Barsotti-Tate groups and p-adic representations of the fundamental group scheme

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Abstract

On a scheme S over a base scheme B we study the category of *locally constant* BT groups i.e. groups over S that are twists, in the flat topology, of BT groups defined over B. These groups generalize p-adic local systems and can be interpreted as integral p-adic representations of the fundamental group scheme of S/B (classifying finite flat torsors on the base scheme) when such a group exists. We generalize to these coefficients the Katz correspondence for p-adic local systems and show that they are closely related to the maximal nilpotent quotient of the fundamental group scheme.

1 Introduction

It is a classical result in analytic geometry (e.g. [5] chapter I) that when S is a complex manifold there are natural equivalences of categories between: (i) complex representations of the fundamental group $\pi_1(S)$; (ii) local systems of $\mathbb{C}$-vector spaces on S and (iii) complex vector bundles on S with an integrable analytic connection.

When S is a scheme of characteristic p, the fundamental group should be replaced by the étale fundamental group in the sense of [SGA1]. It is a profinite group, hence a compact totally disconnected topological group. We should therefore look for continuous representations and the interesting (finite dimensional) ones are those with ℓ-adic coefficients. In the present paper, we are concerned with integral representations, i.e. representations on free finitely generated modules over complete (discrete) valuation rings. We still have an equivalence between ℓ-adic representations of $\pi_1(S)$ and ℓ-adic local systems on S, but, when $\ell \neq p$, there is no "de Rham" side of the correspondence.

By contrast, when $\ell = p$, we have the Katz correspondence between p-adic representations of the fundamental group and the category of unit root F-crystals ([14], proposition 4.1.1). However, there are "too few" p-adic representations of $\pi_1(S)$. There are several reasons for this discrepancy, the most obvious one being the fact that in positive characteristic one should also take into account inseparable morphisms.

When S is a reduced scheme over a field k and b is a fixed k-valued point, Nori [18] defined the *fundamental group scheme* $\boldsymbol{\pi}(S/k; b)$ generalizing Grothendieck's étale fundamental group: it is a profinite k-group scheme which classifies pointed torsors over S under finite group schemes. For instance, when $S = A$ is an abelian variety, $\boldsymbol{\pi}(A/k; 0) = \varprojlim \ker n_{A^t}$, where A^t denotes the dual abelian variety. Glimpses of this "true" fundamental group already appear in [SGA1], e.g. exposé X, remarque 2.5.

Strictly speaking $\boldsymbol{\pi}(S/k; b)$, as a group scheme over k, is only entitled to have representations into k-vector spaces. However, we can define integral p-adic representations of $\boldsymbol{\pi}(S/k; b)$ by taking a suitable generalization of the notion of p-adic local system on S. Of the many generalizations of this concept, the oldest and most immediate is that of p-divisible or, as we shall say, Barsotti-Tate group (BT group for short): a BT group X over S is an inductive system of finite locally free commutative group schemes $X(n)$ over S such that, denoting by p_X the multiplication by p, $X(n) = \ker p_X^n$ (so X is p-torsion) and $p_X : X \to X$ is an epimorphism (so X is p-divisible). The datum of the inductive system is in fact equivalent to the projective system $\{p_X : X(n+1) \to X(n)\}_n$ and from this second description we see that BT groups generalize p-adic local systems, which correspond to the case when all the $X(n)$'s are étale group schemes.

The fibres of a general BT group over S can vary a lot; we are going to focus on those with isotrivial variation. We will say that a BT group X is locally constant if $X(n)$ becomes constant over a flat cover of the base for every n. It turns out then (proposition 3 below) that $X(n)$ in fact trivializes over a torsor under a finite group scheme over S, thereby defining a morphism

$$\boldsymbol{\pi}(S/k; b) \to \underline{Aut}(X_b).$$

Hence a locally constant BT truly is a p-adic representation of the fundamental group scheme.

The notion of locally constant BT group makes sense over any scheme S over a base B while, so far, fundamental group schemes are defined only for reduced flat schemes over a field (Nori [18]) or a Dedekind scheme (Gasbarri [7]). We have therefore chosen to formulate our results for locally constant BT groups (typically B will be a field or a dvr, but very little will be assumed on S) while the applications to the fundamental group scheme are only given in the last section of the paper. However, the philosophy of fundamental groups was very much the source of inspiration for this work.

Our first main result is a generalization of the aforementioned Katz correspondence between p-adic local systems and unit root F-crystals. First, as already noticed by Berthelot and Messing ([2], p. 175), the Katz correspondence can be interpreted in terms of the Dieudonné functor $\mathbb{D}$. Given a p-adic representation $\rho : \pi_1(S) \to GL_n(\mathbb{Z}_p)$, let X be the p-adic local system associated to the *dual* representation, viewed as an étale BT group on S. Then the unit-root F-crystal corresponding to ρ is precisely $\mathbb{D}(X)$.

Recall that a Dieudonné crystal is a locally free F-crystal $\mathcal{M}$ equipped with a Verschiebung operator V such that $FV = VF = p$. We shall say that a locally free crystal $\mathcal{M}$ is locally cloven if $\mathcal{M}/p^n$ becomes constant over a flat cover of the base for every n.

Theorem 3 *Let k be a field of positive characteristic, S a k-scheme over which the Dieudonné functor $\mathbb{D}$ is fully faithful. Then $\mathbb{D}$ establishes an anti-equivalence of categories between locally constant BT groups over S and locally cloven Dieudonné crystals on S.*

The main limitation of our approach is that the category of locally constant BT groups does not have tensor products, as a decent category of representations should. Indeed the tensor product of BT groups (as sheaves of $\mathbb{Z}_p$-modules) is not representable. The obvious way around this problem would be to take locally cloven F-crystals as the category of coefficients, but it does not seem clear that such crystals bear a relationship with the fundamental group scheme e.g. that their reduction mod p^n would trivialize over a torsor under a finite flat group scheme. We hope to be able to address this question in the future.

Our second main result concerns the quotient $\boldsymbol{\pi}_{BT_+}(S/k; b)$ of the fundamental group scheme acting on locally constant BT groups:

Theorem 5 *The connected component of $\boldsymbol{\pi}_{BT_+}(S/k; b)$ is nilpotent and torsion-free.*

When S is proper and smooth over a perfect field k, Nori [19] has given a second construction of the fundamental group scheme as the tannakian fundamental group of a suitable category of vector bundles on S. Using this approach, we can show that $\boldsymbol{\pi}_{BT_+}(S/B; b)$ is “close” to the maximal nilpotent quotient of the fundamental group scheme: see Theorem 6 for a precise statement.

Let us now review in more detail the structure of the paper.

In §2 we study the category of locally constant Barsotti-Tate groups. We show (proposition 3) that, on a scheme over a field or a mixed discrete valuation ring, the p^n kernels of a locally constant BT trivialize over a torsor under a finite flat group scheme. In the process, we show (proposition 2) that if H and H' are BT groups over a field, the fppf sheaf $\mathcal{H}om_{\mathbf{BT}}(H, H')$ is representable by the Tate module of a BT group, a result that may be of independent interest.

When S is a scheme over a perfect field of positive characteristic, we relate the property for a BT group of being locally constant with that of being completely slope divisible, a notion introduced by Zink [24] which plays an important role in Oort's foliation structure on a Newton polygon stratum in the moduli space of abelian varieties [20]. Specifically, we show (theorem 1) that a BT group is completely slope divisible if and only if it is locally constant with completely slope divisible fibre.

Zink [24] (resp. Oort and Zink [21]) show that over a regular (resp. normal) base scheme S a BT group with constant Newton polygon is isogenous to a completely slope divisible BT group. In theorem 2 we show that, over any scheme over a perfect field, a locally constant BT admits an isogeny with locally constant kernel to a completely slope divisible BT.

In §3 we specialize to our situation the crystalline Dieudonné functor of Berthelot, Breen and Messing [1]. For locally constant BT groups, we give an alternative description of $\mathbb{D}$ as a $\mathcal{H}om$ into Witt covectors (rather than as an $\mathcal{E}xt^1$) which allows us to construct (without assumptions on the base) an explicit inverse functor. We conclude this section by giving, over a proper smooth S, a crystalline version of the universal extension for vector bundles.

Finally, in §4 we define and study a quotient $\boldsymbol{\pi}_{BT_+}(S/B; b)$ of the fundamental group scheme corresponding to locally constant BT groups.

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2 Locally constant Barsotti-Tate groups

Notation: B will denote a base scheme and S will be a B-scheme. If X is a Barsotti-Tate group, we will write $X(n) = \ker[\, p^n : X \to X]$.

Definition 1 *A BT group X over S is* locally constant *if there exists a BT group H over B such that for all n there is an fppf covering $f_n : S_n \to S$ and an isomorphism $X(n) \times_S S_n \cong H(n) \times_B S_n$.*

Remark 1 It will be convenient to say that such an X is *locally constant of type H* although the latter group is not uniquely defined (it could be replaced by a twist). In applications to the fundamental group scheme, we will have a marked point $b \in S(B)$ and we can rigidify the situation by taking $H = X_b$.

Example 1 Étale or multiplicative BT groups are locally constant, even in the étale topology. Over a proper smooth scheme over a perfect field of characteristic p, we will construct later (Example 3) a canonical locally constant group, extension of the BT group of the Picard variety by $\hat{\mathbb{G}}_{\mathrm{m}}$.

Example 2 Let E be the Tate curve over $k[[q]]$, where k is a perfect field of characteristic p and put $K = k((q))$. The associated BT group $X = E(p^\infty)$ over $S = \operatorname{Spec} K$ is an extension of an étale by a multiplicative group. A splitting of the extension $X(n)$ is given by a point of exact order p^n of E_K. By Tate's uniformization theorem $E(\bar{K}) = \bar{K}^*/q^{\mathbb{Z}}$, hence $X(n)$ splits over the $\boldsymbol{\mu}_{p^n}$-torsor defined by the equation $T^{p^n} = q$.

Definition 2 *Let* $\mathbf{BT}(S)$ *be the category of Barsotti-Tate groups over* S. *We define the category* $\mathbf{LCBT}(S/B)$ *of* locally constant Barsotti-Tate groups over S *as the full subcategory of* $\mathbf{BT}(S)$ *whose objects are locally constant BT groups in the sense of definition 1.*

We are going to show that when B is the spectrum of a field or a mixed discrete valuation ring, the trivializing maps $f_n : S_n \to S$ of definition 1 can be chosen to be finite and syntomic. For this we need some representability results. The first proposition gathers some well-known facts, which we quote for ease of reference:

Proposition 1 *(1) ([SGA3] XI.3.12(b)) Let* B *be a scheme,* H *a finite locally free* B*-group scheme,* G *an affine* B*-group scheme. The sheaf* $\mathcal{H}om(H,G)$ *(group-homomorphisms) is representable by an affine* B*-group scheme denoted* $\underline{Hom}_B(H,G)$. *If* G *is of finite type (resp. finite presentation), so is* $\underline{Hom}_B(H,G)$.

(2) ([17], chap. III thm. 4.3) Let G *be an affine group scheme flat of finite presentation over* B. *Any fppf sheaf that is a* G*-torsor is representable.*

Remark 2 When both H and G are sheaves of $\mathbb{Z}/p^n\mathbb{Z}$-modules, the subgroup $\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}-\text{modules}}(H,G)$ of $\mathcal{H}om_{\text{groups}}(H,G)$ is also representable, as this condition can be expressed by a diagram involving finitely many fibred products. In order to lighten the notation, when no confusion is possible, we still denote by $\underline{Hom}_B(H,G)$ the group of homomorphisms as $\mathbb{Z}/p^n\mathbb{Z}$-modules.

The following result generalizes a well-known fact when $H = \mathbb{Q}_p/\mathbb{Z}_p$ or $H' = \hat{\mathbb{G}}_{\text{m}}$.

Proposition 2 *Let* k *be a field and* H, H' *BT groups over* k. *The sheaf* $\mathcal{H}om_{\mathbf{BT}}(H,H')$ *is representable by a profinite group scheme* $\underline{Hom}_k(H,H')$ *which is the projective limit of the* p^n*-kernels of a BT group.*

Proof. First notice that, for a given integer n, $\mathcal{H}om(H(n),H') = \mathcal{H}om(H(n),H'(n))$ is representable by a group scheme $\underline{Hom}_k(H(n),H')$ by proposition 1. For a fixed $n \in \mathbb{N}$ put

$$\Gamma(n) = \bigcap_{m\geq 0} \text{im}\left[\underline{Hom}_k(H(n+m),H') \to \underline{Hom}_k(H(n),H')\right],$$

where the maps are restrictions and images are taken in the abelian category of commutative group schemes over k. Clearly $\mathcal{H}om_{\mathbf{BT}}(H,H') = \varprojlim \underline{Hom}_k(H(n),H') = \varprojlim \Gamma(n)$.
For fixed n and m, if $i'_{n,m} : H'(n) \to H'(n+m)$ (resp. $j_{n,m} : H(n+m) \to H(n)$) denotes the canonical injection (resp. projection), we define a map

$$\gamma_{n,m} : \underline{Hom}_k(H(n),H') \longrightarrow \underline{Hom}_k(H(n+m),H') \qquad f \mapsto i'_{n,m} \circ f \circ j_{n,m}.$$

For any k-algebra A and $f \in Hom_A(H(n+m),H')$, one checks immediately that $p^m f = \gamma_{n,m}(f|_{H(n)})$: we are going to show that the system $\{\Gamma(n),\gamma_{n,1}\}$ defines a BT group.
It follows from the definitions that $\gamma_{n,m}$ identifies $\underline{Hom}_k(H(n),H')$ with the kernel of multiplication by p^n in $\underline{Hom}_k(H(n+m),H')$. Therefore the sequence

$$0 \longrightarrow \Gamma(n) \xrightarrow{\gamma_{n,m}} \Gamma(n+m) \xrightarrow{\text{restr.}} \Gamma(m) \longrightarrow 0$$

is exact to the left and it is exact to the right by the definition of the $\Gamma(i)$'s; exactness in the middle follows by inspection of the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \underline{Hom}_k(H(n+1),H') & \xrightarrow{\gamma_{n+1,m}} & \underline{Hom}_k(H(n+m+1),H') & \xrightarrow{\text{restr.}} & \underline{Hom}_k(H(m),H') \\
& \downarrow{\scriptstyle\text{restr.}} & & \downarrow{\scriptstyle\text{restr.}} & & \| \\
0 \longrightarrow & \underline{Hom}_k(H(n),H') & \xrightarrow{\gamma_{n,m}} & \underline{Hom}_k(H(n+m),H') & \xrightarrow{\text{restr.}} & \underline{Hom}_k(H(m),H').
\end{array}$$

It remains to show that $\Gamma(n)$ is a finite group scheme. For this we borrow an idea of F. Oort ([20], lemma 1.5). We can assume that k is algebraically closed. By Dieudonné theory, $\mathcal{H}om_{\mathbf{BT}}(H,H')(k) = \mathrm{Hom}_{\mathbf{BT}(k)}(H,H')$ is a free $\mathbb{Z}_p$-module of finite rank. For all n the image of $\mathrm{Hom}_{\mathbf{BT}(k)}(H,H')$ in $\underline{Hom}_k(H(n),H')(k)$ is a finite subgroup. Denoting by $\Gamma_{n,m}$ the image of $\underline{Hom}_k(H(n+m),H')$ in $\underline{Hom}_k(H(n),H')$ (a closed subgroup scheme), we get a sequence

$$\underline{Hom}_k(H(n),H')(k) \supseteq \Gamma_{n,1}(k) \supseteq \dots \Gamma_{n,m}(k) \supseteq \dots \supseteq \mathrm{Hom}_{\mathbf{BT}(k)}(H,H')/p^n.$$

It follows from prop. 1 that $\underline{Hom}_k(H(n),H')$ is noetherian, hence the topological space $\underline{Hom}_k(H(n),H')(k)$ is noetherian and the sequence stabilizes: $\Gamma_{n,m}(k)$ is finite for large m. So $\Gamma_{n,m}$ is the spectrum of a semi-local ring which is a finitely generated k-algebra, and must therefore be an Artin ring. □

Corollary 1 *Let k be a field and H a BT group over k. The sheaf $\mathcal{A}ut_{\mathbf{BT}}(H)$ is representable by a profinite group scheme denoted $\underline{Aut}_k(H)$.*

Proof. Put $G_n = \underline{Aut}_k(H(n))$ and $E_n = \underline{End}_k(H(n))$. For fixed n and all $m \geq 0$, we have a morphism $G_{n+m} \to G_n$ whose image is representable by the group scheme $G_{n,m}$ of units in $E_{n,m} = \mathrm{im}\,[E_{n+m} \to E_n]$. As seen in the proof of proposition 2, there is an m such that $E_{n,m} = E_{n,m+1} = \dots$ is finite. Hence $\bar{G}_n := G_{n,m} = G_{n,m+1} = \dots$ is finite and clearly $\mathcal{A}ut_{\mathbf{BT}}(H)$ is prorepresented by $\varprojlim \bar{G}_n$. Since the transition maps $\bar{G}_{n+1} \to \bar{G}_n$ are finite, by [EGA] IV 8.2.3 the projective limit $\varprojlim \bar{G}_n$ is an affine scheme. □

Proposition 3 *Let k be a field and H a BT group over k. Let X be a locally constant BT group of type H on a k-scheme S. Then for all n, $X(n)$ becomes isomorphic to $H(n)$ over a scheme S_n which is a torsor under a finite k-group scheme.*

Proof. Put $G_n = \underline{Aut}_k(H(n))$. As $X(n)$ is a twisted form of $H(n)$, it is trivialized by the fppf G_n-torsor $S'_n = \underline{Isom}_S(H(n)_S, X(n))$. If k is of characteristic zero, G_n is finite, so assume $\mathrm{char}\, k = p$.

As in the proof of corollary 1, for fixed n and all $m \geq 0$, let $G_{n,m}$ be the image of G_{n+m} in G_n. For all m we have:

$$S'_n \cong G_n \wedge^{G_{n+m}} S'_{n+m} \cong G_n \wedge^{G_{n,m}} \left(G_{n,m} \wedge^{G_{n+m}} S'_{n+m}\right).$$

As seen in the proof of corollary 1, for large m, $\bar{G}_n = G_{n,m}$ is finite and independent on m. Fix such an m and put $S_n = \bar{G}_n \wedge^{G_{n+m}} S'_{n+m}$. It is a $\bar{G}_n$-torsor, hence finite and a local complete intersection. Moreover, $X(n)$ trivializes over S_n:

$$X(n) \cong H(n) \wedge^{G_n} S'_n \cong H(n) \wedge^{\bar{G}_n} S_n.$$

□

Remark 3 In [20], Oort introduces the notion of *geometrically fibrewise constant* BT group and shows that (under suitable assumptions on S), if X is such a BT, for all $n > 0$ there is a finite surjective morphism $T_n \to S$ such that $X(n) \times_S T_n$ is constant. While it is obvious that a locally constant BT is geometrically fibrewise constant, proposition 3 does not follow from Oort's theorem because the map $T_n \to S$ above is not necessarily flat if S is not regular.

Proposition 4 *Let B be the spectrum of a discrete valuation ring R with mixed characteristic and H a BT group over B. Let X be a locally constant BT group of type H on a flat B-scheme S. Then for all n, $X(n)$ becomes isomorphic to $H(n)$ over a torsor under a finite flat B-group scheme.*

Proof. Recall that an R-algebra is flat iff it is torsion-free. Again, put $G_n = \underline{Aut}_B(H(n))$. Let A_n be its Hopf algebra, T_n its torsion submodule, which is an ideal, and $A'_n = A_n/T_n$, thus a flat R-algebra. Let K be the fraction field of R: since

$$G_n \times_R K = \mathrm{Spec}\,(A_n \otimes K) = \underline{Aut}_K(H(n) \times_R K)$$

is a finite group scheme and $A_n \otimes K = A'_n \otimes K$ we have that A'_n is a finite flat R-algebra; put $G'_n = \mathrm{Spec}\, A'_n$. The restriction to G'_n of the multiplication in G_n is given by a map $A_n \to A'_n \otimes_R A'_n$ which must factor through A'_n, hence G'_n is a subgroup scheme. Moreover, if Λ is a flat R-algebra, $G_n(\Lambda) = G'_n(\Lambda)$.
Let U be a flat S-scheme, hence flat over R and let $g \in G_{n,S}(U)$; by compostion it defines a U-valued point of G_n, which by the above remark factors through a point of G'_n: by the universal property of fibred products, $g : U \to G_{n,S}$ must factor through $G'_{n,S}$. Hence any Čech 1-cocycle for the fppf sheaf $G_{n,S}$ comes from a Čech 1-cocycle for the fppf sheaf $G'_{n,S}$.
As $X(n)$ is a twisted form of $H(n)$, it is trivialized by an fppf G_n-torsor hence by an fppf G'_n-torsor. □

Before discussing in more detail the structure of locally constant BTs over a field, we prove a the following rigidity result.

Proposition 5 *Let $B = \mathrm{Spec}\, R$ be the spectrum of a complete noetherian local ring with perfect residue field k, $\mathfrak{S}$ a formal R-scheme and S_0 its special fibre. The functor* $\mathbf{LCBT}(\mathfrak{S}/R) \to \mathbf{LCBT}(S_0/k)$ *taking a locally constant formal BT group $\mathfrak{X}$ to its special fibre is faithful; if $\mathfrak{S} = \hat{S}$ is the formal completion of a proper R-scheme S, the same holds for the functor* $\mathbf{LCBT}(S/R) \to \mathbf{LCBT}(S_0/k)$.

Proof. The second claim follows from the first and the formal GAGA theorems of [EGA] III 5. Let A be an R-algebra, J a square-zero ideal, $A_o = A/J$ and let $i : \mathrm{Spec}\, A_o \hookrightarrow \mathrm{Spec}\, A$. It suffices to show that, given locally constant BT groups H and H' over A with reductions H_o and H'_o, the natural map of fppf sheaves $\underline{Hom}_A(H, H') \to i_*\underline{Hom}_{A_o}(H_o, H'_o)$ is injective. This is a standard rigidity argument that can be deduced from the Grothendieck-Illusie [10] deformation theory. □

In the remaining part of this section, we assume that B is is the spectrum of a perfect field k of characteristic $p > 0$. In this case the simple-minded notion of locally constant BT group turns out to be related to a subtler one introduced by Zink ([24], definition 10, [21], def. 1.2).

Definition 3 *Let $s \geq r_1 > r_2 > \cdots > r_m \geq 0$ be integers. A BT group X over S is* completely slope divisible *(csd) with slopes $\lambda_1 = \frac{r_1}{s}, \ldots, \lambda_m = \frac{r_m}{s}$, if it admits a filtration*

$$0 = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

by BT subgroups such that

1. *$p^{-r_i}F^s : X_i \to X_i^{(p^s)}$ is an isogeny for $i = 1, \ldots, m$.*
2. *The induced maps $p^{-r_i}F^s : X_i/X_{i-1} \to (X_i/X_{i-1})^{(p^s)}$ are isomorphisms.*

When there is only one slope, X is said to be isoclinic slope divisible.

Let H be a BT group over the spectrum of a perfect field k. Trivially, H is locally constant. On the other hand, by [21] proposition 1.4, over an algebraically closed field, H is completely slope divisible iff it is a direct sum of isoclinic BT groups defined over a finite field. We combine this arithmetic information with the geometric data of def.1.

Definition 4 *A BT group X over S is* locally finite *if it is locally constant of type H where H is a completely slope divisible BT group.*

Let X be a locally finite BT group over S and let $\lambda_1 > \cdots > \lambda_m$ be the slopes of H. In general, $\Phi_m = p^{-r_m}F^s : X \to X^{(p^s)}$ is only a *quasi-isogeny*, i.e. there is an integer $n \geq 0$ such that $p^n\Phi_m$ is an isogeny. Φ_m is an isogeny if and only if $p^n\Phi_m : X(n) \to X^{(p^s)}(n)$ is the zero morphism, a property that can be checked after the base change $S_n \to S$. If there is only one slope, Φ_m is even an isomorphism i.e. X is isoclinic slope divisible. If X has more than one slope, we can apply [21] Corollary 1.9: there is an exact sequence of p-divisible groups

$$0 \longrightarrow X^{\Phi_m\text{-nil}} \longrightarrow X \longrightarrow X^{\Phi_m} \longrightarrow 0$$

such that Φ_m is an isomorphism on the Φ_m*-étale part* X^{Φ_m} and is nilpotent on $X_{m-1} := X^{\Phi_m\text{-nil}}$. The formation of X^{Φ_m} commutes with base change, hence X_{m-1} is locally finite with slopes $\lambda_1, \ldots, \lambda_{m-1}$. Repeating the argument above for X_{m-1} we conclude that a locally finite BT group is completely slope divisible.

Theorem 1 *A BT group X over S is locally finite iff it is completely slope divisible.*

Proof. We have already seen the *if* part. Let X be an isoclinic slope divisible of slope $\lambda = r/s$. We may assume that $\mathbb{F}_{p^s} \subseteq k$. By [21], Corollary 1.10, for all n there is an affine étale S-scheme S_n, a BT_n group $H(n)$ over $\mathbb{F}_{p^s}$ and an isomorphism $X(n) \times_S S_n \cong H(n) \times_{\mathbb{F}_{p^s}} S_n$. In other words, $X(n)$ is an fpqc twist of $H(n) \times S$, so it is trivialized by an $\underline{Aut}_k(H(n))$-torsor, which is of finite presentation by proposition 1. Hence X is locally finite (even for the étale topology). The claim follows then from the following lemma by induction on the number of slopes. □

Lemma 1 *Let E be BT group which is divisible by $\lambda = r/s$ (i.e. $\Phi = p^{-r}F^s : E \to E^{(p^s)}$ is an isogeny). If the Φ-nilpotent part $X = E^{\Phi\text{-nil}}$ is locally constant, then E is locally constant.*

Proof. The Φ-étale part Y of E is isoclinic slope divisible of slope λ. It suffices to show that for all n there is an fppf $S_n \to S$ such that the sequence

$$0 \longrightarrow X(n) \longrightarrow E(n) \longrightarrow Y(n) \longrightarrow 0 \tag{1}$$

splits when pulled back to S_n. First, notice that a splitting exist locally in the fppf topology. This is a standard argument: we follow [24], p. 84, from which we borrow the notation. Write $E(n) = \operatorname{Spec} \mathcal{M}$ and $Y(n) = \operatorname{Spec} \mathcal{L}$. As an $\mathcal{O}_S$-module, $\mathcal{L}$ has a basis fixed by Φ and let $m \geq 1$ be such that Φ^m is zero on the finitely generated $\mathcal{M}/\mathcal{L}$. The sequence splits if $\ker \Phi^m \to \mathcal{M}/\mathcal{L}$ is surjective. Take $x \in \mathcal{M}$ lifting a generator $\bar{x}$ of $\mathcal{M}/\mathcal{L}$ and put $a = \Phi^m(x) \in \mathcal{L}$. The equation $\Phi^m(y) = a$ can be solved in $\mathcal{L}$ after adjunction of a finite number of p^{sm}-th roots to $\mathcal{O}_S$ and $x - y \in \ker \Phi^m$ lifts $\bar{x}$. Incidentally, this also shows that we really need inseparable covers to trivialize non-isoclinc locally finite BT groups.
A splitting of extension (1) is a section of the fppf sheaf $\underline{Hom}_S(Y(n), E(n))$ lifting the identity of $\underline{End}_S(Y(n))$. The set of such sections is then a torsor under the fppf sheaf $\underline{Hom}_S(Y(n), X(n))$. By proposition 1, this sheaf is representable. Moreover, its formation commutes with flat base change. By induction, there is an fppf scheme $S'_n \to S$ and BT_n groups $H(n)$ and $G(n)$ over k such that $Y(n) \times_S S'_n \cong H(n) \times_k S'_n$ and $X(n) \times_S S'_n \cong G(n) \times_k S'_n$, hence

$$\underline{Hom}_S(Y(n), X(n)) \times_S S'_n \cong \underline{Hom}_k(H(n), G(n)) \times_k S'_n.$$

Therefore $\underline{Hom}_S(Y(n), X(n))$ is also flat over S and, by proposition 1.(2), any torsor under it is representable. □

When S is a regular scheme, Zink [24] has shown that any $X \in \mathbf{BT}(S)$ with constant Newton polygon is isogenous to a completely slope divisible BT group; Oort and Zink [21], theorem 2.1 have relaxed the hypothesis to S normal noetherian. We conclude this section by adapting the arguments of [24] to show that, for a perfect field k, a locally constant BT group over any k-scheme S is isogenous to a completely slope divisible BT group by an isogeny whose kernel is locally constant (i.e. becomes a constant group scheme after a finite flat base change). We call such an isogeny locally constant.

Lemma 2 *Let k be a perfect field and X a locally constant BT group over a k-scheme S. Let λ be the smallest slope of X. Then there exists a locally constant BT group Y over S which is divisible by λ and a locally constant isogeny $Y \to X$ over S.*

Proof. For X a BT group we denote by X^t its Serre dual and by $\mathbb{D}(X)$ its crystalline Dieudonné module in the sense of [1] (see also §4). Notice that by [1] theorem 4.2.14, if $H \in \mathbf{BT}(k)$, with classical (contravariant) Dieudonné module M and T is any k-scheme, $\mathbb{D}(H_T) = M^\sigma \otimes_{W(k)} \mathcal{O}_{T/W(k)}$, where M^σ is M with σ-twisted $W(k)$-structure.
Let H be a BT group over k of height h and let $\lambda = r/s$ be the smallest slope of H. Define a BT group Ξ over k by the relation:

$$\mathbb{D}(\Xi^t) = \sum_{i=0}^{h-1} p^{(s-r)i} F^{s(h-i-1)} \mathbb{D}(H^t). \tag{2}$$

This is the group constructed in [24] after lemma 9 (where the covariant Dieudonné module is used, whence the duality and twist). It follows from the construction that Ξ is λ-divisible.
If T is a k-scheme, any automorphism $f : H_T \to H_T$ induces an automorphism of $\mathbb{D}(H_T)$ sending (because of equation (2)) the subcrystal $\mathbb{D}(\Xi_T)$ to itself, hence an automorphism of $\mathbb{D}(\Xi_T)$. If T is a local complete intersection over k, by [12], theorem 3.1 the Dieudonné functor is fully faithful, so f induces an automorphism of Ξ_T.
We therefore get a map $\mathcal{A}ut_{\mathbf{BT}}(H) \to \mathcal{A}ut_{\mathbf{BT}}(\Xi)$ of sheaves on the small syntomic site $\mathrm{Syn}(k)$. By corollary 1, these sheaves are prorepresentable in $\mathrm{Syn}(k)$, whence a group homomorphism

$$\underline{Aut}_k(H) \to \underline{Aut}_k(\Xi) \tag{3}$$

which is in fact injective, because there are no homomorphisms from the p-divisible group Ξ to the finite group $\ker[\,\Xi \to H]$.
Let $X \in \mathbf{BT}(S)$ be locally constant of type H. If $\underline{Aut}_k(H) = \varprojlim \bar{G}_n$, by proposition 3 for all n there is a $\bar{G}_n$-torsor S_n such that $X(n) \cong H(n) \wedge^{\bar{G}_n} S_n$. It suffices to take $Y(n) \cong \Xi(n) \wedge^{\bar{G}_n} S_n$, where $\bar{G}_n$ acts on $\Xi(n)$ via (3). □

Theorem 2 *Let k be a perfect field and X a locally constant BT group over a k-scheme S. Then there exists a completely slope divisible BT group X' over S which is isogenous to X by a locally constant isogeny.*

Proof. We argue by induction on the number of slopes of X. Let λ be the smallest slope of X; by lemma 2 there is a λ-divisible $Y \in \mathbf{LCBT}(S/k)$ and a locally constant isogeny $Y \to X$. If X is isoclinic, we are done by theorem 1. Otherwise, there is an exact sequence

$$0 \longrightarrow Y^{\Phi_\lambda\text{-nil}} \longrightarrow Y \longrightarrow Y^{\Phi_\lambda} \longrightarrow 0 \tag{4}$$

where $\Phi_\lambda = p^{-r}F^s$ if $\lambda = r/s$. By lemma 1, $Y^{\Phi_\lambda\text{-nil}}$ is locally constant with one slope less than X and Y: by induction, there is a locally constant isogeny $Y^{\Phi_\lambda\text{-nil}} \to Z$ with Z completely slope divisible. We take X' to be the pushout of extension (4) by this isogeny:

$$0 \longrightarrow Z \longrightarrow X' \longrightarrow Y^{\Phi_\lambda} \longrightarrow 0.$$

Now Z is slope divisible with respect to slopes which are $> \lambda$, it is also λ-divisible and so is $Y^{\Phi_\lambda\text{-nil}}$ because Y is. From of the exact sequence

$$0 \longrightarrow Y^{\Phi_\lambda\text{-nil}} \longrightarrow Y \times Z \longrightarrow X' \longrightarrow 0$$

we conclude that also X' is completely slope divisible. □

3 The Dieudonné functor

In this section, S is a scheme over a perfect field k of characteristic $p > 0$. We will specialize to $\mathbf{LCBT}(S/k)$ the Dieudonné theory developed by Berthelot, Breen and Messing in [1]. Let $\Sigma = \operatorname{Spec} W(k)$ with Frobenius σ. With the canonical divided power structure on Σ, we work with the big crystalline site $\mathrm{Cris}(S/\Sigma)$ of S endowed with the fppf topology and let $i_{S/\Sigma} : S_{fppf} \to (S/\Sigma)_{\mathrm{Cris}}$ be the canonical immersion of topoi.

Let X be a BT group over S: its *Dieudonné crystal*

$$\mathbb{D}(X) = \mathcal{E}xt^1_{S/\Sigma}(i_{S/\Sigma*}X, \mathcal{O}_{S/\Sigma})$$

is a locally free crystal in $\mathcal{O}_{S/\Sigma}$-modules equipped with a Frobenius and Verschiebung morphisms.

To study the restriction to $\mathbf{LCBT}(S/k)$ of the Dieudonné functor, we use an alternative description via a Dieudonné module à la Barsotti-Fontaine (cf. Berthelot-Messing [2] §2 for the étale and multiplicative cases).

As usual, denote by $D_k = W(k)[F, V]/(FV = VF = p)$ the Dieudonné ring. Recall from [1] §4.1 that on $\mathrm{Cris}(S/\Sigma)$ we have a sheaf of D_k-modules $CW^\sigma_{S/\Sigma}$, the sheaf of Witt covectors, and a canonical extension

$$0 \longrightarrow \mathcal{O}_{S/\Sigma} \longrightarrow \mathcal{E}_{S/\Sigma} \longrightarrow CW^\sigma_{S/\Sigma} \longrightarrow 0 \tag{5}$$

whose construction commutes with arbitrary base change.

Definition 5 *The* Dieudonné module *of a locally constant BT group X is the crystalline sheaf of D_k-modules $\mathbb{M}(X) = \mathcal{H}om_{S/\Sigma}(i_{S/\Sigma*}X, CW^\sigma_{S/\Sigma})$.*

Proposition 6 *Let X be a locally constant BT group over S. The boundary map $\partial : \mathbb{M}(X) \to \mathbb{D}(X)$ obtained by applying the functor $\mathcal{H}om_{S/\Sigma}(i_{S/\Sigma*}X, -)$ to the extension (3) is an isomorphism of D_k-modules.*

Proof. Let us first remark (cf. [1] proof of proposition 2.4.5) that

$$\mathbb{M}(X) = \varprojlim \mathcal{H}om_{S/\Sigma}(i_{S/\Sigma*}X(n), CW^\sigma_{S/\Sigma}) \qquad \text{and} \qquad \mathbb{D}(X) = \varprojlim \mathcal{E}xt^1_{S/\Sigma}(i_{S/\Sigma*}X(n), \mathcal{O}_{S/\Sigma}),$$

so it suffices to prove the statement for $X(n)$ for all n. Let $f_n : S_n \to S$ be such that $X(n) \times_S S_n \cong H(n) \times_k S_n$ Because of [1] 1.1.11, $f_{n,\mathrm{Cris}}$ is a localization functor, so $f^*_{n,\mathrm{Cris}}$ is exact and we only need to check that $f^*_{n,\mathrm{Cris}}(\partial)$ is an isomorphism. Recalling that the extension (3) commutes with base change, we have $f^*_{n,\mathrm{Cris}} CW^\sigma_{S/\Sigma} = CW^\sigma_{S_n/\Sigma}$. Let $\pi_n : S_n \to \operatorname{Spec} k$ be the stucture morphism. Applying [1] 1.3.3 twice we get

$$\begin{array}{ccc}
f^*_{n,\mathrm{Cris}}\mathcal{H}om_{S/\Sigma}(i_{S/\Sigma*}X(n), CW^\sigma_{S/\Sigma}) & \xrightarrow{\partial} & f^*_{n,\mathrm{Cris}}\mathcal{E}xt^1_{S/\Sigma}(i_{S/\Sigma*}X(n), \mathcal{O}_{S/\Sigma}) \\
\cong\downarrow & & \downarrow\cong \\
\mathcal{H}om_{S_n/\Sigma}(i_{S_n/\Sigma*}(H(n) \times S_n), CW^\sigma_{S_n/\Sigma}) & \xrightarrow{\partial} & \mathcal{E}xt^1_{S_n/\Sigma}(i_{S_n/\Sigma*}(H(n) \times S_n), \mathcal{O}_{S_n/\Sigma}) \\
\cong\uparrow & & \uparrow\cong \\
\pi^*_{n,\mathrm{Cris}}\mathbb{M}(H(n)) & \xrightarrow{\cong} & \pi^*_{n,\mathrm{Cris}}\mathbb{D}(H(n))
\end{array}$$

the bottom isomorphism being the comparison between classical and crystalline Dieudonné theory over a perfect field ([1] theorem 4.2.14). Notice that, if M(-) is the classical (controvariant) Dieudonné module, $\mathbb{M}(H(n)) = M(H(n)) \otimes_\sigma W(k)$. □

Definition 6 *Let $\mathcal{M}$ be a finitely presented, locally free, non-degenerate F-crystal and put $\mathcal{M}_n = \mathcal{M}/p^n\mathcal{M}$. We say that $\mathcal{M}$ is a* locally cloven[1] F-crystal *if there exists a non-degenerate F-crystal M over k such that for all n there is an fppf covering $f_n : S_n \to S$, and an isomorphism of $\mathcal{O}_{S_n/\Sigma}[F,V]$-modules $f^*_{n,\mathrm{CRIS}}\mathcal{M}_n \cong M_n \otimes \mathcal{O}_{S_n/\Sigma}$. If $FV = VF = p$ on $\mathcal{M}$, we will call it a* locally cloven Dieudonné crystal.

As in [2] 2.4, let $\mathbf{C}(S)$ be the category of finitely presented $\mathcal{O}_{S/\Sigma}$-modules with Frobenius and Veschiebung operators. We define the category $\mathbf{LCC}(S)$ (resp. $\mathbf{LCD}(S)$) of locally cloven F- (*resp.* Dieudonné) crystals over S *as the full subcategory of $\mathbf{C}(S)$ whose objects are locally cloven F- (resp. Dieudonné) crystals.*

Clearly, if X is a locally constant Barsotti-Tate group, $\mathbb{M}(X)$ (with $\mathcal{O}_{S/\Sigma}$-module structure defined by the isomorphism of proposition 6) is a locally cloven Dieudonné module, whence a contravariant functor

$$\mathbb{M} : \mathbf{LCBT}(S/k)^\circ \longrightarrow \mathbf{LCD}(S)$$

Following the original idea of Grothendieck [8] II.6 we are going to construct an explicit inverse functor, namely

$$\mathbb{E}(\mathcal{M}) = \varinjlim i^*_{S/\Sigma}\mathcal{H}om_{S/\Sigma,D_k}(\mathcal{M}_n, CW^\sigma_{S/\Sigma}), \tag{6}$$

where $\mathcal{H}om_{S/\Sigma,D_k}$ is the sheaf of homomorphisms as D_k-modules. For this to make sense, we need a representability result:

Lemma 3 *The fppf sheaf $\mathbb{E}(\mathcal{M}_n) = i^*_{S/\Sigma}\mathcal{H}om_{S/\Sigma,D_k}(\mathcal{M}_n, CW^\sigma_{S/\Sigma})$ is representable by a locally constant BT_n group.*

Proof. Let $f_n : S_n \to S$ be as in def. 6, with $\pi_n : S_n \to \operatorname{Spec} k$. Then

$$\begin{aligned} f^*_n\, i^*_{S/\Sigma}\mathcal{H}om_{S/\Sigma,D_k}(\mathcal{M}_n, CW^\sigma_{S/\Sigma}) &\cong i^*_{S_n/\Sigma}\mathcal{H}om_{S_n/\Sigma,D_k}(f^*_{n,\mathrm{CRIS}}(\mathcal{M}_n), CW^\sigma_{S_n/\Sigma}) \\ &\cong i^*_{S_n/\Sigma}\mathcal{H}om_{S_n/\Sigma,D_k}(M_n \otimes \mathcal{O}_{S_n/\Sigma}, CW^\sigma_{S_n/\Sigma}) \\ &\cong \pi^*_n\,\mathbb{E}(M_n). \end{aligned}$$

Hence $\mathbb{E}(\mathcal{M}_n)$ is a twisted form of $\mathbb{E}(M_n) \times S$ and therefore representable by proposition 1. □

Theorem 3 *Over an arbitrary k-scheme S the functor $\mathbb{E}$ is a quasi-inverse of $\mathbb{M}$. In particular, if S is such that the Dieudonné functor is fully faithful, its restriction is an anti-equivalence $\mathbb{M} : \mathbf{LCBT}(S/k)^\circ \to \mathbf{LCD}(S)$.*

Proof. It suffices to prove the statement for p^n kernels (resp. cokernels). Recall from [1] 1.1.4.3 that the canonical map $i^*_{S/\Sigma}i_{S/\Sigma*}X(n) \to X(n)$ is an isomorphism. Composing $i^*_{S/\Sigma}$ with the evaluation map we get a canonical morphism

$$X(n) \cong i^*_{S/\Sigma}\, i_{S/\Sigma*}X(n) \to i^*_{S/\Sigma}\mathcal{H}om_{S/\Sigma,D_k}\Big(\mathcal{H}om_{S_n/\Sigma}(i_{S/\Sigma*}X(n), CW^\sigma_{S/\Sigma}), CW^\sigma_{S/\Sigma}\Big) = \mathbb{E}(\mathbb{M}(X(n)))$$

and both are twists of the same constant BT_n, so this map is an isomorphism. Conversely, start with an $\mathcal{M}_n$ and take $f_n : S_n \to S$ such that $f^*_{n,\mathrm{CRIS}}\mathcal{M}_n \cong M_n \otimes \mathcal{O}_{S_n/\Sigma}$. Recall that $f_{n,\mathrm{CRIS}}$ is a localization map and that $f^*_{n,\mathrm{CRIS}} \circ i_{S/\Sigma*} = i_{S_n/\Sigma*} \circ f^*_n$ (cf. [1] 1.1.17.1). Applying $f^*_{n,\mathrm{CRIS}}$

[1] In mineralogy, cleavage is defined as the property for a crystal to split along a symmetry axis or plane.

to the adjoint map $\mathcal{H}om_{S/\Sigma,D_k}(\mathcal{M}_n, CW^\sigma_{S/\Sigma}) \to i_{S/\Sigma *}\mathbb{E}(\mathcal{M}_n)$, we see that it is an isomorphism. Composition with evaluation yields a morphism:

$$\mathcal{M}_n \to \mathcal{H}om_{S/\Sigma}\left(\mathcal{H}om_{S/\Sigma,D_k}(\mathcal{M}_n, CW^\sigma_{S/\Sigma}), CW^\sigma_{S/\Sigma}\right) \stackrel{\cong}{\leftarrow} \mathbb{M}(\mathbb{E}(\mathcal{M}_n))$$

which we again check to be an isomorphism by taking $f^*_{n,\mathrm{CRIS}}$. □

Remark 4 For étale or multiplicative BT groups, this result was proved by Berthelot-Messing [2], Corollary 2.4.10. In the general case, because of torsion in PD envelopes, some assumption on the base scheme S is necessary for the Dieudonné functor to be fully faithful. Berthelot and Ogus [3], appendix, give an example of a *constant* BT group X over $S = \operatorname{Spec} k[x,y]/(x^2, xy, y^2)$ such that $\operatorname{End}_{\mathbf{BT}(S)}(X) \not\cong \operatorname{End}_{\mathbf{C}(S)}(\mathbb{D}(X))$. We refer to de Jong's ICM talk [11] for a summary of current knowledge about the faithfulness of the Dieudonné functor.

We now give a recipe to construct locally cloven F-crystals on a proper smooth scheme mimicking the universal extension for vector bundles.

Example 3 Let S be a proper, smooth scheme over k. Denote by a superscript $^\vee$ the linear dual of a crystal in $\mathcal{O}_{S/\Sigma}$-modules. For $m \geq 0$, let $\mathcal{O}_{S/\Sigma}(-m)$ be the F-crystal whose Frobenius is multiplication by p^m. If $\mathcal{M}$ is an F-crystal of level ℓ (i.e. if $FV = VF = p^\ell$ on $\mathcal{M}$), denote by $\mathcal{M}(-m) = \mathcal{M} \otimes \mathcal{O}_{S/\Sigma}(-m)$ (an F-crystal of level $\ell + m$) and by $\mathcal{M}^\vee(-\ell)$ the F-crystal of level ℓ with underlying crystal $\mathcal{M}^\vee$ and Frobenius $V^t_{\mathcal{M}}$. To lighten the notation, a constant (F-)crystal $M \otimes_{W(k)} \mathcal{O}_{S/\Sigma}$ is simply written M.

Step 1. Let $\mathcal{M}$ be an F-crystal of level ℓ on S and *assume* that the first crystalline cohomology group $H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))$ is free over $W(k)$. Then $H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))$ can be endowed with a F-crystal structure of level $\ell+1$ with Frobenius and Verschiebung induced by those of $\mathcal{M}^\vee(-\ell)$. Thus $N = H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))^\vee(-\ell-1)$ is an F-crystal of level $\ell+1$ over k.

Step 2. Let $\mathcal{U}(\mathcal{M})$ be the locally free crystal corresponding to the identity in

$$\begin{aligned}\operatorname{End}_{W(k)}\left(H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))\right) &= H^1\left(S/\Sigma, \mathcal{M}^\vee(-\ell) \otimes_{W(k)} H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))^\vee\right) \\ &\cong H^1(S/\Sigma, \mathcal{H}om_{\mathcal{O}_{S/\Sigma}}(\mathcal{M}(-1), N)) \\ &= \operatorname{Ext}^1_{\mathcal{O}_{S/\Sigma}}(\mathcal{M}(-1), N).\end{aligned}$$

$\mathcal{U}(\mathcal{M})$ is universal among locally free crystals which are extensions of $\mathcal{M}(-1)$ by a free crystal:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \mathcal{U}(\mathcal{M}) & \xrightarrow{u} & \mathcal{M}(-1) & \longrightarrow & 0 \\ & & \partial\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{M}(-1) & \longrightarrow & 0 \end{array}$$

∂ being the dual of the map $\partial^\vee : \Lambda^\vee \to N^\vee$ induced by the cohomology sequence of $\mathcal{E}^\vee$.

Step 3. From the universal property and the fact that the Frobenius of $H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))$ is induced by that of $\mathcal{M}^\vee(-\ell)$, it follows that $F^*\mathcal{U}(\mathcal{M}) = \mathcal{U}(F^*\mathcal{M})$ and that $\mathcal{U}(\mathcal{M})$ is an F-crystal of level $\ell+1$, with Frobenius structure compatible with the extension.

Step 4. Because of the universal property, the canonical map

$$H^1(S/\Sigma, \mathcal{M}^\vee(-\ell)) \to H^1(S/\Sigma, \mathcal{U}(\mathcal{M})^\vee(-\ell-1))$$

taking an extension to its pullback by u, is zero, hence $H^1(S/\Sigma, \mathcal{U}(\mathcal{M})^\vee(-\ell-1))$ injects into the free $W(k)$-module $H^1(S/\Sigma, N^\vee(-\ell-1))$ and is therefore free. Hence $\mathcal{U}(\mathcal{M})$ is an F-crystal whose dual has a free H^1, so we can restart the procedure.

Step 5. Assume now that $\mathcal{M}$ is locally cloven. To show that $\mathcal{U}(\mathcal{M})$ is locally cloven it suffices to prove that for all $n \geq 1$ there is a finite flat cover $g_n : S_n \to S$ such that the induced map $H^1(S/\Sigma_n, \mathcal{M}^\vee(-\ell)_n) \to H^1(S_n/\Sigma_n, g^*_{n,\mathrm{CRIS}}\mathcal{M}^\vee(-\ell)_n)$ is zero. Indeed, if $h_1, \dots h_r$ is a basis for $H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))$ and $h_1^\vee, \dots h_r^\vee$ is the dual basis, the class of the extension $\mathcal{U}(\mathcal{M})$ is given by $\sum h_i \otimes h_i^\vee$, hence the class of $g^*_{n,\mathrm{CRIS}}\mathcal{U}(\mathcal{M})$ is $\sum g_n^*(h_i) \otimes h_i^\vee$.
First, since S is proper and smooth, by [9] II 3.11, $H^1(S/\Sigma, \mathcal{O}_{S/\Sigma})$ is the Dieudonné module of the BT group of the Albanese variety A of S; hence, if $S'_n \to S$ is the cover of S corresponding to the isogeny p^n of A,

$$H^1(S/\Sigma_n, \mathcal{O}_{S/\Sigma_n}) = \mathbb{D}(A(n)) \xrightarrow{p^n} \mathbb{D}(A(n)) \to H^1(S'_n/\Sigma_n, \mathcal{O}_{S'_n/\Sigma_n})$$

is the zero map. So, if $\mathcal{M}$ is constant, this cover splits $\mathcal{U}(\mathcal{M})/p^n$. If $\mathcal{M}$ is locally cloven, by definition there is a finite flat cover $f_n : S''_n \to S_n$ such that $f^*_{n,\mathrm{CRIS}}\mathcal{M}_n \cong M_n \otimes \mathcal{O}_{S''_n/\Sigma_n}$. Taking $S_n = S' \times_S S''_n$ we get:

$$H^1(S/\Sigma_n, \mathcal{M}^\vee(-\ell)_n) \to H^1(S''_n/\Sigma_n, M^\vee(-\ell)_n) \xrightarrow{p^n} H^1(S_n/\Sigma_n, M^\vee(-\ell)_n).$$

Hence $\mathcal{U}(\mathcal{M})$ is a locally cloven F-crystal if $\mathcal{M}$ is.

In particular, if $H^1(S, \mathcal{O}_S) \neq 0$, starting from the trivial crystal $\mathcal{O}_{S/\Sigma}$ we can construct an infinite sequence of locally cloven F-crystals $\mathcal{U}(\mathcal{O}_{S/\Sigma})$, $\mathcal{U}(\mathcal{U}(\mathcal{O}_{S/\Sigma})), \dots$, each time increasing the level by 1. By theorem 3, $\mathcal{U}(\mathcal{O}_{S/\Sigma})$ is the Dieudonné crystal of a locally constant BT group, a nontrivial extension of the BT of the Picard variety of S, whose Dieudonné module is $H^1(\mathcal{O}_{S/\Sigma})^\vee(-1)$, by $\hat{\mathbb{G}}_{\mathrm{m}}$.

Remark 5 We have shown in step 4 above that the F-crystal $H^1(S/\Sigma, \mathcal{U}(\mathcal{M})^\vee(-\ell-1))$ injects into $H^1(S/\Sigma, N^\vee(-\ell-1)) = H^1(S/\Sigma, \mathcal{O}_{S/\Sigma}) \otimes H^1(S/\Sigma, \mathcal{M}^\vee(-\ell))$. It therefore belongs to the $\otimes$-subcategory of the category of F-crystals over k generated by the last two.
In particular, the constant F-crystals associated to the higher extensions $\mathcal{U}^i(\mathcal{O}_{S/\Sigma})$ are subquotients of the $\otimes$-category generated by the Dieudonné modules of $Pic^0_{\mathrm{red}}(S)$ and $\hat{\mathbb{G}}_{\mathrm{m}}$.

Remark 6 We can recover from $\mathcal{U}(\mathcal{M})$ the universal extension $\mathcal{U}(\mathcal{V})$ of the vector bundle $\mathcal{V} = i^*_{S/\Sigma}\mathcal{M}$: in view of the definitions of both universal extensions, it is immediate to check that the class of the extension $i^*_{S/\Sigma}\mathcal{U}(\mathcal{M})$ in $H^1(S, \mathcal{H}om_{\mathcal{O}_S}(\mathcal{V}, H^1_{dR}(\mathcal{V}^\vee)^\vee)) = H^1(S, \mathcal{V}^\vee) \otimes H^1_{dR}(\mathcal{V}^\vee)^\vee$ is the image of the class $[\mathcal{U}(\mathcal{V})] = id \in \mathrm{End}(H^1(S, \mathcal{V}^\vee))$ by the canonical map $H^1_{dR}(S, \mathcal{V}^\vee) \to H^1(S, \mathcal{V}^\vee)$.

4 p-adic representations of the fundamental group scheme

Let S a reduced, connected, flat scheme over a base scheme B and $b \in S(B)$ a fixed B-valued point. Nori [18], chap. 2 (when B the spectrum of a field) and Gasbarri [7] (when B is a Dedekind scheme) have defined the *fundamental group scheme* $\boldsymbol{\pi}(S/B; b)$: it is a profinite B-group scheme which classifies torsors over S under finite, flat B-group schemes with a fixed B-valued point over b. When B is the spectrum of an algebraically closed field and S is complete, $\boldsymbol{\pi}(S/B; b)$ coincides with the fundamental group of the tannakian category of essentially finite vector bundles on S (Nori [19] chap. 1, reproduced in [18]).

$\boldsymbol{\pi}(S/B; b)$ thus generalizes the étale fundamental group of S in the sense of [SGA1], which can be recovered as the maximal étale quotient of $\boldsymbol{\pi}(S/B; b)$, up to a point. Indeed, Grothendieck's fundamental group is constructed using *geometric* base points. So let $\bar{\eta}$ be the spectrum of an algebraic closure of the fraction field of B; the choice of $b \in S(B)$ defines a retraction β of the canonical surjection $\pi_1(S, \bar{\eta}) \to \pi_1(B, \bar{\eta})$ and the maximal étale quotient of $\boldsymbol{\pi}(S/B; b)$ is then the quotient of $\pi_1(S, \bar{\eta})$ by the normal subgroup generated by $\mathrm{im}\,\beta$.

Some of the classical properties of the étale fundamental group proved in [SGA1] carry over to the fundamental group scheme while others fail. The following theorem, a generalization of the specialization theorem [SGA1] corollaire 2.3 under the assumption that S is normal, is a good illustration of the subtleties arising in the torsor context. Notice that the stronger corollaire 2.4 cannot at present be generalized because, over a field, the fundamental group scheme does not commute with extensions of the ground field, as shown by Mehta and Subramanian [15].

Theorem 4 *Let S be a normal scheme, proper and flat over the spectrum B of a discrete valuation ring with fraction field K and let $b \in S(B)$. Then the canonical morphism $\boldsymbol{\pi}(S_K/K; b_K) \to \boldsymbol{\pi}(S/B; b) \times_B K$ is an epimorphism.*

Proof. We may assume that S is connected. Let $T \to S$ be a torsor under a finite flat B-group scheme G such that $\boldsymbol{\pi}(S/B; b) \to G$ is an epimorphism; we have to show that $\boldsymbol{\pi}(S_K/K; b_K) \to G_K$ is also epimorphic. The point is that this is not guaranteed by the connectedness of T_K. So let G'_K be the image of $\boldsymbol{\pi}(S_K/K; b_K)$ in G_K. The scheme-theoretic closure G' of G'_K in G is a finite flat group scheme over B by [22] 2.1 (arguments of this type are the reason why Gasbarri [7] has to stick to Dedekind bases in his construction of the fundamental group scheme). Consider the S-scheme $T' = T \wedge^G (G/G')$: according to [6], chap. III, §4 prop. 4.6, the class of T is in the image of $H^1(S, G') \to H^1(S, G)$ if and only if $T' \to S$ has a section. By assumption, we have a section $s_K : S_K \to T'_K$; let S' be the scheme-theoretic closure of the image of s_K: it is finite and birational over S. Since S is normal, we must have $S' \cong S$. Whence the contradiction that $T \to S$ admits a reduction of the structure group. □

Let B be the spectrum of a field k (resp. of a mixed discrete valuation ring R) and let X be a locally constant BT group on S. By proposition 3 (resp. 4) for all n, $X(n)$ trivializes over a torsor under a finite flat B-group scheme. If we choose the type of X to be $H = X_b$, this torsor is also pointed at b, hence corresponds to a finite quotient of $\boldsymbol{\pi}(S/B; b)$.

Recall that, if $\pi_1(S)$ is the fundamental group of S in the sense of [SGA1] (based at some geometric point of S), there is an equivalence of categories between p-adic local systems on S and representations of $\pi_1(S)$ into finite free $\mathbb{Z}_p$-modules (e.g. [23], chap. VI, §1). We will regard locally constant BT groups on S as *integral p-adic representations* of the fundamental group scheme $\boldsymbol{\pi}(S/B; b)$. In this respect, theorem 3 can be viewed as a generalization of the classical Katz correspondence between p-adic representations of $\pi_1(S)$ and unit-root F-crystals on S (conf. [14], proposition 4.1.1, [23], proposition 3.1.2.1 and [4] theorem 2.2, actually only stated for a smooth S but valid in general as remarked in [2], Corollary 2.4.10). From this point of view, it is impossible not to mention de Jong and Oort's purity theorems [13] generalizing to BT groups Zariski-Nagata's purity theorem for étale covers ([SGA1], exposé X, théorème 3.1), although it should be mentioned that the purity theorem for the fundamental group scheme had already been proved by Nori ([18], chap. II, proposition 7).

$\boldsymbol{\pi}(S/B; b)$ is the limit of the filtered inverse system of its finite quotients ([7] §2). If X_1 and X_2 are locally constant BTs of type H_1 and H_2, they correspond to two "representations" of $\boldsymbol{\pi}(S/B; b)$ into $\underline{Aut}_B(H_i) = \varprojlim \underline{Aut}_B(H_i(n))$: they are dominated by the representation into $\underline{Aut}_B(H_1 \oplus H_1)$ associated to the locally constant BT group $X_1 \oplus X_2$ (we will find it convenient to denote by $\oplus$ the product of BT groups).

Definition 7 *The (weak) BT quotient $\boldsymbol{\pi}_{BT_+}(S/B; b)$ of the fundamental group scheme is the limit of the filtered inverse system of quotients of $\boldsymbol{\pi}(S/B; b)$ associated to locally constant BT groups.*

Remark 7 The qualification of weak quotient and the + in the notation are meant to remind that for BT groups we can only take direct sums. The "right" BT quotient should correspond to a suitable $\otimes$-subcategory of the category of locally cloven F-crystals equipped with an action of $\boldsymbol{\pi}(S/B; b)$. We hope to investigate this question in the future.

From now on we assume that B is the spectrum of a perfect field k of characteristic $p > 0$. By theorem 2 then, $\boldsymbol{\pi}_{BT_+}(S/k;b)$ is the limit of the filtered inverse system of quotients of $\boldsymbol{\pi}(S/B;b)$ associated to completely slope divisible BT groups.

Theorem 5 *If $B = \operatorname{Spec} k$, the connected component $\boldsymbol{\pi}^0_{BT_+}(S/k;b)$ of $\boldsymbol{\pi}_{BT_+}(S/k;b)$ is nilpotent and torsion-free.*

Proof. Let X be a completely slope divisible BT group over S: by definition 3 it has a slope filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_m = X$$

with slopes $1 \geq \lambda_1 > \lambda_2 > \cdots > \lambda_m \geq 0$. Let $H = X_b$ and recall that, this being a csd BT group over a perfect field, $H = \bigoplus_{i=1}^m H_{\lambda_i}$, where H_{λ_i} is an isoclinic slope divisible BT group.
Let G be the quotient of $\boldsymbol{\pi}(S/B;b)$ acting on H i.e. a projective limit of subgroups $G_n \subset \underline{Aut}_k(H(n))$ such that there exists a finite flat G_n-torsor S_n over which $X(n)_{S_n} \simeq H(n)_{S_n}$.
Notice that X_i is locally constant of type $H_i = \bigoplus_{j=1}^i H_{\lambda_j}$ and that an isomorphism $X_T \simeq H_T$ over some flat B-scheme T induces an isomorphism $X_{i,T} \simeq H_{i,T}$ because X_i (resp. H_i) is the Φ_i-nilpotent part of X (resp. H). Hence G stabilizes H_i.
Assume moreover that G is connected. Then it acts trivially on $H_{\lambda_i} = H_i/H_{i-1}$ because X_i/X_{i-1} is isoclinic slope divisible, hence trivializes over an étale torsor. Define, for $0 \leq i \leq m$ a subgroup scheme G^i of G by setting, for all k-algebra Λ,

$$G^i(\Lambda) = \{g \in G(\Lambda) |\, (g-1)(H_{i,\Lambda}) \subseteq H_{j-i,\Lambda},\ j = 1, \ldots, m\}$$

(where $H_l = 0$ for $l < 0$). By the above remark, $G^1 = G$. Moreover, one checks immediately that $G^i = 1$ for $i \geq m$ and $\left[G^i, G^j\right] \subseteq G^{i+j}$. Hence G is nilpotent. The fact that it is torsion-free follows from the following remark. □

Remark 8 With notations as in the proof, the subgroup G^i stabilizes H_{i+1} (as G does) and acts trivially on H_i. Hence G^i/G^{i+1} acts on H_{i+1}, with trivial action on both H_i and $H_i/H_{i+1} = H_{\lambda_{i+1}}$ whence a representation

$$G^i/G^{i+1} \longrightarrow \underline{Hom}_k(H_{\lambda_{i+1}}, H_i) = \bigoplus_{j=1}^{i} \underline{Hom}_k(H_{\lambda_{i+1}}, H_{\lambda_j}).$$

We can therefore visualize the action of G on the "flag" H as given by unitriangular matrices with entries $a_{i,j} \in \underline{Hom}_k(H_{\lambda_j}, H_{\lambda_i})$ for $i < j$ (hence $\lambda_i > \lambda_j$). Being the Tate modules of BT groups (proposition 2), these groups are torsion-free. Notice however that G is not necessarily unipotent, for $\underline{Hom}_k(\mathbb{Q}_p/\mathbb{Z}_p, \hat{\mathbb{G}}_{\mathrm{m}}) = \varprojlim \boldsymbol{\mu}_{p^n}$.

To understand how far $\boldsymbol{\pi}^0_{BT_+}(S/k;b)$ is from the connected component of the *nilpotent fundamental group scheme* $\boldsymbol{\pi}^{\mathrm{nil}}(S/k;b)$ (i.e. the quotient of $\boldsymbol{\pi}(S/k;b)$ classifying pointed torsors over S under finite nilpotent group schemes), consider the case when S is proper and smooth. Then, according to Nori [18], chap. IV, prop. 6, the *abelian* quotient of the fundamental group scheme is

$$\boldsymbol{\pi}^{\mathrm{ab}}(S/k;b) = \varprojlim Alb(S)(n)$$

where $Alb(S)$ is the Albanese variety of S (here n is not necessarily a p-power). By example 3, there is a locally constant BT group X, defined by $\mathbb{D}(X) = \mathcal{U}(\mathcal{O}_{S/\Sigma})$, such that $X(n)$ trivializes over the cover of S defined by the isogeny p^n of $Alb(S)$. Hence $\boldsymbol{\pi}(S/k;b) \to \boldsymbol{\pi}^{\mathrm{ab},p}(S/k;b)$ (maximal pro-p-quotient) factors through $\boldsymbol{\pi}_{BT_+}(S/k;b)$

Example 3 suggest that, in order to study non abelian quotients, we should take into account not only BT groups but locally cloven crystals as well. However, it is not clear that (the reductions mod p^n of) such crystals trivialize over finite *torsors*. Still, when S is proper and smooth, we can use Nori's interpretation [19] of $\boldsymbol{\pi}(S/k;b)$ as the Tannaka fundamental group of the category $\mathbf{EF}(S,b)$ of *essentially finite vector bundles* on S. The functor $i^*_{S/\Sigma} : (S/\Sigma)_{\mathrm{CRIS}} \to S_{fppf}$, defined by $i^*_{S/\Sigma}(\mathcal{F})(U) = \mathcal{F}(U,U)$ induces a functor

$$i^*_{S/\Sigma} : \mathbf{LCC}(S) \longrightarrow \mathbf{EF}(S,b)$$

because the vector bundle $i^*_{S/\Sigma}(\mathcal{F})$ trivializes over a finite flat cover. By composition we get a functor

$$\mathbf{LCBT}(S/k)^\circ \xrightarrow{\mathbb{M}} \mathbf{LCC}(S) \xrightarrow{i^*_{S/\Sigma}} \mathbf{EF}(S,b) \cong Rep_k\left(\boldsymbol{\pi}(S/k;b)\right).$$

Recall that a vector bundle is called *unipotent* if it is a successive extension of trivial bundles (beware that Nori [18], chap. IV calls them nilpotent).

Theorem 6 *Let S be a proper smooth scheme over k. The tannakian subcategory of $\mathbf{EF}(S,b)$ generated by the image of $i^*_{S/\Sigma} \circ \mathbb{M}$ contains the category $Rep_k\left(\boldsymbol{\pi}^{\mathrm{un}}(S/k;b)\right)$ of unipotent vector bundles.*

Proof. Let $\mathcal{U}^n = \mathcal{U}\left(\ldots \mathcal{U}(\mathcal{O}_S)\right)$ be the n-th universal extension of $\mathcal{O}_S$: it is a unipotent vector bundle trivializing over a torsor under a finite unipotent group scheme N_n. According to [18], chap. IV, lemma 7,

$$\boldsymbol{\pi}^{\mathrm{un}}(S/k;b) = \varprojlim N_n.$$

The claim now follows from example 3 and remark 6. □

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