On the “Galois closure” for finite morphisms

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Abstract

We give necessary and sufficient conditions for a finite flat morphism of schemes of characteristic $p > 0$ to be dominated by a torsor under a finite group scheme. We show that schemes satisfying this property constitute the category of covers for the fundamental group scheme.

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Introduction

The fundamental construction in Galois theory is that any separable field extension can be embedded in a Galois extension. Grothendieck [7] has generalized Galois theory to schemes (and potentially to even more abstract situations: Galois categories). Again, the basic step is, starting from a finite étale morphism $\pi : X \to S$, to construct a finite group G, a subgroup $H \leq G$ and a diagram

$$\begin{array}{ccc} Y & \xrightarrow{h} & X \\ & \searrow^{g} & \downarrow{\scriptstyle \pi} \\ & & S \end{array} \qquad (1)$$

where g and h are finite étale Galois covers of groups G and H respectively. Recall that a finite étale morphism $X \to S$ is a Galois cover if a finite group G acts on X without fixed points and S is identified with the quotient of X by this action (cf. [10], §7). This is equivalent to saying that X is a principal homogenous space (or torsor) over S under G, i.e., that the map $G \times X \to X \times_S X$ given by $(\gamma, x) \mapsto (\gamma x, x)$ is an isomorphism.

In characteristic $p > 0$ or in an arithmetic context it is often necessary to consider not only actions by abstract groups but infinitesimal actions as well. For instance an isogeny between abelian varieties may have an inseparable component (or degenerate to one). One is then led to consider torsors under finite flat group schemes (cf. [10], §12).

In this note, we start with a finite flat morphism $\pi : X \to S$ of schemes of characteristic $p > 0$ and we try to find a “Galois closure” as in diagram (1), where g and h are torsors under group schemes G and $H \leq G$ defined over the prime field $\mathbb{F}_p$.

First of all, not any finite flat morphism π will do: indeed, if a “Galois closure” Y as above can be found at all, X will be a twisted form (in the flat topology) of the homogeneous scheme G/H, so π will have to be a local complete intersection morphism. It turns out that the right class of morphism, namely the *differentially homogeneous* morphisms, has been studied thoroughly by Sancho de Salas [13], who has developed a differential calculus extending Grothendieck’s for smooth and étale morphisms. As for smoothness and étaleness, this is a local notion.

Our first result (Theorem 2.3) is that any finite differentially homogeneous morphism $\pi : X \to S$ of schemes in characteristic $p > 0$ fits in a diagram as in (1) above, where g and h are torsors under group schemes G and $H \leq G$ defined over the prime field $\mathbb{F}_p$.

As we shall explain shortly, Grothendieck's construction of the Galois closure for finite étale morphisms does not apply when one drops the étaleness assumption. We thus have to give a direct construction of a universal torsor dominating π: in many cases, it will much larger than the actual "Galois closure".

Let us describe our construction in the case of fields: a separable extension $L = K[x]/f(x)$ of degree n can be seen as a twist of K^n. The automorphism group of the geometric fibre of $K \subseteq L$ (i.e., the set of roots of f in an algebraic closure of K) is the symmetric group $\mathfrak{S}_n$, so L defines a Galois cohomology class in $H^1(K, \mathfrak{S}_n)$, represented by a Galois K-algebra A such that $A \otimes_K L \simeq A^n$. Any étale K-algebra B such that $B \otimes_K L \simeq B^n$ receives a map from A, and in particular the Galois closure of L/K is a direct summand of A. Moreover $L \subseteq A$ consists of elements fixed by the stabilizer $\mathfrak{S}_{n-1}$ of a given root of f.

Unfortunately, the group schemes acting on our universal torsor are not finite in general; for instance they are not in the case of the Frobenius morphism $\pi : \mathbb{P}^1 \to \mathbb{P}^1$. The reason is that, in contrast with the étale case, the automorphism group scheme of a fibre of π is not finite.

Our main result, Theorem 2.11, gives necessary and sufficient conditions for the existence of a finite Galois closure Y as in (1). In contrast with the étale case, these conditions are of a global nature, as can be expected from the counterexample above.

Except when one can reduce to the case of field extensions (e.g., when all schemes involved are normal), Grothendieck's construction of the Galois closure of an étale morphism is indirect and relies on his theory of the fundamental group [7], V §4. Let us now briefly review it, disregarding base points for simplicity. Grothendieck first proves that the category of finite étale covers of a given scheme is filtered: this relies on the fact that fibred products of étale morphisms are again étale. In fact, existence of finite fibred products is the first axiom that any Galois category should satisfy. This fails dismally for arbitrary finite flat morphisms.

Grothendieck's second step is formal: being filtered, the category of étale covers has a projective limit, which is the universal cover. He then turns his attention to connected covers, as any cover breaks down as a disjoint union of connected ones. Since any endomorphism of a connected cover is an automorphism, he defines the Galois objects as the simple, connected covers. Tautologically, these form a filtering subsystem, thus any cover is dominated by a smallest Galois cover, which is the Galois closure. Obviously, this process cannot be replicated with flat covers: a trivial torsor under any infinitesimal group scheme is connected.

The arithmetic fundamental group $\pi_1(S)$ is the projective limit of all the Galois groups over S i.e., the automorphism groups of the Galois covers of S. If S is given over a base scheme B, later in his seminar (X 2.5), Grothendieck suggested to look for a profinite B-group scheme classifying torsors over S under finite flat B-group schemes. This *fundamental group scheme* $\boldsymbol{\pi}(S/B)$ should be the projective limit of all finite group schemes occurring as structure groups of torsors over S. In terms of Galois theory as outlined above, this approach forgets the general category of covers to focus solely on Galois objects.

This program has been pursued by Nori [11] (over a base field) and Gasbarri [4] (over a Dedekind base). Much progress has been made recently on the fundamental group scheme. This is especially true in the case of proper reduced schemes over a field, where again Nori [11] gave a Tannakian interpretation of the fundamental group scheme in terms of vector bundles, whence a connection with motivic fundamental groups.

The basic existence criterion for the fundamental group scheme is that the category of torsors should admit finite fibred products: a formal argument due to Nori shows then that the category of torsors is filtered and the universal cover is just the limit of this category. As is to be expected from the above mentioned pathologies, the existence of fibred products can only be proven under quite restrictive assumptions on S and B. In Theorem 4.5, as a consequence of our main result, we improve slightly on previously known existence results for the fundamental group scheme.

The conceptual significance of the Galois closure problem is that it pinpoints the essential property of covers for abstract fundamental groups: for the flat topology, it allows us to trace Grothendieck's steps backwards, from Galois objects to covers. "Covers" should indeed be taken to mean morphisms that can be dominated by a finite torsor. A formal argument (Theorem 4.13) shows that the fundamental group scheme exists if and only if the category of "covers" admits fibred products, and that the universal cover is indeed the initial object among covers. The merit of Theorem 2.11 is to show these speculations to be non-vacuous. In fact, it allows us to determine completely the category of covers for flat schemes over a perfect field in positive characteristic. What is sorely missing is a similar characterization of "covers" for arithmetic schemes.

Let us now review in more detail the structure of the paper. Until the last section, we work in characteristic $p > 0$.

In § 1, after reviewing Sancho de Salas' work [13] on differentially homogeneous morphisms, we focus on the subcategory of finite differentially homogeneous morphisms. We show that a finite morphism is differentially homogeneous if and only if it is a twisted form in the flat topology of a finite $\mathbb{F}_p$-scheme, completely determined by the differential structure of the morphism.

In § 2, we first prove that any finite differentially homogeneous morphism can be dominated by a torsor under a flat, but not necessarily finite, $\mathbb{F}_p$-group scheme. We next prove our main result, Theorem 2.11, giving necessary and sufficient conditions for a finite morphism to admit a finite Galois closure. A morphism with this property is called F-constant.

M. Antei and M. Emsalem have introduced in [1] another class of finite flat morphisms (called essentially finite), admitting a Galois closure. Their construction is based on Nori's tannakian approach to the fundamental group scheme: it is thus restricted to reduced schemes proper over a field, but provides a description of the Galois group. In § 3, we show that, whenever they may be compared, essentially finite and F-constant morphisms are equivalent (Theorem 3.5).

Finally, in § 4 we give applications to the fundamental group scheme. We first give an existence result (Theorem 4.5): let S be a flat scheme over a Dedekind base which has a fundamental group scheme, then if $X \to S$ is a finite flat map with étale or F-constant generic fibre, X has a fundamental group scheme too. If moreover X itself can be dominated by a finite torsor, then its fundamental group scheme injects into that of S (Theorem 4.9). The remainder of the section is devoted to speculations on Galois theory for the flat topology.

I am indebted to Pedro Sancho de Salas for pointing out a mistake in an earlier version of this paper, providing example 1.7 below. It is a pleasure to thank Noriyuki Suwa for many interesting conversations and useful comments.

1 Differentially homogeneous morphisms

Notations and conventions: After example 1.2 below and until §4 all schemes are assumed to be noetherian of characteristic $p > 0$. We fix a separated scheme of finite type S.

If Z is a scheme of characteristic p, denote $F_Z : Z \to Z$ the absolute Frobenius. If U is a Z-scheme, $U^{(i/Z)}$ denotes the pullback of U by the i-th iterate of F_Z and $F_{U/Z} : U \to U^{(1/Z)}$ the relative Frobenius, a morphism of Z-schemes. We shall simplify and write $U^{(i)}$ for $U^{(i/\mathbb{F}_p)}$.

If G is an $\mathbb{F}_p$-group scheme, we denote by ${}_{F^i}G \trianglelefteq G$ the kernel of $F^i_{G/\mathbb{F}_p} : G \to G^{(i)}$.

Definition 1.1 *An S-scheme X of finite type is* differentially homogeneous[1] *if it is flat and for all $r \geq 0$ the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X/\mathcal{I}^{r+1}$ is coherent and locally free, where $\mathcal{I}$ is the sheaf of ideals defined by the diagonal map $X \to X \times_S X$.*

[1] Or *normally flat along the diagonal* in the EGA lingo: [5] IV.6.10.1.

A morphism $\pi : X \to S$ is said to be differentially homogeneous at $x \in X$ if $\operatorname{Spec} \mathcal{O}_{X,x}$ is differentially homogeneous over $\operatorname{Spec} \mathcal{O}_{S,\pi(x)}$. From the definition (and the behaviour of the differential sheaves) it follows immediately that this property is local on the source and stable under base change and faithfully flat descent. For any $\pi : X \to S$, the set of points $x \in X$ such that π is differentially homogeneous at x is open.

Example 1.2 Smooth morphisms are differentially homogeneous. Twisted forms in the flat topology of differentially homogeneous schemes are differentially homogeneous. If k is a field and S is a k-scheme, torsors over S under an algebraic k-group scheme are differentially homogeneous.

Differentially homogeneous morphisms have been investigated by Sancho de Salas [13]. In characteristic zero, a morphism is differentially homogeneous if and only if it is smooth. In characteristic $p > 0$, differentially homogeneous schemes can be characterized in terms of p-th powers. For any $n \geq 0$, let X_{p^n} be the scheme with the same underlying topological space as X and whose structure sheaf is $\mathcal{O}_S[\mathcal{O}_X^{p^n}]$, the $\mathcal{O}_S$-subalgebra of $\mathcal{O}_X$ generated by p^n-th powers of sections of $\mathcal{O}_X$.

Proposition 1.3 (Sancho de Salas [13]) *Let S be a connected scheme and $\pi : X \to S$ a flat morphism of finite type.*

1) *X is differentially homogeneous if and only if $\Omega^1_{X_{p^r}/S}$ is a flat $\mathcal{O}_{X_{p^r}}$-module for any $r \geq 0$ ([13], proposition 2.4).*

2) *X is differentially homogeneous if and only if for every $x \in X$ there are affine neighborhoods $V = \operatorname{Spec} B$ of x and $U = \operatorname{Spec} A$ of $\pi(x)$ such that $\pi(V) \subseteq U$, and there exists a chain $B_0 \subset B_1 \subset \cdots \subset B_n = B$, where B_0 is a smooth A-algebra and $B_{i+1} = B_i[x_i]/(x_i^{p^{e_i}} - b_i)$ for some $b_i \in A[B_i^{p^{e_i}}]$ ([13], Theorem 3.4).*

3) *If X is differentially homogeneous over S then X is finite and differentially homogeneous over X_{p^n} for all n and X_{p^n} is smooth over S for $n \gg 0$ ([13], Corollary 2.5 and Theorem 2.6).*

Remark 1.4 The condition $b_i \in A[B_i^{p^{e_i}}]$ in prop. 1.3.2 has the unpleasant consequence that if Y is differentially homogeneous over a scheme X that is differentially homogeneous (even smooth) over S then Y may not be differentially homogeneous over S. For instance, the affine curve Y given by $y^p = x^{p+1}$ is differentially homogeneous over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[x]$, but $\Omega^1_{Y/\mathbb{F}_p}$ is not flat at the origin, so Y is not differentially homogeneous over $\mathbb{F}_p$.

Definition 1.5 *We will use the acronym* qfdh *(respectively* fdh*) to indicate a quasi-finite (resp. finite) differentially homogeneous morphism $X \to S$.*

Example 1.6 A flat S-group scheme of finite height (i.e $G = \ker F^h_{G/S}$ for some $h \geq 0$) is qfdh. Indeed its fibres are fdh and $G_{p^i} = \ker F^{h-i}_{G^{(i/S)}/S}$, hence $G \to G_{p^i}$ is faithfully flat. We can apply [13], proposition 2.8: X is differentially homogeneous if and only if its fibres are differentially homogeneous and $X \to X_{p^i}$ is faithfully flat for all $i > 0$.

Example 1.7 (Sancho de Salas) Unfortunately, qfdh morphisms are not composable: let $A = \mathbb{F}_p[x](x^p)$, $B = A[u]/(u^p)$ and $C = B[v](v^p - xu)$. Then $X = \operatorname{Spec} B$ is fdh over $S = \operatorname{Spec} A$ and $Y = \operatorname{Spec} C$ is fdh over X, by the criterion 1.3.2, but Y is not differentially homogeneous over S since $\Omega^1_{C/A} = Cdu \oplus Cdv/(xdu)$ is not a flat C-module.

Remark 1.8 J.-M. Fontaine (unpublished) defined quiet morphisms as the smallest class of syntomic morphisms closed under composition and containing étale maps and morphisms of the type $\operatorname{Spec} A[x]/(x^p - a) \to \operatorname{Spec} A$. All such morphisms are qfdh and, by [13], prop. 1.7, a differentially homogeneous morphism is a complete intersection morphism. Therefore, qfdh morphisms are the building blocks of Fontaine's quiet topology.

In the following, we will show that any scheme X qfdh over a connected scheme S of characteristic p is a twisted form in the flat topology of a "constant" scheme defined over the prime field $\mathbb{F}_p$. The first step is to attach to $X \to S$ such "typical fibre". The starting point is the following remark.

Lemma 1.9 *If $X \to S$ is a qfdh morphism of connected schemes, $\operatorname{rk} \Omega^1_{X/S} \geq \operatorname{rk} \Omega^1_{X_p/S}$.*

Proof We may assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are local. Let $dz_1, \dots, dz_r$ be a basis of $\Omega^1_{B/A}$ and define a map $\varphi : C = A[Z_1, \dots, Z_r] \to B$ by $Z_i \mapsto z_i$. Since $d\varphi : B \otimes_C \Omega^1_{C/A} \to \Omega^1_{B/A}$ is an isomorphism, φ induces an isomorphism at the level of tangent spaces and is therefore surjective. φ maps the subalgebra $A[C^p] = A[Z_1^p, \dots, Z_r^p]$ to the subalgebra $A[B^p]$. Let $\bar{f} \in A[B^p]$ and $f \in C$ such that $\varphi(f) = \bar{f}$. Since $d\bar{f} = 0$ in $\Omega^1_{B/A} = \Omega^1_{B/A[B^p]}$ and $d\varphi$ is an isomorphism, $df = 0$ hence $f \in A[Z_1^p, \dots, Z_r^p]$. Therefore $\varphi : A[C^p] \to A[B^p]$ is again surjective and so $\Omega^1_{A[B^p]/A}$ is generated by the $d\varphi(Z_i^p) = d(z_i^p)$ and has thus rank $\leq r$. $\square$

Definition 1.10 *Let X be a qfdh, connected S-scheme and consider the factorization*

$$X \to X_p \cdots \to X_{p^i} \cdots \to S.$$

We shall say that an integer $\nu \geq 1$ is a break *if $\operatorname{rk} \Omega^1_{X_{p^\nu}/S} \lneq \operatorname{rk} \Omega^1_{X_{p^{\nu-1}}/S}$.*

Definition 1.11 *Let X be a qfdh, connected S-scheme and $r = \operatorname{rk} \Omega^1_{X/S}$. To $X \to S$ we associate the following data:*

1. *The r-tuple $\boldsymbol{\nu} = (\nu_1, \dots, \nu_r)$ of breaks, each one repeated $\operatorname{rk} \Omega^1_{X_{p^{\nu-1}}/S} - \operatorname{rk} \Omega^1_{X_{p^\nu}/S}$ times, arranged in increasing order.*
2. *The scheme $\Sigma^{\boldsymbol{\nu}} = \operatorname{Spec} \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}})$.*
3. *If $X \to S$ is finite, the degree $d = \deg(X_{p^{\nu_r}}/S)$ of the étale subcover.*

Proposition 1.12 *A finite scheme X over a connected scheme S is fdh if and only if, locally for the flat topology on S, it is isomorphic to $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}_S$.*

Proof The if part is clear. We may assume that $S = \operatorname{Spec} A$ is local. Replacing A by its strict henselization, we may assume that $X = \coprod_{i=1}^d \operatorname{Spec} B$ with B/A fdh and radicial. Let thus $d = 1$. We may also assume that $X \to S$ has a section: indeed, by [13], corollary 3.5, there is a section over the pullback by a qfdh A-algebra A'. The kernel $J = \ker[B \to A]$ of this section is a nilpotent ideal, since X and S have the same topological space. By [13], Theorem 1.6, there is a faithfully flat base change $A \to A''$ such that $B'' = A'' \otimes_A B \cong A''[t_1, \dots, t_r]/(t_1^{p^{e_1}}, \dots, t_r^{p^{e_r}})$, for suitable integers $e_1 \leq \cdots \leq e_r$. Computing the breaks of $\Omega^1_{B''/A''} = B'' \otimes_B \Omega^1_{B/A}$, one checks immediately that $(e_1, \dots, e_r) = (\nu_1, \dots, \nu_r)$. $\square$

2 Galois closures

Definition 2.1 *Let $X \to S$ be a finite flat morphism. We shall say that a torsor T/S under a group scheme G* dominates *X if $T \to S$ factors through a flat morphism $T \to X$ which is a torsor under a suitable subgroup $H \subseteq G$.*

$$\begin{array}{ccc} T & \xrightarrow{H} & X \\ & \underset{G}{\searrow} & \downarrow \\ & & S \end{array}$$

In the previous section we have established that an fdh scheme $X \to S$ is a twisted form of a disjoint sum of “constant” schemes $\Sigma^{\boldsymbol{\nu}}$. In order to construct a torsor T dominating X, we should investigate the automorphisms of $\Sigma^{\boldsymbol{\nu}}$ as a sheaf for the flat topology. The idea is to mimic the following process: the symmetric group $\mathfrak{S}_n$ is the automorphism group of the set $\Sigma = \{1, \dots, n\}$. Evaluation at $1 \in \Sigma$ yields a surjective map $\mathfrak{S}_n \to \Sigma$ identifying the latter as the homogeneous space $\mathfrak{S}_n/\mathfrak{S}_{n-1}$.
By [2] II §1, 2.7 (see also the proof of the following lemma), the sheaf of automorphisms of $\Sigma^{\boldsymbol{\nu}}$ is representable by an affine group scheme $\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$ of finite type over $\mathbb{F}_p$. Let $o \in \Sigma^{\boldsymbol{\nu}}(\mathbb{F}_p)$ be the origin. We denote by $\underline{Aut}_{\,o}(\Sigma^{\boldsymbol{\nu}})$ its stabilizer and by $q : \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ the canonical morphism defined, for any $\mathbb{F}_p$-algebra A, by mapping an automorphism g of $\Sigma^{\boldsymbol{\nu}}_A$ to $g(o) \in \Sigma^{\boldsymbol{\nu}}(A)$.

The following lemma gathers the information we will need about $\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$ and some of its subgroups. It is probably well known, but we include it for lack of references.

Lemma 2.2 *The morphism $q : \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ is faithfully flat. For any integer $n \geq \nu_r$ it induces an isomorphism ${}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{Aut}_{\,o}(\Sigma^{\boldsymbol{\nu}}) \cong \Sigma^{\boldsymbol{\nu}}$.*

Proof Let $N = \{[0, p^{\nu_1} - 1] \times \cdots \times [0, p^{\nu_r} - 1]\} \cap \mathbb{N}^r$ and let $N_i = \{J \in N |\, p^{\nu_i} J \in N\}$.
The t^J, with $J \in N$ form a basis of the $\mathbb{F}_p$-vector space $\mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}})$. The functor on $\mathbb{F}_p$-algebras $A \mapsto Hom_{A-sch}(\Sigma^{\boldsymbol{\nu}}_A, \mathbb{A}^r_A)$ is represented by $\mathbb{A}^{r|N|} = \operatorname{Spec} R[x_{i,J}]$, a morphism $\Sigma^{\boldsymbol{\nu}}_A \to \mathbb{A}^r_A$ being defined by a map

$$\begin{array}{rl} A[t_1, \dots, t_r] & \longrightarrow A \otimes \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) \\ t_i & \longmapsto \sum_J x_{i,J} \otimes t^J. \end{array} \tag{2}$$

This map factors through $\Sigma^{\boldsymbol{\nu}}_A$ if and only if

$$\left(\sum_J x_{i,J} \otimes t_1^{j_1} \dots t_r^{j_r} \right)^{p^{\nu_i}} = \sum_J x_{i,J}^{p^{\nu_i}} \otimes t_1^{p^{\nu_i} j_1} \dots t_r^{p^{\nu_i} j_r} = 0$$

for $i = 1, \dots, r$. Hence the sheaf of monoids $A \mapsto End_{A-sch}(\Sigma^{\boldsymbol{\nu}}_A)$ is represented by $\underline{End}\,(\Sigma^{\boldsymbol{\nu}}) = \operatorname{Spec} \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} |\; i = 1, \dots, r; J \in N_i)$.
From (2) we infer that the action $\underline{End}\,(\Sigma^{\boldsymbol{\nu}}) \times \Sigma^{\boldsymbol{\nu}} \to \Sigma^{\boldsymbol{\nu}}$ (described on A-valued points by $(g, x) \mapsto g(x)$) is given by

$$\begin{array}{rl} \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) & \longrightarrow \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} |\, J \in N_i) \otimes \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) \\ t_i & \longmapsto \sum_J x_{i,J} \otimes t^J \end{array}$$

and therefore $q : \underline{End}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$, given by

$$\begin{array}{rl} \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) & \longrightarrow \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} |\, J \in N_i) \\ t_i & \longmapsto x_{i,0} \end{array} \tag{3}$$

is faithfully flat (since $0 \in N_i\ \forall i$) and so is the restriction to the open subscheme $\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}}) \subset \underline{End}\,(\Sigma^{\boldsymbol{\nu}})$.
For any $n \geq 0$ the endomorphisms whose pull-back by the n-th iterate of Frobenius is the identity form a submonoid ${}_{F^n}\underline{End}\,(\Sigma^{\boldsymbol{\nu}}) \subseteq \underline{End}\,(\Sigma^{\boldsymbol{\nu}})$. If $n \geq \nu_r$, from (2), we deduce that

$$ {}_{F^n}\underline{End}\,(\Sigma^{\boldsymbol{\nu}}) = \operatorname{Spec} \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} \,|\, J \in N_i;\ x_{i,J}^{p^n} \,|\, J \notin N_i) $$

and from (3) that the induced map $q_n : {}_{F^n}\underline{End}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ is faithfully flat for all $n \geq \nu_r$. Therefore, so is the restriction to the open subscheme ${}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$. Let us consider the diagram:

$$ \begin{array}{ccc} {}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}}) & \longrightarrow & {}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{Aut}\,_o(\Sigma^{\boldsymbol{\nu}}) \\ & \searrow^{q_n} & \downarrow{\scriptstyle \iota_n} \\ & & \Sigma^{\boldsymbol{\nu}} \end{array} $$

By [2] III §3 5.2, the quotient ${}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{Aut}\,_o(\Sigma^{\boldsymbol{\nu}})$ is representable and the canonical map ι_n is an immersion. By [2] III §3 2.5, the horizontal projection is faithfully flat. Hence ι_n is flat and is thus an open immersion. Since $\Sigma^{\boldsymbol{\nu}}$ is local, ι_n is an isomorphism. □

Theorem 2.3 *Let S be a connected scheme, $X \to S$ an fdh morphism. There exists a torsor T/S in the fppf topology under an affine $\mathbb{F}_p$-group scheme of finite type, dominating X and such that $T \times_S X \cong \coprod_{i=1}^d \Sigma_T^{\boldsymbol{\nu}}$.*

Proof As an automorphism of a scheme induces an automorphism of the set of connected components, $\underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ is a (split) extension of the symmetric group $\mathfrak{S}_d$ by $\prod_{i=1}^d \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$. The fppf sheaf $\underline{Isom}_S(\coprod_{i=1}^d \Sigma_S^{\boldsymbol{\nu}}, X)$ is an $\underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor over S and is thus representable (e.g., [9], III, 4.3) by a scheme T.
Let o_1 be the origin of the first connected component of $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ and $\underline{Aut}\,_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ its stabilizer (an extension of $\mathfrak{S}_{d-1}$ by $\underline{Aut}\,_o(\Sigma^{\boldsymbol{\nu}}) \times \prod_{i=2}^d \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$). If U is any S-scheme, to any $\varphi_U : \coprod_{i=1}^d \Sigma_U^{\boldsymbol{\nu}} \to X_U$ we can associate $\varphi_U(o_1) \in X(U)$. These data define an $\underline{Aut}\,_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-equivariant morphism

$$ f : T = \underline{Isom}_S(\coprod_{i=1}^d \Sigma_S^{\boldsymbol{\nu}}, X) \to X. $$

Around any closed point of X, locally for the flat topology, f is isomorphic to the "evaluation at o_1" map $q : \underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) \to \coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ followed by the projection onto the first factor. Hence f is faithfully flat by lemma 2.2. Finally, one checks immediately that the diagram

$$ \begin{array}{ccc} T \times \underline{Aut}\,_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) & \longrightarrow & T \times_X T \\ \downarrow & & \downarrow \\ T \times \underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) & \longrightarrow & T \times_S T \end{array} $$

where the horizontal maps are given by $(\varphi_U, g_U) \mapsto (\varphi_U, \varphi_U \circ g_U)$, is cartesian. Since T is an $\underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor, the bottom map is an isomorphism, hence so is the top map. □

Remark 2.4 The datum of an isomorphism $X \times_S X \cong \Sigma_X^{\boldsymbol{\nu}}$ as X-schemes is equivalent to a section $X \to T = \underline{Isom}_S(\Sigma_S^{\boldsymbol{\nu}}, X)$ of $f : T \to X$; in such a situation, T is a trivial torsor over X. This is the case in particular when X is itself a torsor over S.

Being a torsor under an algebraic group scheme, T is differentially homogeneous but never finite: as seen in the proof of lemma 2.2, the reduced connected component of $\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$ is positive-dimensional. The remainder of this section is devoted to the following question: is it possible to find a torsor Y/S under a *finite* group scheme dominating X? In other words, when does T admit a reduction of the structure group to a finite subgroup?

Proposition 2.5 *Locally on S for the Zariski topology, an fdh morphism $X \to S$ is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme.*

Proof If $S = \operatorname{Spec} A$ is local then $X = \operatorname{Spec} B$ admits a chain $B_0 \subset B_1 \subset \cdots \subset B_n = B$ as in proposition 1.3. Since $B_{i+1} = B_i[x_i]/(x_i^{p^{e_i}} - b_i)$ can be seen as an $\boldsymbol{\alpha}_{p^{e_i}}$-torsor over B_i, replacing B_0 by its finite étale Galois closure over A, we get a factorization of X as a tower of finite torsors. By [3], Theorem 2, X is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme. □

Another explanation for the fact that locally on the base an fdh morphism can be dominated by a finite torsor will be provided in proposition 3.8 in the next section. In general however, it is not possible to dominate an fdh morphism by a finite torsor, as shown in example 2.7 below. The example and the subsequent results are based on the following remark.

Remark 2.6 Let Σ be a finite $\mathbb{F}_p$-scheme, $G = \underline{Aut}\,(\Sigma)$ and let $X \to S$ be a twisted form of Σ_S. The Frobenius morphism $F^n_{G/\mathbb{F}_p} : G \to G^{(n)}$ induces an exact sequence in flat cohomology

$$\check{H}^1(S, {}_{F^n}G) \longrightarrow \check{H}^1(S, G) \longrightarrow \check{H}^1(S, G^{(n)}).$$

The second map sends the class of $T = \underline{Isom}_S(\Sigma_S, X)$ to that of $\underline{Isom}_S(\Sigma_S, X^{(n/S)})$. Hence $X^{(n/S)}$ is isomorphic to Σ_S if and only if T is induced from a torsor Y under the finite subgroup ${}_{F^n}G$. The canonical map $Y \to Y \times G \to Y \wedge^{F^n G} G \cong T$ gives a point in $T(Y) = \mathrm{Isom}_Y(\Sigma_Y, X_Y)$, hence X becomes isomorphic to Σ over Y.

Example 2.7 Let k be a perfect field, $X = S = \mathbb{P}^1_k$ and $\pi : X \to S$ be the relative (k-linear) Frobenius. X is a twisted form of $\Sigma^1_S = S \times \operatorname{Spec} \mathbb{F}_p[t]/t^p$. Suppose that X trivializes over a torsor under a finite subgroup $H \leq G = \underline{Aut}(\Sigma^1)$. As there are no étale covers of $\mathbb{P}^1$, there is no loss in generality in assuming H connected and thus $H \leq {}_{F^n}G$ for a suitable integer n. In other words, X would become isomorphic to Σ^1_S over the n-th iterate $F^n_S : S \to S$ of the absolute Frobenius. In particular the pullback $p_2^*\Omega^1_{X/S} = p_2^*\Omega^1_X$ would have to be constant over $S \times_{S,F^n_S} X$ and so would then be the pullback $F^{n*}_X\Omega^1_X$. This is absurd, since $\Omega^1_X = \mathcal{O}(-2)$ and $F^{n*}_X\Omega^1_X = \mathcal{O}(-2p^n)$ is never constant.

Definition 2.8 *Let $X \to X^{ét} \to S$ be an fdh morphism, factored into a radicial and an étale morphism. We will say that X is F-*constant *over S if the pull-back of X over a suitable iterate of the absolute Frobenius $F_S : S \to S$ becomes isomorphic to $\Sigma^{\boldsymbol{\nu}}_{X^{ét}}$.*

Remark 2.9 Notice that since $X^{ét} \to S$ is étale, the diagram

$$\begin{array}{ccc} X^{ét} & \xrightarrow{F_{X^{ét}}} & X^{ét} \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

is cartesian, so X is F-constant over S if and only if it is F-constant over $X^{ét}$.

Remark 2.10 F-constance can be checked after finite étale base change: X is F-constant over S if and only if, for any finite étale base $S' \to S$ the scheme $X' = S' \times_S X$ is F-constant over S'. By the above remark, we may assume $S = X^{ét}$. Composing a section over S' with the projection yields a finite S-morphism $\sigma : S' \to X' \to X$. Since S'/S is étale while X/S is radicial, one checks immediately that the image of σ is isomorphic to S, thus providing a section to π.

Theorem 2.11 *Let S be a connected scheme and X a finite S-scheme. The following conditions are equivalent:*

1. *X is F-constant;*
2. *there are finite $\mathbb{F}_p$-group schemes $H \leq G$ and an X-scheme Y which is a G-torsor over S and an H-torsor over X;*
3. *there exists a torsor Y/S under a finite $\mathbb{F}_p$-group scheme such that $Y \times_S X$ is a finite disjoint union of copies of $\Sigma^{\boldsymbol{\nu}}_Y$.*

Proof We By [3], Theorem 2, X is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme.
1) $\Rightarrow$ 2) By Theorem 2.3, X becomes isomorphic to $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}_T$ over the $\underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor $T = \underline{Isom}_S(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}_S, X)$. Since $\underline{Aut}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ is an extension of the étale group $\mathfrak{S}_d$ by the connected component $\prod_{i=1}^d \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$, we can factor $T \to S$ through an étale $\mathfrak{S}_d$-cover $Z \to S$, which we can interpret as a disjoint union of $[Gal(X^{ét}/S) : \mathfrak{S}_d]$ copies of the Galois closure of the maximal étale subcover $X^{ét} \to S$. We have to show that the connected torsor $T \to Z$ is induced by a finite subgroup of the structure group $\prod_{i=1}^d \underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$ so, replacing S by Z and X by a connected component of $Z \times_{X^{ét}} X$ we may assume that X is radicial over S.
Since X is F-constant, $X^{(p^n/S)} \cong \Sigma^{\boldsymbol{\nu}}_S$ for $n \gg 0$. Hence, by remark 2.6, there is an ${}_{F^n}\underline{Aut}\,(\Sigma^{\boldsymbol{\nu}})$-torsor Y such that $X \times_S Y = \Sigma^{\boldsymbol{\nu}}_Y$. Taking $n \geq \nu_r$, so that lemma 2.2 applies, the same argument as in Theorem 2.3 shows that Y is an ${}_{F^n}\underline{Aut}_{\,o}(\Sigma^{\boldsymbol{\nu}})$-torsor over X.

2) $\Rightarrow$ 3) Denoting by $\mu : Y \times G \to Y$ the action and by m the multiplication in G, we have a commutative diagram

$$\begin{array}{ccc} Y \times G \times H & \xrightarrow{id_Y \times m} & Y \times G \\ {\scriptstyle id_Y \times \mu \times id_H}\downarrow & & \downarrow{\scriptstyle id_Y \times \mu} \\ Y \times_S Y \times H & \xrightarrow{id_Y \times \mu} & Y \times_S Y \end{array}$$

whose vertical arrows are isomorphisms because Y is a G-torsor over S. Hence the quotient $Y \times (G/H)$ by the top action is isomorphic, as an Y-scheme, to the quotient $Y \times_S X$ by the bottom one. Therefore X becomes isomorphic over Y to G/H and the latter, by [2], III §3, 6.1, is a scheme of type $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$.

3) $\Rightarrow$ 1) Being a twisted form of $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ in the flat topology, X certainly is differentially homogeneous, and we can factor it as $X \to X^{ét} \to S$ as the composition of a radicial and an étale morphism. According to remark 2.9, to check that X is F-constant we may assume that $X^{ét} = S$. Since G is an extension of an étale group $G^{ét}$ by a connected one G^0, we can also factor the cover $Y \to Z \to S$, where the first is G^0-torsor and the second a Galois étale cover. By [10] II, §7, proposition 2, there is an equivalence of categories between coherent sheaves on S and coherent $G^{ét}$-sheaves on Z. Since the absolute Frobenius commutes with automorphisms, $X \times_S Z$ is F-constant over Z if and only if X is F-constant over S.
We may therefore assume that Y/S is a torsor under G^0. The latter is a finite connected group scheme, hence has finite Frobenius height $\leq h$. Therefore Y is an fdh S-scheme with $Y_{p^h} = S$ and we have a factorization of F^h_S as $S \to Y \to S$. From the isomorphism $Y \times_S X \cong \Sigma^{\boldsymbol{\nu}}_Y$ we then deduce that $S \times_{F^h_S} X \cong \Sigma^{\boldsymbol{\nu}}_S$. □

Corollary 2.12 *Let k be a field of characteristic $p > 0$, S a connected k-scheme and X a finite S-scheme. Then in conditions 2 and 3 in Theorem 2.11 we may replace $\mathbb{F}_p$-group schemes by k-group schemes.*

Proof This is just a little dévissage. It suffices to prove 3) $\Rightarrow$ 1). Let thus G be a finite k-group scheme and Y/S a G-torsor such that X trivializes over Y. We may replace S by $Y^{ét}$, the maximal étale subcover of $Y \to S$ and X by $Y^{ét} \times_S X$. The group G is then replaced by its connected component, whose Hopf algebra we denote by R. If $r = \dim_k R$, we have an embedding $G \subseteq {}_{F^n}GL(R) = {}_{F^n}GL_r \times_{\mathbb{F}_p} k$, for a suitable integer n. Let Y' be the ${}_{F^n}GL_r$-torsor over S induced by this embedding. Since $Y \times_S X = \Sigma_Y^{\boldsymbol{\nu}}$, a fortiori $Y' \times_S X = \Sigma_{Y'}^{\boldsymbol{\nu}}$. We can now conclude by Theorem 2.11. □

3 Essentially finite morphisms

In this section, k is a perfect field of characteristic $p > 0$. When S is a connected and reduced scheme, proper over k, Antei and Emsalem [1] have introduced another class of finite flat morphisms $X \to S$ that can be dominated by a finite torsor. Their construction is based on the tannakian approach to Nori's fundamental group scheme ([11], chapter I).

Definition 3.1 (Nori [11]) *Let S be a connected, reduced, proper k-scheme.*

1) *A vector bundle $\mathcal{V}$ on S is* finite *if there exist polynomials $f(t) \neq g(t)$ in $\mathbb{N}[t]$ such that $f(\mathcal{V}) = g(\mathcal{V})$.*
2) *Let $SS(S)$ be the category of semistable vector bundles on S. The category $EF(S)$ of* essentially finite *vector bundles on S is the full subcategory of $SS(S)$ whose objects are sub-quotients of finite bundles. In other words, a vector bundle $\mathcal{E}$ is essentially finite if there exists a finite bundle $\mathcal{V}$ and sub-bundles $\mathcal{V}'' \subset \mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{E} \simeq \mathcal{V}'/\mathcal{V}''$.*

Of course, definition 3.1.2 relies on the fact that every finite vector bundle is semistable ([11], corollary I.3.1).

If S has a rational point $s \in S(k)$, the fibre functor $\mathcal{E} \mapsto \mathcal{E}_s$ from $EF(S)$ to k-vector spaces makes $EF(S)$ into a neutral tannakian category ([11], §I.3). It is thus equivalent to the category of representations of an affine group scheme of finite type $\boldsymbol{\pi}(S/k; s)$, the *fundamental group scheme* of S. The crucial result is then:

Proposition 3.2 (Nori [11], I.3.10) *If $\mathcal{E}$ is any essentially finite vector bundle, the representation $\boldsymbol{\pi}(S/k; s) \to GL(\mathcal{E}_s)$ factors through a finite quotient of $\boldsymbol{\pi}(S/k; s)$.*

It follows from this that $\boldsymbol{\pi}(S/k; s)$ is a profinite group scheme.

Definition 3.3 (Antei-Emsalem [1]) *Let S be a connected, reduced, proper k-scheme. A finite flat morphism $\pi : X \to S$ is* essentially finite *if the vector bundle $\pi_*\mathcal{O}_X$ is essentially finite.*

Proposition 3.4 (Antei-Emsalem [1], 3.2) *Let S be a connected, reduced, proper k-scheme with a rational point $s \in S(k)$. Let $\pi : X \to S$ be an essentially finite morphism. Assume that $H^0(S, \pi_*\mathcal{O}_X) = k$ and that there exists a point $x \in X(k)$ above s. Then X is dominated by a torsor under a finite k-group scheme.*

As a matter of fact, the main result of [1] is much more precise: it describes the actual "Galois group" of X/S as the quotient of $\boldsymbol{\pi}(S/k; s)$ determined by $\pi_*\mathcal{O}_X$, as in proposition 3.2.

Theorem 3.5 *Let S be a connected, reduced, proper k-scheme, $\pi : X \to S$ a finite flat morphism.*

1) *If X is F-constant, then π is essentially finite.*

2) *If π is essentially finite and $H^0(X, \mathcal{O}_X)$ is an étale k-algebra, then X is F-constant over S.*

Proof 1) If X is F-constant, by Theorem 2.11 there is a torsor Y/S under a finite flat group scheme such that the pullback to Y of $\pi_*\mathcal{O}_X$ becomes constant as a sheaf of $\mathcal{O}_Y$-algebras and therefore as an $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{O}_X$ is essentially finite by [11], proposition I.3.8.

2) Replacing k by a finite extension and X by a connected component, we may assume that the hypotheses of proposition 3.4 are satisfied. Then X is dominated by a torsor $Y \to S$ under a finite k-group scheme G. Since Y is a torsor over X under a subgroup $H \subseteq G$ we have that $Y \times_S X \cong (G/H)_Y$. Then X is F-constant by corollary 2.12. □

Remark 3.6 The condition on $H^0(X, \mathcal{O}_X)$ in proposition 3.4 ensures not only that X is connected but also reduced (in the sense of covers, cf. [11] definition II.3). Specifically, it guarantees that the action of the Galois group G on the fibre X_s is transitive ([1] lemma 3.18). As a consequence $X_s \cong G/G_x$, where G_x is the stabilizer at x. In particular it implies that X is fdh. Hence this global condition in Antei-Emsalem's construction translates into a local one in ours.

We would like now to address the apparent inconsistency between the F-constance condition, requiring that a pullback of $\pi_*\mathcal{O}_X$ trivializes as a sheaf of algebras, and essential finiteness, requiring only a trivialization as a sheaf of modules. This becomes even more glaring if we recall the following fact, whose proof inspired remark 2.6 above.

Proposition 3.7 (Mehta-Subramanian [8], §2) *A vector bundle $\mathcal{E}$ on a k-scheme S trivializes over a torsor under a finite local k-group scheme if and only if $(F_S^n)^* \mathcal{E}$ is the trivial bundle for some integer $n > 0$ (such a bundle is called* F-finite*).*

Let $\pi : X \to S$ be an essentially finite morphism and let $f : Y \to S$ be a torsor under a finite group scheme trivializing the vector bundle $\pi_*\mathcal{O}_X$. We can factor the finite cover $Y \to S' \to S$ into a radicial torsor followed by an étale one. Then $\pi' : X' = S' \times_S X \to S'$ is essentially finite and the bundle $\pi'_*\mathcal{O}_{X'}$ trivializes over a torsor under a finite local group scheme, namely $Y \to S'$ (we could call such a morphism *F-finite*). Summarizing:

- $\pi : X \to S$ is essentially finite $\Longleftrightarrow \exists$ an integer $n > 0$ and a finite étale cover $S' \to S$ such that $\left(F_{S'}^n\right)^* \pi'_*\mathcal{O}_{X'}$ is a free $\mathcal{O}_{S'}$-module.

- $\pi : X \to S$ is F-constant $\Longleftrightarrow \exists$ an integer $n > 0$ and a finite étale cover $S' \to S$ such that $\left(F_{S'}^n\right)^* \pi'_*\mathcal{O}_{X'} \cong \mathcal{O}_{S'}[t_1, \ldots, t_r]/(t_1^{p^{\nu_1}}, \ldots, t_r^{p^{\nu_r}})$.

Yet, according to Theorem 3.5, on a proper reduced scheme, the weaker first condition is equivalent to the second. To clarify this point we shall see that on an arbitrary scheme of characteristic p, the F-constance of a morphism is equivalent to the trivialization of a suitable subquotient of the direct image of the structure sheaf. Therefore, in cases where it is possible to apply the tannakian formalism, the two notions coincide. We will only treat the simplest situation, the general case being conceptually similar but notationally messy.

Let $\pi : X \to S$ be an fdh morphism such that $X_p = S$. Then the relative Frobenius $F_{X/S}$

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X^{(1/S)} & \longrightarrow & X \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi^{(1)}} & & \downarrow{\scriptstyle \pi} \\ & & S & \xrightarrow{F_S} & S \end{array}$$

factors through a section $\varepsilon : S \to X^{(1/S)}$ of $\pi^{(1)}$. Let $\omega_{\pi^{(1)}} = \varepsilon^* \Omega^1_{X^{(1/S)}/S}$.

Proposition 3.8 *Let S be a scheme of characteristic $p > 0$ and $\pi : X \to S$ an fdh morphism such that $X_p = S$. Then π is F-constant if and only if $\omega_{\pi^{(1)}}$ is a free $\mathcal{O}_S$-module.*

Proof If π is F-constant, $\Omega^1_{X^{(1/S)}/S}$ is free and so does $\omega_{\pi^{(p)}}$. Conversely, let $\mathcal{I} \subset \pi^{(1)}_* \mathcal{O}_{X^{(1/S)}}$ be the ideal defined by the closed embedding ε. We have a canonical surjection from the conormal bundle of ε to $\omega_{\pi^{(1)}}$:

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \omega_{\pi^{(1)}} \longrightarrow 0. \tag{4}$$

If $\omega_{\pi^{(1)}}$ is free, any lifting to $\mathcal{I}$ of a basis of $\omega_{\pi^{(1)}}$ defines a surjection of algebras

$$\vartheta : \mathcal{O}_S[t_1, \ldots, t_r] = Sym\,(\omega_{\pi^{(1)}}) \longrightarrow \pi^{(1)}_* \mathcal{O}_{X^{(1/S)}}. \tag{5}$$

Any section $z \in \mathcal{I}$ satisfies $z^p = 0$. Therefore ϑ factors through a surjection:

$$\mathcal{O}_S[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \longrightarrow \pi^{(1)}_* \mathcal{O}_{X^{(p/S)}}.$$

Since π is fdh, this is a nontrivial map between twists and is thus an isomorphism. □

Example 3.9 In the situation of example 2.7, we have $\omega_{\pi^{(1)}} = \mathcal{O}(-2)$. This shows again that the k-linear Frobenius $\pi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is not F-constant.

If S is reduced and proper over a perfect field, from surjections (4) and (5) above we see that $\omega_{\pi^{(1)}}$ generates the same tannakian subcategory of $EF(S)$ as $F_S^* \pi_* \mathcal{O}_X = \pi^{(1)}_* \mathcal{O}_{X^{(1/S)}}$. Therefore, if the latter is the trivial bundle, so is $\omega_{\pi^{(1)}}$ and thus $\pi : X \to S$ if F-constant.

4 Fundamental group schemes

Notations and conventions: Let B be a fixed base scheme. In this section all schemes are assumed to be B-schemes of finite type. We fix a separated flat B-scheme S with a marked rational point $s \in S(B)$.

Definition 4.1 (Nori [11]) *Let $\mathfrak{C}(S/B; s)$ be the category whose objects are triples (X, G, x) consisting of a finite flat B-group scheme G, a G-torsor $f : X \to S$ and a rational point $x \in X(B)$ such that $f(x) = s$. A morphism $(X', G', x') \to (X, G, x)$ in $\mathfrak{C}(S/B; s)$ is the datum of an S-morphism $\alpha : X' \to X$ such that $\alpha(x') = x$ and a B-group scheme homomorphism $\beta : G' \to G$ making the following diagram, where the horizontal arrows are the group actions, commute:*

$$\begin{array}{ccc} G' \times X' & \xrightarrow{\mu'} & X' \\ {\scriptstyle \beta\times\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times X & \xrightarrow{\mu} & X \end{array}$$

Definition 4.2 (Nori [11]) *A scheme S has a fundamental group scheme $\boldsymbol{\pi}(S/B;s)$ if the category* $\mathrm{Pro}(\mathfrak{C}(S/B;s))$ *has an initial object* $(\tilde{S}, \boldsymbol{\pi}(S/B;s), \tilde{s})$.

Nori [11], proposition II.9 (resp. Gasbarri [4], §2) have shown that if S is reduced and B is the spectrum of a field (resp. a Dedekind scheme) then S has a fundamental group scheme. If S is reduced and proper over a perfect field, its fundamental group scheme in the sense of def. 4.2 is identical to the tannakian group considered in § 3. If B is a Dedekind scheme, S has a fundamental group scheme and X/S is a torsor under a finite flat group scheme, then X admits a fundamental group scheme ([3], Theorem 3).

All of the above results are proved using the following criterion:

Proposition 4.3 (Nori [11], proposition II.1, Gasbarri [4], 2.1) *A flat B-scheme S has a fundamental group scheme if and only if $\mathfrak{C}(S/B;s)$ admits finite fibered products, i.e., for any $(Y,G,y) \in \mathfrak{C}(S/B;s)$ and any pair of morphisms $\alpha_i : (Y_i, G_i, y_i) \to (Y, G, y)$ in $\mathfrak{C}(S/B;s)$, the triple $(Y_1 \times_Y Y_2, G_1 \times_G G_2, (y_1, y_2))$ belongs to $\mathfrak{C}(S/B;s)$.*

Remark 4.4 (Nori [11], lemma II.1) For any given torsor $(Y,G,y) \in \mathfrak{C}(S/B;s)$ and any pair of morphisms $\alpha_i : (Y_i, G_i, y_i) \to (Y, G, y)$ in $\mathfrak{C}(S/B;s)$, the triple $Y_1 \times_Y Y_2$ is a $G_1 \times_G G_2$-torsor over a closed subscheme of S containing s. So it is a torsor over S if and only if it is faithfully flat over S.

Theorem 4.5 *Let B be a Dedekind scheme and η its generic point. Let (S,b) a flat pointed B-scheme which has a fundamental group scheme. Let $\pi : X \to S$ be a finite flat B-morphism, equipped with a point $x \in X(B)$ such that $\pi(x) = s$. If the generic fibre $\pi_\eta : X_\eta \to S_\eta$ is étale or F-constant, then also (X,x) has a fundamental group scheme.*

Proof We will apply the criterion above. Let thus (Y_i, G_i, y_i), for $i = 0, 1, 2$, be three torsors in $\mathfrak{C}(X,x)$ and $\alpha_i : (Y_i, H_i, y_i) \to (Y_0, H_0, y_0)$, for $i = 1, 2$, be two morphisms in $\mathfrak{C}(X,x)$. We have to show that the triple $(Y_1 \times_{Y_0} Y_2, H_1 \times_{H_0} H_2, (y_1, y_2))$ belongs to $\mathfrak{C}(X,x)$.

In light of remark 4.4, it suffices to prove this when B is the spectrum of a field. Indeed, since X is the closure of its generic fibre X_η, by [5] IV.2.8.5, the case of a general Dedekind scheme follows by taking the scheme theoretic closure of the objects defined over η: the proof of [4], proposition 2.1 goes through verbatim.

Let thus B be the spectrum of a field. By Grothendieck's Galois theory [7], chap. V (in characteristic 0) or by Theorem 2.11 (in positive characteristic) we can dominate X by a finite torsor:

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ & \searrow & \downarrow{\scriptstyle \pi} \\ & & S \end{array}$$

Pullback via f provides us with the H_i-torsors $Y_i' = X' \times_X Y_i$. Since X'/S is a finite torsor, by [3] Theorem 3, it has a fundamental group scheme. Hence $Y_1' \times_{Y_0'} Y_2'$ is an $H_1 \times_{H_0} H_2$-torsor over X'. In particular, it is faithfully flat over X'. Also f is faithfully flat: by descent we get that $Y_1 \times_{Y_0} Y_2$ is faithfully flat over X, and we conclude by remark 4.4. □

Having established that X has a fundamental group scheme, by functoriality from $\pi : (X,x) \to (S,s)$ we obtain a group homomorphism $\boldsymbol{\pi}(X/B,x) \to \boldsymbol{\pi}(S/B,s)$. If X is a torsor over S, this is an embedding of $\boldsymbol{\pi}(X/B,x)$ as a closed normal subgroup of $\boldsymbol{\pi}(S/B,s)$ ([3], Theorem 4). More generally, we show below that it is an injection if π admits a Galois closure. In order not to have to spell out this condition every time, we introduce the following definition, which should not be taken too seriously.

Definition 4.6 *A morphism $\pi : (X, x) \to (S, s)$ of pointed B-schemes will be called* submissive *if it is finite, flat and it can be dominated, in the sense of definition 2.1, by a torsor under a finite flat B-group scheme with a marked B-point lying over x.*

Proposition 4.7 *Let $\pi : (X, x) \to (S, s)$ be a finite flat morphism of pointed B-schemes. Then π is submissive in the following cases:*

1. *π is étale;*
2. *π is F-constant and B is the spectrum of a perfect field.*

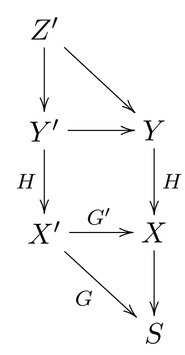
Proof The domination property is guaranteed for an étale cover by Grothendieck's Galois theory and by Theorem 2.11 for an F-constant morphism (even for imperfect fields). The issue is to deal with base points. Let X'/S be a torsor under a finite flat group scheme G dominating X and denote G' the group of X'/X. It may happen that X' has no integral points over x, but only acquires one over a finite étale (since B is perfect) extension $\widetilde{B}$ of B. In this case, denoting $\widetilde{T}$ the base change of a B-scheme T, we may replace G and G' by the Weil restrictions $\Re_{\widetilde{B}/B}(\widetilde{G})$ and $\Re_{\widetilde{B}/B}(\widetilde{G}')$ and X' by $\Re_{\widetilde{S}/S}(\widetilde{X}') = \Re_{\widetilde{X}/X}(\widetilde{X}')$. □

Remark 4.8 The perfectness assumption is needed in the proof because Weil restriction only behaves nicely with respect to étale morphisms. The reason to invoke Weil restriction, instead of descent theory, is the nasty behaviour of fundamental group schemes under base change. If $\widetilde{B}/B$ is a faithfully flat extension, functoriality yields a morphism $\boldsymbol{\pi}(\widetilde{S}/\widetilde{B}) \to \boldsymbol{\pi}(S/B) \times_B \widetilde{B}$, but this is by no means an isomorphism: see [8], §3 for a counterexample with S an integral projective curve and B and $\widetilde{B}$ algebraically closed fields. A counterexample with S a smooth curve has been given by Pauly in [12].

Theorem 4.9 *Let B be a Dedekind scheme, (S, b) and (X, x) flat pointed B-schemes admitting a fundamental group scheme. Let $\pi : X \to S$ be a submissive B-morphism with $\pi(x) = s$. Then π induces a closed immersion $\boldsymbol{\pi}(X/B, x) \to \boldsymbol{\pi}(S/B, s)$ of fundamental group schemes.*

Proof Let X'/S be a marked torsor under a finite flat group scheme G dominating X and denote G' the group of X'/X.

Any quotient H of $\boldsymbol{\pi}(X/B, x)$ corresponds to a marked H-torsor (Y, y) over (X, x). Let $Y' = X' \times_X Y$. By [3], Theorem 2 (if $\dim B = 1$ one has to repeat the scheme-theoretic closure argument above) we can find a finite flat B-group scheme $\Phi = \Phi(G, H)$ and a scheme Z' which is a Φ-torsor over X' dominating Y'. Moreover, Φ is equipped with an action of G and Z' is a $\Phi \rtimes G$-torsor over S. It follows from this that Z' is a $\Phi \rtimes G'$-torsor over X.



In other words, any quotient H of $\boldsymbol{\pi}(X/B, x)$ fits in a diagram:

$$\begin{array}{ccc} \boldsymbol{\pi}(X/B, x) & \longrightarrow & \boldsymbol{\pi}(S/B, s) \\ \downarrow & & \downarrow \\ \Phi & \longrightarrow & \Phi \rtimes G' \\ \downarrow & & \\ H & & \end{array}$$

Since $\boldsymbol{\pi}(X/B,x)$ is the projective limit of such H's and the bottom horizontal arrow is a closed immersion, the top one is a monomorphism, and it is a closed immersion by [6] IV.8.10.5. □

The previous theorem suggests that submissive morphisms play, for the fundamental group scheme, the role that covers have for the étale fundamental group. The remainder of this section is devoted to making this hunch more precise.

Definition 4.10 *Let (S,s) be a pointed B-scheme. Let $\mathfrak{Sub}(S/B;s)$ be the category whose objects are pairs (X,x) consisting of a submissive B-scheme $\pi : X \to S$ and a point $x \in X(B)$ such that $\pi(x)=s$. A morphism $(X',x') \to (X,x)$ is a morphism of pointed (S,s)-schemes.*

The forgetful functor $(X,G,x) \mapsto (X,x)$ embeds $\mathfrak{C}(S/B;s)$ into $\mathfrak{Sub}(S/B;s)$ [though not as a full subcategory: if B is a perfect field k of characteristic $p>0$, A is a k-algebra and $a \in A^\times$, then $X = \operatorname{Spec} A[x]/(x^p - a)$ can be given both an $\boldsymbol{\alpha}_p$ and a $\boldsymbol{\mu}_p$-torsor structure over $S = \operatorname{Spec} A$; as there are no nonzero morphisms over k between these group schemes, the identity on X does not come from a morphism $(X,\boldsymbol{\alpha}_p) \to (X,\boldsymbol{\mu}_p)$].

Proposition 4.11 *Let (S,s) be a flat pointed B-scheme. Finite fibred products exist in the category $\mathfrak{C}(S/B;s)$ if and only if they exist in $\mathfrak{Sub}(S/B;s)$.*

Proof The if part follows from remark 4.4: given three torsors $(Y_i,G_i,y_i) \in \mathfrak{C}(S/B;s)$, if $Y_1 \times_{Y_0} Y_2$ exists in $\mathfrak{Sub}(S/B;s)$ it is in particular flat over S, and therefore a $G_1 \times_{G_0} G_2$-torsor over the whole of S.
For the converse, let (X_i,x_i) be three submissive schemes over (S,s) and let $(Y_i,G_i,y_i) \in \mathfrak{C}(S/B;s)$ dominate (X_i,x_i). Denote by H_i the group of Y_i/X_i. Let us furthermore assume that these schemes fit in a diagram in $\mathfrak{Sub}(S/B;s)$

$$\begin{array}{ccccc} Y_1 & \xrightarrow{\alpha_1} & Y_0 & \xleftarrow{\alpha_2} & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 & \longleftarrow & X_2 \end{array}$$

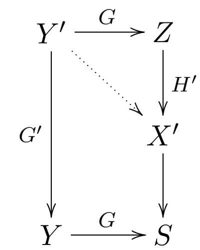
where $(\alpha_i,\beta_i) : (Y_i,G_i,y_i) \to (Y_0,G_0,y_0)$ are in $\mathfrak{C}(S/B;s)$: that such a construction is possible, will be proved in the following lemma 4.12.
If finite fibred products exist in $\mathfrak{C}(S/B;s)$, then $Y_1 \times_{Y_0} Y_2$ is a $G_1 \times_{G_0} G_2$-torsor over S. One checks immediately that the following diagram is cartesian:

$$\begin{array}{ccc} (H_1 \times_{H_0} H_2) \times_B (Y_1 \times_{Y_0} Y_2) & \xrightarrow{(\mu,id)} & (Y_1 \times_{Y_0} Y_2) \times_{X_1 \times_{X_0} X_2} (Y_1 \times_{Y_0} Y_2) \\ {\scriptstyle (\iota,id)}\downarrow & & \downarrow{\scriptstyle (id,id)} \\ (G_1 \times_{G_0} G_2) \times_B (Y_1 \times_{Y_0} Y_2) & \xrightarrow{(\mu,id)} & (Y_1 \times_{Y_0} Y_2) \times_S (Y_1 \times_{Y_0} Y_2) \end{array}$$

where μ is the group action and $\iota : H_1 \times_{H_0} H_2 \to G_1 \times_{G_0} G_2$ the inclusion. Since the bottom arrow is an isomorphism, so is the top one. Hence $Y_1 \times_{Y_0} Y_2$ is an $H_1 \times_{H_0} H_2$-torsor over $X_1 \times_{X_0} X_2$. Therefore the latter is finite and flat over S and dominated by a torsor. □

Lemma 4.12 *Let $f : X' \to X$ be a morphism of submissive S-schemes, Y a finite torsor over S dominating X. Then there exists a finite torsor Y'/S dominating both X' and Y.*

Proof Let G be the group of Y/S. By assumption, there exists a scheme Z which is a torsor over S under a finite flat B-group scheme G' and a torsor over X' under a subgroup $H' \subseteq G'$. Put $Y' = Y \times_S Z$: by construction, it is a $G \times_B G'$-torsor over S, a G'-torsor over Y and a G-torsor over Z. Therefore, it is a $G \times_B H'$-torsor over X'. □



Theorem 4.13 *A flat B-scheme S has a fundamental group scheme if and only if the category $\mathfrak{Sub}(S/B;s)$ admits finite fibered products. The universal cover is the initial object in* Pro $(\mathfrak{Sub}(S/B;s))$.

Proof Nori's proof that $\mathfrak{C}(S/B;s)$ is filtered if and only if it has finite fibered products ([11], proposition II.1) is formal and can be repeated verbatim for $\mathfrak{Sub}(S/B;s)$.
By proposition 4.11, the projective limit $(\hat{S},\hat{s})$ of $\mathfrak{Sub}(S/B;s)$ exists if and only if the universal cover $(\tilde{S}, \boldsymbol{\pi}(S/B;s), \tilde{s})$, which is the projective limit of $\mathfrak{C}(S/B;s)$, exists. Since $\mathfrak{C}(S/B;s)$ is a subcategory of $\mathfrak{Sub}(S/B;s)$, there is a canonical morphism $\hat{S} \to \tilde{S}$ in Pro $(\mathfrak{Sub}(S/B;s))$. On the other hand, any object in $\mathfrak{Sub}(S/B;s)$ receives a morphism from $\tilde{S}$ and, by lemma 4.12, we can build a compatible system of such maps. Therefore also $\tilde{S}$ is a projective limit in $\mathfrak{Sub}(S/B;s)$, and we conclude by uniqueness of the limit. □

Remark 4.14 When B is the spectrum of a perfect field of positive characteristic, by proposition 4.7 the category $\mathfrak{Sub}(S/B;s)$ coincides with the category of pointed F-constant S-schemes. Let $\mathfrak{FDH}(S,s)$ be the category of pointed fdh S-schemes; it contains $\mathfrak{Sub}(S/B;s)$ as a full subcategory. Then $\mathfrak{FDH}(S,s)$ has finite fibred products if and only if either $\mathfrak{Sub}(S/B;s)$ or $\mathfrak{C}(S/B;s)$ do. This is a simple consequence of remark 4.4 (existence of products is a local problem on the base) and proposition 2.5 (locally on the base every fdh morphism is submissive).

Remark 4.15 It would be interesting to have a characterization for submissive morphisms of arithmetic schemes. The differentially homogeneous condition is too strong: if $\Omega^1_{X/S}$ is locally free, it vanishes on the generic fibre (a submissive morphism in characteristic zero is étale), hence it is zero altogether. A necessary condition is that the fibres should be submissive (i.e., F-constant or étale).

References

[1] M. Antei - M. Emsalem, *Galois closure of essentially finite morphisms*, preprint, `arXiv:09011551 v1 [math.AG]`.

[2] M. Demazure - P. Gabriel, *Groupes Algébriques*, Masson, Paris (1970).

[3] M.A. Garuti, *On the "Galois closure" for torsors*, Proc. Amer. Math. Soc. 137, 3575-3583 (2009).

[4] C. Gasbarri, *Heights of vector bundles and the fundamental group scheme of a curve*, Duke Math. J. 117, 287-311 (2003).

[5] A. Grothendieck, *Élements de géométrie algébrique IV$_2$*, Publ. Math. IHES 24 (1965).

[6] A. Grothendieck, *Élements de géométrie algébrique IV$_3$*, Publ. Math. IHES 28 (1966).

[7] A. GROTHENDIECK, *Revêtements étales et groupe fondamental*, Lecture Notes in Math. 224 Springer (1971).

[8] V.B. MEHTA, S. SUBRAMANIAN, *On the fundamental group scheme*, Inventiones Math. 148, 143-150 (2002).

[9] J.S. MILNE, *Étale cohomology*, Princeton Univ. Press (1980).

[10] D. MUMFORD, *Abelian Varieties*, Oxford University Press, Oxford (1982).

[11] M. NORI, *The fundamental group scheme*, Proc. Indian Acad. Sci. (Math. Sci.) 91, 73-122 (1982).

[12] C. PAULY, *A smooth counterexample to Nori's conjecture on the fundamental group scheme*, Proc. Amer. Math. Soc. 135, 2707-2711 (2007).

[13] P.J. SANCHO DE SALAS, *Differentially homogeneous schemes*, Journal of Algebra, 221(1), 279-292 (1999).

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