

**MATEMATICA A, Cenni sulle soluzioni degli esercizi di autovalutazione bis, 4**  
(ATTENZIONE: in un compito non basta scrivere come nel seguito, vanno giustificati tutti i passaggi!)

Vicenza, novembre 2007.

**Limiti di successione e di funzione**

1.

$$\lim_n \frac{n^n}{e^{n^2}} = \lim_n e^{n \log n - n^2} = 0.$$

2.

$$\lim_n \frac{n^a - \cos n}{3n^2 - n^2 \sin(n^3) + \sin(\sqrt{n})} = \lim_n \frac{n^a}{n^2(3 - \sin(n^3))} = \lim_n \frac{n^{a-2}}{(3 - \sin(n^3))}.$$

Poichè  $2 \leq 3 - \sin(n^3) \leq 4$ , e dunque  $1/4 \leq 1/(3 - \sin(n^3)) \leq 1/3$ , per  $a = 1$  il limite è 0 e per  $a = 3$  è  $+\infty$ .

3. (Svolto a lezione)

$$\lim_n \frac{n^{\sqrt{n}} + (\sqrt{n})^n}{2^{n+\sqrt{n}}} = +\infty.$$

4.

$$\lim_n \left(1 + \frac{1}{\sqrt{n}}\right)^{(n^{1/3} \sin n + (-1)^n)} = \lim_n e^{(n^{1/3} \sin n + (-1)^n) \log\left(1 + \frac{1}{\sqrt{n}}\right)},$$

dove per le asintoticità

$$\lim_n (n^{1/3} \sin n + (-1)^n) \log\left(1 + \frac{1}{\sqrt{n}}\right) = \lim_n (n^{1/3} \sin n + (-1)^n) \frac{1}{\sqrt{n}} =$$

$$\lim_n \left(\frac{\sin n}{n^{1/2-1/3}} + \frac{(-1)^n}{n^{1/2}}\right) = 0.$$

Quindi, risulta  $e^0 = 1$ .

5. (Svolto a lezione)

$$\lim_{x \rightarrow \pi^-} \frac{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}{1 - \cos^2 x} = +\infty.$$

6. (Svolto a lezione)

$$\lim_{x \rightarrow -\infty} \left(\sqrt{3x^2 - x} - \sqrt{3x^2 + x + 1}\right) = \frac{\sqrt{3}}{3}.$$

7.

$$\lim_{x \rightarrow +\infty} \frac{1 + 3 \sin x - x \sin(2x)}{x^2 - 1} = \lim_{x \rightarrow +\infty} \frac{-x \sin(2x)}{x^2} = 0.$$

## Limiti di successione e di funzione (da appelli)

1. Usando le scale (e giustificando gli "o-piccolo" usati..)

$$\lim_{n \rightarrow +\infty} \frac{5n! - \sqrt{5}n^{2+n^3}}{n^3 \log\left(1 + \frac{1}{\sqrt{n}}\right) - 2n^2 \log(1+n)} n^{-n^3} = \lim_{n \rightarrow +\infty} \frac{-\sqrt{5}n^{2+n^3} n^{-n^3}}{n^3 \log\left(1 + \frac{1}{\sqrt{n}}\right)} = \lim_{n \rightarrow +\infty} \frac{-\sqrt{5}n^2}{\frac{n^3}{\sqrt{n}}} = 0.$$

2. (Svolto a lezione: si una Mac-Laurin)

$$\lim_{x \rightarrow 0^+} \frac{2^x - \sin(\alpha x) - 1 + x^3 \sin \frac{1}{x}}{1 - \cos(\sqrt{x}) - \frac{1}{2} \log(x+1)}.$$

Risulta:  $+\infty$  se  $\alpha < \log 2$ ;  $-\infty$  se  $\alpha > \log 2$ ;  $\frac{12}{5} \log^2 2$  se  $\alpha = \log 2$ .

3. (Usando Mac-Laurin)

$$\lim_n \frac{1 + \tan^3\left(\frac{1}{n}\right) - e^{\sin^3\left(\frac{1}{n}\right)}}{\frac{1}{n^{3+\alpha}} \left( e^{\sin^2\left(\frac{2}{n}\right)} - e^{\frac{1}{n^2}} \right)} = \lim_n \frac{\frac{3}{2} \frac{1}{n^5}}{\frac{1}{n^{3+\alpha}} \left( \frac{3}{n^2} \right)} = \lim_n \frac{1}{2} n^\alpha,$$

perchè

$$\tan^3\left(\frac{1}{n}\right) = \left( \frac{1}{n} + \frac{1}{3} \frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \right)^3 = \frac{1}{n^3} + \frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^3\left(\frac{1}{n}\right) = \left( \frac{1}{n} - \frac{1}{6} \frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \right)^3 = \frac{1}{n^3} - \frac{1}{2} \frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^2\left(\frac{2}{n}\right) = \left( \frac{2}{n} + o\left(\frac{1}{n}\right) \right)^2 = \frac{4}{n^2} + o\left(\frac{1}{n^2}\right),$$

$$e^{\frac{1}{n^2}} = 1 + \frac{1}{n^2} + \frac{1}{2} \frac{1}{n^4} + o\left(\frac{1}{n^4}\right),$$

$$e^{\sin^3\left(\frac{1}{n}\right)} = 1 + \sin^3\left(\frac{1}{n}\right) + \frac{1}{2} \sin^6\left(\frac{1}{n}\right) + o\left(\sin^6\left(\frac{1}{n}\right)\right) = 1 + \frac{1}{n^3} - \frac{1}{2} \frac{1}{n^5} + o\left(\frac{1}{n^5}\right).$$

$$e^{\sin^2\left(\frac{2}{n}\right)} = 1 + \sin^2\left(\frac{2}{n}\right) + o\left(\sin^2\left(\frac{2}{n}\right)\right) = 1 + \frac{4}{n^2} + o\left(\frac{1}{n^2}\right).$$

Risulta:  $+\infty$  se  $\alpha > 0$ ;  $\frac{1}{2}$  se  $\alpha = 0$ ;  $0$  se  $\alpha < 0$ .

**3. modificato** Se si considera al posto dell' es. 3 l'esercizio seguente

$$\lim_n \frac{1 + \tan^3\left(\frac{1}{n}\right) - e^{\sin^3\left(\frac{1}{n}\right)}}{\frac{1}{n^{3+\alpha}} \left( e^{\sin^2\left(\frac{1}{n}\right)} - e^{\frac{1}{n^2}} \right)} = \lim_n \frac{\frac{3}{2} \frac{1}{n^5}}{\frac{1}{n^{3+\alpha}} \left( -\frac{1}{3} \frac{1}{n^4} \right)} = \lim_n -\frac{9}{2} n^{2+\alpha},$$

usando Mac-Laurin si ha quanto sopra, perchè

$$\tan^3\left(\frac{1}{n}\right) = \left( \frac{1}{n} + \frac{1}{3} \frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \right)^3 = \frac{1}{n^3} + \frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^3\left(\frac{1}{n}\right) = \left( \frac{1}{n} - \frac{1}{6} \frac{1}{n^3} + o\left(\frac{1}{n^3}\right) \right)^3 = \frac{1}{n^3} - \frac{1}{2} \frac{1}{n^5} + o\left(\frac{1}{n^5}\right),$$

$$\sin^2\left(\frac{1}{n}\right) = \left(\frac{1}{n} - \frac{1}{6} \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right)^2 = \frac{1}{n^2} - \frac{1}{3} \frac{1}{n^4} + o\left(\frac{1}{n^4}\right),$$

$$e^{\frac{1}{n^2}} = 1 + \frac{1}{n^2} + \frac{1}{2} \frac{1}{n^4} + o\left(\frac{1}{n^4}\right),$$

$$e^{\sin^3\left(\frac{1}{n}\right)} = 1 + \sin^3\left(\frac{1}{n}\right) + \frac{1}{2} \sin^6\left(\frac{1}{n}\right) + o\left(\sin^6\left(\frac{1}{n}\right)\right) = 1 + \frac{1}{n^3} - \frac{1}{2} \frac{1}{n^5} + o\left(\frac{1}{n^5}\right).$$

$$e^{\sin^2\left(\frac{1}{n}\right)} = 1 + \sin^2\left(\frac{1}{n}\right) + \frac{1}{2} \sin^4\left(\frac{1}{n}\right) + o\left(\sin^4\left(\frac{1}{n}\right)\right) = 1 + \frac{1}{n^2} - \frac{1}{3} \frac{1}{n^4} + \frac{1}{2} \frac{1}{n^4} + o\left(\frac{1}{n^4}\right).$$

Risulta:  $-\infty$  se  $\alpha > -2$ ;  $-\frac{9}{2}$  se  $\alpha = -2$ ;  $0$  se  $\alpha < -2$ .

4. (Si usa Mac-Laurin, dopo la sost.  $y = 1/x$ )

$$L = \lim_{x \rightarrow +\infty} \frac{\left(\frac{1}{x}\right)^{1/x} - 2e^{1/x} + \cos\left(\frac{1}{x}\right) + \frac{a}{x} \log\left(\frac{1}{x}\right)}{\left(\sqrt{1 + \sinh\left(\frac{1}{x}\right)} - \sqrt{1 + \sin\left(\frac{1}{x}\right)}\right)^{1/3}} = \lim_{y \rightarrow 0^+} \frac{y^y - 2e^y + \cos(y) + ay \log(y)}{\left(\sqrt{1 + \sinh(y)} - \sqrt{1 + \sin(y)}\right)^{1/3}}$$

dove

$$y^y = e^{y \log y} = 1 + y \log y + \frac{1}{2} y^2 \log^2 y + o(y^2 \log^2 y) \quad (\text{perchè } y \log y \rightarrow 0..),$$

$$\sqrt{1 + \sinh(y)} = 1 + \frac{1}{2} \sinh y - \frac{1}{8} \sinh^2 y + \frac{1}{16} \sinh^3 y + o(\sinh^3 y) = 1 + \frac{1}{2} \left(y + \frac{1}{6} y^3\right) - \frac{1}{8} y^2 + \frac{1}{16} y^3 + o(y^3)$$

$$\sqrt{1 + \sin(y)} = 1 + \frac{1}{2} \sin y - \frac{1}{8} \sin^2 y + \frac{1}{16} \sin^3 y + o(\sin^3 y) = 1 + \frac{1}{2} \left(y - \frac{1}{6} y^3\right) - \frac{1}{8} y^2 + \frac{1}{16} y^3 + o(y^3).$$

Quindi

$$L = \lim_{y \rightarrow 0^+} \frac{(1+a)y \log y - 2y + o(y)}{\left(\frac{1}{12} y^3 + \frac{1}{12} y^3 + o(y^3)\right)^{1/3}} = \lim_{y \rightarrow 0^+} \frac{(1+a)y \log y - 2y}{\frac{1}{\sqrt[3]{6}} y}$$

e risulta  $-\infty$  se  $a > -1$ ;  $+\infty$  se  $a < -1$ ;  $-2\sqrt[3]{6}$  se  $a = -1$ .