \cap A = (T-1)$. By the universal property of fractions, either $A_0 \subseteq B$ or $A_1 \subseteq B$. Since A_i is a valuation ring, we conclude that either $A_0 = B$ or $A_1 = B$. (Notice: if $V_1 \subseteq V_2$ are DVRs with $\mathrm{Frac}\, V_1 = \mathrm{Frac}\, V_2$, then $V_1 = V_2$: if $x \in V_2$, $x \notin V_1$, then $x^{-1} \in \mathfrak{m}_{V_1}$ and $V_1[x] = \mathrm{Frac}\, V_1$ is then contained in V_2.)

Exercise 4. Write $|\ |$ for the absolute value. By definition the series converges if the sequence $s_n = \sum_{k=0}^{n} a_k$ converges i.e. is a Cauchy sequence. Therefore, if the series converges, for every $\varepsilon > 0$ there is an N_ε such that $|a_n| = |s_n - s_{n-1}| < \varepsilon$ for all $n \geq N_\varepsilon$, hence $\lim\limits_{n\to\infty} a_n = 0$. Conversely, if $\lim\limits_{n\to\infty} a_n = 0$, fix $\varepsilon > 0$ and pick N_ε such that $|a_n| < \varepsilon$ for all $n \geq N_\varepsilon$. Then

$$|s_{n+m} - s_n| = \left| \sum_{k=1}^{m} a_{n+k} \right| \leq \max_{1 \leq k \leq m} |a_{n+k}| < \varepsilon$$

hence $\{s_n\}$ is a Cauchy sequence and the series converges.

Let $f(X) \in K[X]$. For $x_0 \in K$, expanding $f(X) = f(x_0) + \sum_{k=1}^{\deg f} a_k (X - x_0)^k$ we have that for every $x \in K$ with $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| = \left| \sum_{k=1}^{\deg f} a_k (x - x_0)^k \right| \leq \max_{1 \leq k \leq \deg f} |a_k| \delta^k.$$

Thus, for any $0 < \varepsilon < 1$, taking $\delta = \min\{\varepsilon, \frac{\varepsilon}{m}\}$, with $m = \max_{1 \leq k \leq \deg f} |a_k|$, we get that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Hence $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

If $m \in \mathbb{N}$, the binomial coefficient $\binom{m}{n} \in \mathbb{Z} \subseteq \mathbb{Z}_p$ hence $|\binom{m}{n}|_p \leq 1$. The binomial coefficient $\binom{x}{n}$ is polynomial, hence continuous. Every $\alpha \in \mathbb{Z}_p$ can be expanded as $\alpha = \sum_{k=0}^{\infty} s_k p^k$ with $s_k \in \{0, \ldots, p-1\}$. Hence, setting $b_m = \sum_{k=0}^{m-1} s_k p^k \in \mathbb{N}$, we have $|\alpha - b_m|_p < |p^m|_p = \frac{1}{p^m}$. Taking $\varepsilon = 1$ we get that, for $m \in \mathbb{N}$ sufficiently large, there exists a $b_m \in \mathbb{Z}$ such that $|\binom{\alpha}{n} - \binom{b_m}{n}|_p < 1$. So

$$\left|\binom{\alpha}{n}\right|_p = \left|\binom{\alpha}{n} - \binom{b_m}{n} + \binom{b_m}{n}\right|_p \leq \max\left\{\left|\binom{\alpha}{n} - \binom{b_m}{n}\right|_p, \left|\binom{b_m}{n}\right|_p\right\} \leq 1.$$

The binomial series $(1+p)^{\frac{1}{p^2-1}} = \sum_{n=0}^{\infty} \binom{\frac{1}{p^2-1}}{n} p^n$ converges in $\mathbb{Z}_p$ because $\binom{\frac{1}{p^2-1}}{n} \in \mathbb{Z}_p$ hence

$$\lim_{n\to\infty} \left| \binom{\frac{1}{p^2-1}}{n} p^n \right|_p = \lim_{n\to\infty} |p^n|_p = \lim_{n\to\infty} \frac{1}{p^n} = 0.$$

We couldn't get this from Hensel's Lemma as $X^{p^2-1} - p - 1 \equiv X^{p^2-1} \bmod p$ which doesn't factor into a product of coprime polynomials in $\mathbb{F}_p[X]$. But we could have obtained a root appliyng Newton's method (exercise 3.9).