On Poincaré-Birkhoff periodic orbits for mechanical Hamiltonian systems on $T^*\mathbb{T}^n$

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Abstract

A version of the Arnol'd conjecture, first studied by Conley, Zehnder, giving a generalization of the Poincaré-Birkhoff last geometrical theorem, is here proved inside Viterbo's framework of the generating functions quadratic at infinity. By making this goal, we give short recalls on some tools here exploited, often utilized in symplectic topology.

Keywords: Symplectic Topology, Symplectic Geometry, Hamiltonian Mechanics, Calculus of Variations.

1 Introduction

Henri Poincaré has been the main pioneer of the modern dynamical systems theory. Among the large multitude of his contributes, he formulated the nowadays said 'Poincaré's last geometrical theorem' in order to schematize a crucial class of problems related to the search of period solutions in Hamiltonian dynamics:

{P} *Any area preserving diffeomorphism of the annulus $A = \{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$ into itself, uniformly rotating the two boundary circles of radius a and b in opposite directions, admits at least two geometrically distinct fixed points.*

The first rigorous proof of this statement was given in the twenties of the past century by Birkhoff by means of a technique which seems not easily extendible to greater dimensional systems. In a following paper [4], he remarked the power of “maximum-minimum considerations” in the existence of periodic orbits. Nowadays, these aspects are well ruled in the Lusternik-Schnirelman setting: in this framework, one can select minimax critical values (connected to periodic orbits) of suitable generating functions –quadratic at infinity, see below.
In the sixties, in a series of papers Arnol'd proposed his celebrated conjecture:
{A} *Any Hamiltonian diffeomorphism of a compact symplectic manifold* (M, ω) *possesses at least many fixed points as a function* $f : M \to \mathbb{R}$ *on* M *possesses critical points.*
This new and intriguing topological question has been answered by Conley and Zehnder [10] in the case $M = \mathbb{T}^{2n}$; in that same paper they also proved that
{C-Z} *For a Hamiltonian* $H : \mathbb{R} \times T^*\mathbb{T}^n \to \mathbb{R}$*, such that for* $|p| \geq C$ *the related vector field* X_H *is p-linear and independent of* $q \in \mathbb{T}^n$ *and* $t \in \mathbb{R}$*, the time-one flow* ϕ^1_H *of* X_H *admits at least many fixed points as a function* $f : \mathbb{T}^n \to \mathbb{R}$ *on* $\mathbb{T}^n$ *possesses critical points.*
It is interesting to notice that this last statement, directly descending from Poincaré's last geometrical theorem, in a sense, comes back to the original setting of Analytical Mechanics in which it arose. E.g., the above Hamiltonians are at once interpreted as describing a physical landscape in which a number of particles does interact among them only under a suitable energy threshold (low energy scattering):

$$H(q, p) = \frac{1}{2}|p|^2 + f(q, p), \quad q \in \mathbb{T}^n, \quad f \in O(1).$$

Incidentally, we can note that this is quite near to a typical Hamiltonian setting of Nekhoroshev perturbation theory: $H(q, p) = 1/2|p|^2 + \varepsilon f(q, p)$.

Conley and Zehnder introduced a sort of Liapunov-Schmidt reduction technique, now known as *Amann-Conley-Zehnder reduction*, based on a suitable Fourier cut-off on the loop space and giving, at last, a finite dimensional variational problem. Chaperon –see [5]– proposed few time later his new ingenious *broken geodesics reduction*, showing it is not indispensable to start from the infinite dimensional formulation of the problem. In both cases, the estimates on fixed points of ϕ^1_H are proved using the isolated invariant sets and the Morse index, as presented by Conley [9].

More recently, Golé [13], [12], gave an alternative proof of the statement {C-Z}, extending $\mathbb{T}^n$ to any compact manifold and using a variation of Chaperon's argument. The finite variational problem which in such a way he obtained was solved by utilizing techniques based on Conley index and further results on it by Floer. Furthermore, the author pointed out that his function, defining the above finite variational problem, was not a generating function quadratic at infinity, an essential property in order to apply agreeably Lusternik-Schnirelman theory.

Nowadays, a short and nice proof of this theorem can be built up using the fine papers [6] and [7] by Chaperon.

After the impressive paper [22], there exists a rather common growing prejudge that the framework of the generating functions quadratic at infinity and Lusternik-Schnirelman theory should be a right environment to better understand many actual aspects of symplectic topology, as Arnol'd conjecture, see p.e. [15], p. 216.

In this paper, by assuming this point of view, we restart from the original statement {C-Z}, for $\mathbb{T}^n$. In genuine framework of the generating functions quadratic at infinity, and then using now classical results by Chaperon, Chekanov, Laudenbach, Sikorav and Viterbo, we propose a finite variational problem consisting of a generating function quadratic at infinity: a suitable application of Lusternik-Schnirelman theory in the degenerate case, and Morse theory in the nondegenerate one, produces the expected result. By making this goal, we give short recalls on some tools here exploited, often involved in symplectic topology.

2 Preliminaries

2.1 Generating functions

Let N be a compact manifold and $L \subset T^*N$ a Lagrangian submanifold. If $L = \operatorname{im}(df) = L_f$, where $f : N \to \mathbb{R}$ is a C^2 function, then the set $\operatorname{crit}(f)$ of the critical points of f coincides with the intersection of L_f with the zero section $0_N \subset T^*N$:

$$\operatorname{crit}(f) = L_f \cap 0_N.$$

In the more general case, Lagrangian submanifolds have not the above graph structure L_f, and a classical argument by Maslov and Hörmander shows that, at least locally, every Lagrangian submanifold is described by some generating function like $S : N \times \mathbb{R}^k \longrightarrow \mathbb{R}$, $(x, \xi) \longmapsto S(x, \xi)$, in the following way:

$$L_S := \{(x, \frac{\partial S}{\partial x}(x, \xi)) \ : \ \frac{\partial S}{\partial \xi}(x, \xi) = 0\},$$

where 0 is a regular value of the map $(x, \xi) \longmapsto \frac{\partial S}{\partial \xi}(x, \xi)$.

Some authors (e.g. Benenti, Tulczyjew, Weinstein) say that in this case the generating function S is a *Morse family*. In order to apply the Calculus of Variations to generating functions, one needs a condition implying the existence of critical points. In particular, the following class of generating functions has been decisive in many issues:

Definition 2.1 *A generating function $S : N \times \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity (GFQI) if for $|\xi| > C$*

$$S(x, \xi) = \xi^T Q \xi, \tag{1}$$

where $\xi^T Q \xi$ is a nondegenerate quadratic form.

There were known in literature (see e.g. [23], [16]) two main operations on the generating functions which leave invariant the corresponding Lagrangian submanifolds. The Lemma 2.2 and 2.3 below recollect these facts. The globalization was realized by Viterbo (see [20]).

Lemma 2.2 *Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a GFQI and $N \times \mathbb{R}^k \ni (x, \xi) \mapsto (x, \phi(x, \xi)) \in N \times \mathbb{R}^k$ a map such that, $\forall x \in N$,*

$$\mathbb{R}^k \ni \xi \longmapsto \phi(x, \xi) \in \mathbb{R}^k$$

is a diffeomorphism. Then $S_1(x, \xi) := S(x, \phi(x, \xi))$ generates the same Lagrangian submanifold: $L_{S_1} = L_S$.

Proof. Since ϕ is a diffeomorphism, $\frac{\partial S_1}{\partial \xi} = \frac{\partial S}{\partial \xi}\frac{\partial \phi}{\partial \xi} = 0$ if and only if $\frac{\partial S}{\partial \xi} = 0$. Moreover, $\frac{\partial S_1}{\partial x} = \frac{\partial S}{\partial x} + \frac{\partial S}{\partial \xi}\frac{\partial \phi}{\partial x}$ and it is immediately verified that 0 is a regular value for $\frac{\partial S_1}{\partial \xi}(x, \xi)$. $\square$

Lemma 2.3 *Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a GFQI. Then*

$$S_1(x, \xi, \eta) := S(x, \xi) + \eta^T B \eta,$$

where $\eta \in \mathbb{R}^l$ and $\eta^T B \eta$ is a nondegenerate quadratic form, generates the same Lagrangian submanifold: $L_{S_1} = L_S$.

Proof. $\frac{\partial S_1}{\partial \xi}(x, \xi, \eta) = 0$ if and only if $\frac{\partial S}{\partial \xi}(x, \xi) = 0$. Moreover, $\frac{\partial S_1}{\partial \eta}(x, \xi, \eta) = 0$ if and only if $B\eta = 0$, that is $\eta = 0$. Thus $\frac{\partial S_1}{\partial x}|_{\frac{\partial S_1}{\partial \xi}=0, \frac{\partial S_1}{\partial \eta}=0} = \frac{\partial S}{\partial x}|_{\frac{\partial S}{\partial \xi}=0}$. $\square$

Finally, as a third –although trivial– invariant operation, we observe that by adding to a generating function S any arbitrary constant $c \in \mathbb{R}$ the described Lagrangian submanifold is invariant: $L_{S+c} = L_S$. The problems 1 and 2 below have been crucial in the global theory of Lagrangian submanifolds and their parameterizations.

1. When does a Lagrangian submanifold $L \subset T^*N$ admit a FGQI?
2. If L admits a GFQI, when can we state the uniqueness of it (up to the operations described above)?

The following theorem –see [18]– answers partially to the first question.

Theorem 2.4 *(Chaperon-Chekanov-Laudenbach-Sikorav) Let 0_N be the zero section of T^*N and $(\phi_t)_{t\in[0,1]}$ a Hamiltonian isotopy. Then the Lagrangian submanifold $\phi_1(0_N)$ admits a GFQI.*

The answer to the second problem is due to Viterbo:

Theorem 2.5 *(Viterbo) Let 0_N be the zero section of T^*N and $(\phi_t)_{t\in[0,1]}$ a Hamiltonian isotopy. Then the Lagrangian submanifold $\phi_1(0_N)$ admits a unique (up to the operations described above) GFQI.*

The theorems above –see also [21]– still hold in $T^*\mathbb{R}^n$, provided that $(\phi_t)_{t\in[0,1]}$ is a flow of a compactly supported Hamiltonian vector field.

2.2 Lusternik-Schnirelman theory

Let $f : N \to \mathbb{R}$ be a C^2 function. We shall assume that either N is compact or f satisfies the Palais-Smale (PS) condition:
(PS) Any sequence $\{x_n\}$ such that $\nabla f(x_n) \to 0$ and $f(x_n)$ is bounded, admits a converging subsequence.
We recall now some results of the Lusternik-Schnirelman theory, which allows us to associate critical values of f to non-vanishing relative cohomology classes and to give a lower bound to the number of critical points of f in terms of the topological complexity of N.
Let us define the sublevel sets

$$N^\nu := \{x \in N : \ f(x) \leq \nu\}. \tag{2}$$

(PS) condition guarantees the well-defined gradient vector field ∇f, whose flow realizes a diffeomorphism between N^μ and N^ν whenever no critical values exist in $[\mu, \nu]$:

Proposition 2.6 *Let $\mu < \nu$. If f has no critical points in $N^\nu \setminus N^\mu$, then $H^*(N^\nu, N^\mu) = 0$.*

Thus if $H^*(N^\nu, N^\mu) \neq 0$, then in $N^\nu \setminus N^\mu$ there exists at least one critical point of f, with critical value in $[\mu, \nu]$. For $\lambda \in [\mu, \nu]$, let $i_\lambda : N^\lambda \hookrightarrow N^\nu$ be the inclusion.

Definition 2.7 *For every $u \in H^*(N^\nu, N^\mu)$, $u \neq 0$, we define:*

$$c(u, f) =: \inf \{\lambda \in [\mu, \nu] : \; i_\lambda^* u \neq 0\},$$

where

$$i_\lambda^* : \; H^*(N^\nu, N^\mu) \longrightarrow H^*(N^\lambda, N^\mu)$$

denotes the pull-back of the inclusion.

This Definition provides a tool to detect critical values, indeed:

Theorem 2.8 *$c(u, f)$ is a critical value of f.*

The main result of this construction consists in the following

Theorem 2.9 *(Cohomological Lusternik-Schnirelman theory) Let $0 \neq u \in H^*(N^\nu, N^\mu)$ and $v \in H^*(N^\nu) \setminus H^0(N^\nu)$.*

1. $$c(u \wedge v, f) \geq c(u, f). \tag{3}$$

2. *If (3) is an equality ($c(u \wedge v, f) = c(u, f) =: c$), set $K_c = \{x \; : \; df(x) = 0, \; f(x) = c\}$, then, for every neighbourhood U of K_c, v is not vanishing in $H^*(U)$, and the common critical level contains infinitely many critical points.*

Corollary 2.10 *Let N be a compact manifold. The function $f : N \to \mathbb{R}$ has at least a number of critical points equal to the cup-lenght of N:*

$$\mathrm{cl}(N) := \max \left\{k : \; \exists v_1, \ldots, v_{k-1} \in H^*(N) \setminus H^0(N) \; s.\,t. \; v_1 \wedge \ldots \wedge v_{k-1} \neq 0\right\}. \tag{4}$$

Proof. Apply Theorem 2.9 with $\mu < \inf f$, $\sup f < \nu$ and $u = 1 \in H^*(N, \emptyset) = H^*(N)$. □

By Corollary 2.11 below, we verify that the preceding estimate on the number of critical points of f still holds in the non-compact case whenever GFQI f are taken into account.

Corollary 2.11 *Let N be a compact manifold and $f : N \times \mathbb{R}^n \to \mathbb{R}$ be a GFQI, $f(x, \xi) = Q(\xi)$ out of a compact set in the parameters ξ. Then, for $c > 0$ large enough, there exist $0 \neq u \in H^*(f^c, f^{-c})$ and $v_1, \ldots, v_{k-1}$ as in (4) such that*

$$u \wedge p^* v_1 \wedge \ldots \wedge p^* v_{k-1} \neq 0,$$

where $p : N \times \mathbb{R}^n \to N$ is the canonical projection. Consequently, the GFQI $f : N \times \mathbb{R}^n \to \mathbb{R}$ has at least $\mathrm{cl}(N)$ critical points.

Proof. Let us first observe that for $c > 0$ large enough, the sublevel sets of f are invariant from a homotopical point of view: $f^{\pm c} = N \times Q^{\pm c}$, and $f^{\pm \bar{c}}$ retracts on $f^{\pm c}$ for any $\bar{c} > c$. Let $A := Q^{-(c+\epsilon)}$, $\epsilon > 0$ small. Then the isomorphisms below (the first one by excision and the second one by retraction) hold:

$$H^*(Q^c, Q^{-c}) \cong H^*(Q^c \setminus \mathring{A}, Q^{-c} \setminus \mathring{A}) \cong H^*(D^i, \partial D^i),$$

where i is the index of the quadratic form Q and D^i denotes the disk (of radius $\sqrt{c}$) in $\mathbb{R}^i$. Consequently

$$H^h(Q^c, Q^{-c}) \cong H^h(D^i, \partial D^i) = \begin{cases} 0 & \text{if } h \neq i \\ \alpha\,\mathbb{R} & \text{if } h = i \end{cases}$$

To conclude in the non-compact case $N \times \mathbb{R}^n$, by Künneth isomorphism

$$H^*(N) \cong H_c^{*+i}(N \times \mathbb{R}^i)$$

and the homotopy argument

$$H_c^*(N \times \mathbb{R}^i) \cong H^*(N \times D^i, N \times \partial D^i),$$

the following isomorphism

$$H^*(N) \ni v \longmapsto q^*\alpha \wedge p^* v \in H^{*+i}(N \times D^i, N \times \partial D^i)$$

holds, where $p : N \times \mathbb{R}^n \to N$, $q = (q_1, q_2) : (N \times D^i, N \times \partial D^i) \to (D^i, \partial D^i)$ are the standard projections. Now we apply the Theorem 2.9 with $u = q^*\alpha$; since $q^*\alpha \wedge p^* v_1 \wedge \ldots \wedge p^* v_{k-1} = q^*\alpha \wedge p^*(v_1 \wedge \ldots \wedge v_{k-1}) \neq 0$ whenever $v_1 \wedge \ldots v_{k-1} \neq 0$, then the number of critical points of the GFQI $f : N \times \mathbb{R}^n \to \mathbb{R}$ is at least $\mathrm{cl}(N)$. $\square$

3 The Hamiltonian setting

Let $T^*\mathbb{R}^n \equiv \mathbb{R}^{2n} = \{(q,p) :\ q \in \mathbb{R}^n,\ p \in \mathbb{R}^n\}$ be endowed with the standard symplectic form $\omega = dp \wedge dq = \sum_{i=1}^n dp_i \wedge dq^i$.
On $(\mathbb{R}^{2n}, \omega)$ we consider the-time dependent globally Hamiltonian vector field X_H given by

$$H(t,q,p) \in C^2(\mathbb{R} \times \mathbb{R}^{2n}; \mathbb{R}),$$

periodic in q of period 2π and

$$H(t,q,p) = \frac{1}{2}|p|^2 \quad \text{if} \quad |p| \geq C > 0. \tag{5}$$

Our aim is to draw a new proof of a popular version, due to Arnol'd, of Poincaré's last geometrical theorem (see [10],[6], [7], [12]) inside Viterbo's framework of symplectic topology [22].

3.1 Properties of flows on the cotangent of the torus

In connection with the above Hamiltonian H, let[1] ϕ_H^t be the flow of the Hamiltonian vector field X_H, $\omega(X_H, \eta) = -dH(\eta)$, so that $X_H = J\nabla H$, where J is the symplectic $2n$-matrix. The n-torus is denoted by $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Therefore a Hamiltonian $\bar{H}$ and the related flow $\phi_{\bar{H}}^t$ are well defined on $T^*\mathbb{T}^n$, see Corollary 3.2 below:

$$\begin{tikzcd}
\mathbb{R}\times T^*\mathbb{R}^n \arrow[r, "H"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathbb{R} \arrow[d, "id_{\mathbb{R}}"] \\
\mathbb{R}\times T^*\mathbb{T}^n \arrow[r, "\bar{H}"'] & \mathbb{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*\mathbb{R}^n \arrow[r, "\phi_H^t"] \arrow[d, "\pi"'] & T^*\mathbb{R}^n \arrow[d, "\pi"] \\
T^*\mathbb{T}^n \arrow[r, "\phi_{\bar{H}}^t"'] & T^*\mathbb{T}^n
\end{tikzcd}$$

It is standard matter to see that

Proposition 3.1 *The flow ϕ_H^t associated to H satisfies:*

$$\begin{aligned}
(\phi_H^t)_q(q+2\pi k, p) &= (\phi_H^t)_q(q,p) + 2\pi k, \\
(\phi_H^t)_p(q+2\pi k, p) &= (\phi_H^t)_p(q,p),
\end{aligned}$$

$\forall k \in \mathbb{Z}^n$ *and* $\forall (q,p) \in \mathbb{R}^{2n}$.

We denote by $[q] \in \mathbb{T}^n := \mathbb{R}^n/2\pi\mathbb{Z}^n$ the class of $q \in \mathbb{R}^n$. From the above deductions it follows the

Corollary 3.2 *The flow of $X_{\bar{H}}$ is*

$$\phi_{X_{\bar{H}}}^t([q],p) = \left(\left[\phi_{X_H,q}^t(q,p)\right], \phi_{X_H,p}^t(q,p)\right). \tag{6}$$

3.2 The splitting $H = H_0 + f$

We remind that the Hamiltonian H coincides with $\frac{1}{2}|p|^2$ if $|p| \geq C > 0$. Consequently, outside of this compact set (in the p variables) the flow associated to the Hamiltonian H reduces to

$$\mathbb{R}^n \times \{p:\ |p| \geq C\} \longrightarrow \mathbb{R}^n \times \{p:\ |p| \geq C\}$$

$$(q,p) \longmapsto \phi_H^t(q,p) = (q+tp, p).$$

[1]Here, as in other analogous circumstances, we mean $\phi_H^t := \phi_H^{t,0}$.

We split H as the sum of the Hamiltonian $H_0 := \frac{1}{2}|p|^2$ and a Hamiltonian f, hence necessarily compactly supported in the p variables,

$$H = H_0 + f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$(t, q, p) \longmapsto H(t, q, p) = H_0(p) + f(t, q, p).$$

Denoting by ϕ_0^t the flow related to H_0, we define the Hamiltonian K as the pull-back of f with respect to ϕ_0^t:

$$K : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$K := (\phi_0^t)^* f, \text{ i.e. } \quad K(t, q, p) = H(t, q + tp, p) - \frac{|p|^2}{2}.$$

This Hamiltonian K, which is compactly supported in the p variables like f, it will be essential in the next sections. We indicate now ϕ_K^t the flow of K and write down the following proposition, which is, essentially, a result of Hamilton [14] (see also [11]).

Proposition 3.3 *Let ϕ_H^t, ϕ_0^t and ϕ_K^t be the flows of $H = H_0 + (H - H_0)$, H_0 and $K = (\phi_0^t)^*(H - H_0)$ respectively. We have:*

$$\phi_H^t(q, p) = \phi_0^t \circ \phi_K^t(q, p),$$

$\forall (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\forall t \in \mathbb{R}$.

Proof.

$$\begin{aligned}
\frac{d}{dt}(\phi_0^t \circ \phi_K^t)(q, p) &= X_{H_0}(\phi_0^t \circ \phi_K^t(q, p)) + d\phi_0^t(\phi_K^t(q, p))X_K(\phi_K^t(q, p)), \\
&= X_{H_0}(\phi_0^t \circ \phi_K^t(q, p)) + \\
&\quad d\phi_0^t(\phi_0^{-t} \circ \phi_0^t \circ \phi_K^t(q, p))X_K(\phi_0^{-t} \circ \phi_0^t \circ \phi_K^t(q, p)), \\
&= X_{H_0}(\phi_0^t \circ \phi_K^t(q, p)) + (\phi_0^t)_* X_K(\phi_0^t \circ \phi_K^t(q, p)), \\
&= [X_{H_0} + X_{(\phi_0^t)_* K}](\phi_0^t \circ \phi_K^t(q, p)) = X_{H_0 + f}(\phi_0^t \circ \phi_K^t(q, p)), \\
&= X_H(\phi_0^t \circ \phi_K^t(q, p)). \ \Box
\end{aligned}$$

3.3 The 'graph' and the 'cotangent' structures of $\mathbb{R}^{4n}$

We introduce now the linear symplectic isomorphism h, from the 'graph'-structure to the 'cotangent'-structure:

$$h : (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n}) \longrightarrow (T^*(T^*\mathbb{R}^n), \omega_{\mathbb{R}^{2n}})$$

$$(q,p,Q,P) \longmapsto (q,P,p-P,Q+P-q). \tag{7}$$

The following Lagrangian submanifold F of $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n})$,

$$F := \{(q,p,q-p,p) : \ (q,p) \in T^*\mathbb{R}^n\}, \tag{8}$$

is mapped by h to the zero section $0_{\mathbb{R}^{2n}}$: $h(F) = 0_{\mathbb{R}^{2n}} \subset \mathbb{R}^{4n}$.
Since we are looking for fixed points of ϕ^1_H, we denote by Γ_H and Γ_K the graphs of ϕ^1_H and ϕ^1_K in $(T^*\mathbb{R}^n \times T^*\mathbb{R}^n \ , \ \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n})$ respectively, and by Δ the diagonal of $T^*\mathbb{R}^n \times T^*\mathbb{R}^n = \mathbb{R}^{4n}$. It comes out that:

$$(\bar q, \bar p) \in T^*\mathbb{R}^n \text{ is a fixed point of } \phi^1_H,$$

that is, by Prop. 3.3,

$$(\bar q, \bar p, (\phi^1_0 \circ \phi^1_K)_q(\bar q,\bar p), (\phi^1_0 \circ \phi^1_K)_p(\bar q,\bar p)) \in \Gamma_H \cap \Delta,$$

if and only if, setting:

$$\hat\phi_0^{-1}(q,p,Q,P) := id_{\mathbb{R}^{2n}} \times \phi_0^{-1}(q,p,Q,P) = (q,p,Q-P,P),$$

and using F in (8),

$$(\bar q, \bar p, (\phi^1_K)_q(\bar q,\bar p), (\phi^1_K)_p(\bar q,\bar p)) \in \hat\phi_0^{-1}(\Gamma_H) \cap \hat\phi_0^{-1}(\Delta) = \Gamma_K \cap F,$$

if and only if, using h,

$$h(\bar q, \bar p, (\phi^1_K)_q(\bar q,\bar p), (\phi^1_K)_p(\bar q,\bar p)) \in h(\Gamma_K) \cap h(F) = h(\Gamma_K) \cap 0_{\mathbb{R}^{2n}}.$$

Thus, we claim that the periodic time-one solutions, corresponding to fixed points of ϕ^1_H, are caught by the critical points of a (possible) generating function for $h(\Gamma_K)$. Furthermore, they are contained in the region $\mathbb{T}^n \times \{p : |p| < C\}$. In fact, on $\mathbb{T}^n \times \{p : \ |p| \geq C\}$ the Hamiltonian system is trivially integrable and in such a case the tori $\mathbb{T}^n \times \{p\}$ are invariant under the flow $\phi^t_H : (q,p) \mapsto (q+tp, p)$. Consequently, the non trivial periodic solutions of $\phi^1_{\bar H}$, corresponding precisely to the fixed points of ϕ^1_H, must lie in $\mathbb{T}^n \times \{p : \ |p| < C\}$ and are contractible loops on $\mathbb{T}^n$.

4 Existence for generating functions

Our original problem has been translated into the investigation of $h(\Gamma_K) \cap 0_{\mathbb{R}^{2n}}$. The Lagrangian submanifold $h(\Gamma_K)$,

$$h(\Gamma_K) = \{(q, (\phi^1_K)_p(q,p), p - (\phi^1_K)_p(q,p), (\phi^1_K)_p(q,p) + (\phi^1_K)_q(q,p) - q),$$

$$\forall (q,p) \in T^*\mathbb{R}^n\} \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}),$$

in a neighbourhood of infinity (in the p variables) results:

$$h(\Gamma_K) = \{(q,p,0,p),\ \forall q \in \mathbb{R}^n,\ \forall p \in \mathbb{R}^n\ :\ |p| \geq C\}.$$

In this section we study its structure, proving that it is the image (through a suitable symplectic isomorphism ψ of $(T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$) of another Lagrangian submanifold, denoted by $\bar{h}(\Gamma_K)$, which is isotopic to the zero section of $T^*\mathbb{R}^{2n}$, so that it admits a GFQI (Theorem 2.4). This is crucial in order to gain the existence of a generating function for $h(\Gamma_K)$. In fact, by means of a natural composition of the above generating functions for $\bar{h}(\Gamma_K)$ and for ψ, we will be able to construct a GFQI for $h(\Gamma_K)$.

4.1 The factorization of the map h

We introduce the following linear two maps $\bar{h}$ (introduced by Sikorav in [19] and used by Viterbo in [22]) and ψ:

$$\bar{h} : (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, \omega_{\mathbb{R}^n} \ominus \omega_{\mathbb{R}^n}) \longrightarrow (T^*(T^*\mathbb{R}^n), \omega_{\mathbb{R}^{2n}}) \tag{9}$$

$$(q,p,Q,P) \longmapsto (\frac{q+Q}{2}, \frac{p+P}{2}, p-P, Q-q),$$

$$\psi : (T^*(T^*\mathbb{R}^n) = T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \longrightarrow (T^*(T^*\mathbb{R}^n) = T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}) \tag{10}$$

$$(q,p,Q,P) := (x_0,y_0) \longmapsto (x_1,y_1) =: (\frac{2q-P}{2}, \frac{2p-Q}{2}, Q, \frac{2P+2p-Q}{2}).$$

It results well-defined the following map on the quotient tori structures

$$\widetilde{\psi} : T^*(T^*\mathbb{T}^n) \longrightarrow T^*(T^*\mathbb{T}^n)$$

$$([q],p,Q,P) \longmapsto ([\frac{2q-P}{2}], \frac{2p-Q}{2}, Q, \frac{2P+2p-Q}{2})$$

and the following diagram is commutative

$$\begin{tikzcd} T^*(T^*\mathbb{R}^n) \arrow[r, "\psi"] \arrow[d, "\pi"'] & T^*(T^*\mathbb{R}^n) \arrow[d, "\pi"] \\ T^*(T^*\mathbb{T}^n) \arrow[r, "\widetilde{\psi}"'] & T^*(T^*\mathbb{T}^n) \end{tikzcd}$$

It is standard matter to see that the maps ψ and $\bar{h}$ are symplectic isomorphisms and it is easy to check that the factorization $h = \psi \circ \bar{h}$ holds:

$$\begin{tikzcd} T^*\mathbb{R}^n \times T^*\mathbb{R}^n \arrow[r, "h"] \arrow[dr, "\bar{h}"'] & T^*(T^*\mathbb{R}^n) \\ & T^*(T^*\mathbb{R}^n) \arrow[u, "\psi"'] \end{tikzcd}$$

4.2 The Lagrangian submanifold $\bar{h}(\Gamma_K)$

This section is devoted to the proof of the following

Proposition 4.1 *The Lagrangian submanifold $\bar{h}(\Gamma_K) \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ admits a GFQI, $S_1(q,p;\xi)$, 2π-periodic in the q variables.*

Proof. We observe that like H also the Hamiltonian K is periodic of 2π-period in the q variables. Moreover the flow $\phi_K^t = \phi_0^{-t} \circ \phi_H^t$ inherits from the flow ϕ_H^t (see Prop. 3.3) the following properties

$$\begin{aligned}(\phi_K^t)_q(q+2\pi k, p) &= (\phi_K^t)_q(q,p) + 2\pi k,\\ (\phi_K^t)_p(q+2\pi k, p) &= (\phi_K^t)_p(q,p)\end{aligned}$$

$\forall (q,p) \in \mathbb{R}^{2n},\ \forall k \in \mathbb{Z}^n$.
Consequently, for all fixed $t \in \mathbb{R}$ a flow $\widetilde{\phi}_K^{t,0}$ in $T^*\mathbb{T}^n$ results well-defined, in particular, the following definition is independent of the choice of q in the class $[q]$:

$$\widetilde{\phi}_K^{t,0}([q],p) = ((\widetilde{\phi}_K^{t,0})_q([q],p), (\widetilde{\phi}_K^{t,0})_p([q],p)) := ([(\phi_K^t)_q(q,p)], (\phi_K^t)_p(q,p)),$$

$$\begin{array}{ccc} T^*\mathbb{R}^n & \xrightarrow{\phi_K^t} & T^*\mathbb{R}^n \\ \pi \downarrow & & \downarrow \pi \\ T^*\mathbb{T}^n & \xrightarrow[\widetilde{\phi}_K^{t,0}]{} & T^*\mathbb{T}^n \end{array}$$

Here we mean $\pi : (q,p) \to ([q],p)$.
Similarly to Γ_K, we indicate by $\widetilde{\Gamma}_K$ the graph of $\widetilde{\phi}_K^{1,0}$:

$$\widetilde{\Gamma}_K \subset (T^*\mathbb{T}^n \times T^*\mathbb{T}^n, \omega_{\mathbb{T}^n} \ominus \omega_{\mathbb{T}^n}).$$

The Lagrangian submanifold $\bar{h}(\Gamma_K)$:

$$\bar{h}(\Gamma_K) = \Big\{\Big(\frac{q + (\phi_K^1)_q(q,p)}{2}, \frac{p + (\phi_K^1)_p(q,p)}{2}, p - (\phi_K^1)_p(q,p), (\phi_K^1)_q(q,p) - q\Big),$$
$$\forall (q,p) \in T^*\mathbb{R}^n\Big\} \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}}),$$

in a neighbourhood of infinity (in the p variables) results:

$$\bar{h}(\Gamma_K) = \{(q,p,0,0),\ \forall q \in \mathbb{R}^n,\ \forall p \in \mathbb{R}^n\ :\ |p| \geq C\}.$$

It is easy to verify that if $(q,p,Q,P) \in \bar{h}(\Gamma_K)$, then $\forall k \in \mathbb{Z}^n$ $(q+2\pi k, p, Q, P) \in \bar{h}(\Gamma_K)$. Therefore the Lagrangian submanifold $\bar{h}(\Gamma_K) \subset (T^*\mathbb{R}^{2n}, \omega_{\mathbb{R}^{2n}})$ has a natural inclusion into $(T^*(\mathbb{T}^n \times \mathbb{R}^n), \omega_{\mathbb{T}^n\times\mathbb{R}^n})$. Now, we prove that $\bar{h}(\Gamma_K)$ coincides, up to the symplectic morphism $\widetilde{h}$ below from $\widetilde{\Gamma}_K$ to $T^*(\mathbb{T}^n \times \mathbb{R}^n)$, with the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}_K^{1,0}$. In order to see this, we introduce the following well-defined (independent of the choice of q in $[q]$)[2] map

$$\widetilde{h} : \widetilde{\Gamma}_K \longrightarrow T^*(\mathbb{T}^n \times \mathbb{R}^n)$$
$$([q], p, [(\phi_K^1)_q(q,p)], (\phi_K^1)_p(q,p)) \longmapsto$$
$$([\frac{q + (\phi_K^1)_q(q,p)}{2}], \frac{p + (\phi_K^1)_p(q,p)}{2}, p - (\phi_K^1)_p(q,p), (\phi_K^1)_q(q,p) - q).$$

Therefore the following diagram results commutative

$$\begin{tikzcd} \Gamma_K \arrow[r, "\bar{h}"] \arrow[d, "\pi_1"'] & T^*\mathbb{R}^{2n} \arrow[d, "\pi_2"] \\ \widetilde{\Gamma}_K \arrow[r, "\widetilde{h}"'] & T^*(\mathbb{T}^n \times \mathbb{R}^n) \end{tikzcd}$$

here we mean $\pi_1 : (q, p, Q, P) \to ([q], p, [Q], P)$, $\pi_2 : (q, p, Q, P) \to ([q], p, Q, P)$. Thus we have proved that $\bar{h}(\Gamma_K)$ results, up to the symplectic diffeomorphism $\widetilde{h}$, the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}_K^{1,0}$. On the other hand, the manifold $\widetilde{h}(\widetilde{\Gamma}_K)$ is essentially the image of the zero section $\mathbb{T}^n \times \mathbb{R}^n$ through $\widetilde{\phi}_K^{1,0}$. In such hypothesis (see Theorem 2.4) the manifold $\widetilde{h}(\widetilde{\Gamma}_K)$ admits a GFQI, say $s([q], p, \xi)$. Then a GFQI for $\bar{h}(\Gamma_K)$, say $S_1(q, p, \xi)$, can be obtained extending periodically (in the q variables) $s([q], p, \xi)$. $\square$

4.3 A generating function for $h(\Gamma_K)$

In this section we build (see Lemma 4.2 below) a generating function for the linear symplectomorphism ψ. Combining it with the one above (see Proposition 4.1), we will state the existence of a generating function for $h(\Gamma_K)$ (see Proposition 4.3).

The following composition rule is popular in symplectic geometry and mechanics, see e.g. [2], [3], and it has been handled by Laudenbach and Sikorav in meaningful problems in symplectic topology (see [18]).

[2]We note that, unlike the map $\bar{h}$, it does not exist a natural definition of $\widetilde{h}$ from $T^*\mathbb{T}^n \times T^*\mathbb{T}^n$ in $T^*(\mathbb{T}^n \times \mathbb{R}^n)$, since it is essential the property: $(\phi_K^1)_q(q + 2\pi k, p) = (\phi_K^1)_q(q,p) + 2\pi k$.

Lemma 4.2 *The linear symplectomorphism ψ –see (10)– admits the generating function $S_2(x_0, x_1)$:*

$$S_2(x_0,x_1) = \frac{1}{2}\left\langle x_0, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} x_0 \right\rangle - \left\langle x_0, \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} x_1 \right\rangle +$$

$$+\frac{1}{2}\left\langle x_1, \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix} x_1 \right\rangle .$$

(See also [17], p. 280).

Proof. Recalling the map ψ in (10), we proceed to verify by direct computation:

$$-\frac{\partial S_2}{\partial x_0}(x_0,x_1)\big|_{x_0=(q,p),\, x_1=(\frac{2q-P}{2},\frac{2p-Q}{2})} = -\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \frac{2q-P}{2} \\ \frac{2p-Q}{2} \end{pmatrix} =$$

$$= (2p, 2q) + (Q-2p, P-2q) = (Q,P) = y_0,$$

$$\frac{\partial S_2}{\partial x_1}(x_0,x_1)\big|_{x_0=(q,p),\, x_1=(\frac{2q-P}{2},\frac{2p-Q}{2})} = -(q,p)\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \frac{2q-P}{2} \\ \frac{2p-Q}{2} \end{pmatrix} =$$

$$= (2p, 2q) + (Q-2p, P-2q+\frac{2p-Q}{2}) = (Q, \frac{2P+2p-Q}{2}) = y_1. \ \square$$

We can now prove the following

Proposition 4.3 *The Lagrangian submanifold $h(\Gamma_K)$ admits the generating function $S(x_1; x_0, \xi)$:*

$$S(x_1;x_0,\xi) = S_1(x_0,\xi) + S_2(x_0,x_1).$$

Remark. Note that the variables x_0 now are interpreted as auxiliary parameters, at the same level of ξ.

Proof. The symplectomorphism ψ is generated by $S_2(x_0, x_1)$, that is

$$\psi(x_0,y_0) = (x_1,y_1) \text{ iff } \begin{cases} y_0 = -\frac{\partial S_2}{\partial x_0}(x_0,x_1) \\ y_1 = \frac{\partial S_2}{\partial x_1}(x_0,x_1) \end{cases}$$

$\dfrac{\partial S}{\partial x_0}(x_1;x_0,\xi) = 0$ means $\dfrac{\partial S_1}{\partial x_0}(x_0,\xi) + \dfrac{\partial S_2}{\partial x_0}(x_0,x_1) = 0$, that is, $y_0 = \dfrac{\partial S_1}{\partial x_0}(x_0,\xi)$.

Furthermore,

$$\frac{\partial S}{\partial \xi}(x_1;x_0,\xi) = 0 \quad \text{iff} \quad \frac{\partial S_1}{\partial \xi}(x_0,\xi) = 0.$$

Therefore the Lagrangian submanifold generated by $S(x_1; x_0, \xi)$ results

$$\left\{(x_1, y_1) = (x_1, \frac{\partial S}{\partial x_1}(x_1; x_0, \xi)) : \ \frac{\partial S}{\partial x_0}(x_1; x_0, \xi) = 0 \ , \ \frac{\partial S}{\partial \xi}(x_1; x_0, \xi) = 0\right\} =$$

$$= \left\{(x_1, y_1) = (x_1, \frac{\partial S}{\partial x_1}(x_1; x_0, \xi)) : \ y_0 = \frac{\partial S_1}{\partial x_0}(x_0, \xi) \ , \ \frac{\partial S_1}{\partial \xi}(x_0, \xi) = 0\right\} =$$

$$= \left\{(x_1, y_1) = (x_1, \frac{\partial S_2}{\partial x_1}(x_0, x_1)) : \ y_0 = \frac{\partial S_1}{\partial x_0}(x_0, \xi) \ , \ \frac{\partial S_1}{\partial \xi}(x_0, \xi) = 0\right\} =$$

$$= \left\{(x_1, y_1) : \ (x_1, y_1) = \psi(x_0, y_0) \text{ with } (x_0, y_0) \in \bar{h}(\Gamma_K)\right\} = \psi(\bar{h}(\Gamma_K)) = h(\Gamma_K). \ \square$$

4.4 The Quadratic at Infinity property

We are ready to look for fixed points of ϕ^1_H, that is, to estimate

$$\# \ (h(\Gamma_K) \cap 0_{\mathbb{R}^{2n}}).$$

These intersection points are exactly the critical points, with respect *all* the variables, of the generating function S for $h(\Gamma_K)$. More precisely, by the Proposition 4.4 below, we show that they are essentially (that is to say, up to periodicity) the critical points for a GFQI f defined on a domain contracting to the torus $\mathbb{T}^n$: this is crucial in order to gain, in the Lusternik-Schnirelman format, a lower bound estimate of the number of fixed points of ϕ^1_H.
Although in the previous Section we managed with a formal expression of S_2, by a straightforward computation we easily find out the simplified structure[3] of it:

$$S_2(x_0, x_1) = S_2(q_0, p_0, q_1, p_1) = 2(p_0 - p_1) \cdot (q_1 - q_0) + \frac{p_1^2}{2}.$$

Proposition 4.4 *The fixed points of ϕ^1_H correspond to the critical points of the GFQI*

$$f : \mathbb{T}^n \times \mathbb{R}^{3n+k} \longrightarrow \mathbb{R}$$

$$([q_1], p_1, v, p_0, \xi) \xrightarrow{f} S_1([q_1 - v], p_0 + p_1, \xi) + 2p_0 \cdot v + \frac{p_1^2}{2} \qquad (11)$$

Proof. Using the notation $x_1 = (q_1, p_1)$ and $x_0 = (q_0, p_0)$ we can rewrite S as

$$S : \mathbb{R}^{4n+k} \longrightarrow \mathbb{R}$$

[3] here, for opportunity, we write $S_2(q_0, p_0, ...)$ instead of $S_2(q, p, ...)$

$$(q_1,p_1,q_0,p_0,\xi) \longmapsto S_1(q_0,p_0,\xi) + 2(p_0-p_1)\cdot(q_1-q_0) + \frac{p_1^2}{2}.$$

There is an evident invariance property:

$$S(q_1+2\pi k, p_1, q_0+2\pi k, p_0, \xi) = S(q_1,p_1;q_0,p_0,\xi)$$

$\forall (q_1,p_1,q_0,p_0,\xi) \in \mathbb{R}^{4n+k}$ and $\forall k \in \mathbb{Z}^n$. This fact is the same as saying that S is constant over the fibers of the surjective map Π below, thus it results well-defined the following real-valued function $\widetilde{S}$:

$$\begin{tikzcd} \mathbb{R}^{4n+k} \arrow[r, "S"] \arrow[d, "\Pi"'] & \mathbb{R} \\ \mathbb{T}^n \times \mathbb{R}^{3n+k} \arrow[ur, "\widetilde{S}"'] & \end{tikzcd} \tag{12}$$

$$\begin{tikzcd} (q_1,p_1,q_0,p_0,\xi) \arrow[r, "S"] \arrow[d, "\Pi"'] & \mathbb{R} \\ ([q_1],p_1,q_1-q_0,p_0,\xi) \arrow[ur, "\widetilde{S}"'] & \end{tikzcd} \tag{13}$$

$$\Pi^{-1}([q_1],p_1,v,p_0,\xi) = \{(q_1+2\pi k, p_1, q_1-v+2\pi k, p_1, \xi) : \; k \in \mathbb{Z}^n\} \tag{14}$$

$$\widetilde{S} : \mathbb{T}^n \times \mathbb{R}^{3n+k} \longrightarrow \mathbb{R}$$

$$([q_1],p_1,v,p_0,\xi) \longmapsto S_1([q_1-v],p_0,\xi) + 2(p_0-p_1)\cdot v + \frac{p_1^2}{2} \tag{15}$$

satisfying the property:

$$\widetilde{S} \circ \Pi = S \tag{16}$$

Furthermore, since $d\widetilde{S}(y)_{|_{y=\Pi(x)}} \circ d\Pi(x) = dS(x)$, we have that ($rk\ d\Pi =$ max): $\Pi^{-1}(\text{Crit } (\widetilde{S})) = \text{Crit } (S)$. Now $S_1([q_1-v],p_0+p_1,\xi)$ coincides for $|\xi| > C$ with a nondegenerate quadratic form $(A\xi,\xi)$, then for $|p_1|$, $|v|$, $|p_0|$, $|\xi| > C$ and for any fixed $[q_1] \in \mathbb{T}^n$, $f([q_1],p_1,v,p_0,\xi) = Q(p_1,v,p_0,\xi)$ where $Q(p_1,v,p_0,\xi)$ is the nondegenerate quadratic form

$$Q(p_1,v,p_0,\xi) := \begin{pmatrix} \frac{1}{2} & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & A \end{pmatrix} \begin{pmatrix} p_1 \\ v \\ p_0 \\ \xi \end{pmatrix} \begin{pmatrix} p_1 \\ v \\ p_0 \\ \xi \end{pmatrix}.$$

Therefore f is a GFQI. $\square$

4.5 Fixed points: Degenerate case

We conclude this Section with the estimate in the possible degenerate case, first proved by Conley and Zehnder [10], of which we propose a proof based on the Quadratic at Infinity property of the generating function f.

Theorem 4.5 *Let ϕ_H^1 be the time-one map of a time-dependent Hamiltonian $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ satisfying*

$$H(t, q + 2\pi k, p) = H(t, q, p), \quad \forall (t, q, p) \in \mathbb{R} \times \mathbb{R}^{2n},\ \forall k \in \mathbb{Z}^n,$$

and

$$H(t, q, p) = \frac{1}{2}|p|^2 \quad \textit{if} \quad |p| \geq C > 0.$$

Then ϕ_H^1 has at least $n+1$ fixed points and they correspond to homotopically trivial closed orbits of the Hamiltonian flow.

Proof. Fixed points of ϕ_H^1 correspond to critical points of f (see Proposition 4.4). Moreover, via the Lusternik-Schnirelman theory (see Theorem 2.9), critical values of f can be detected involving non-vanishing relative cohomology classes in $H^*(f^c, f^{-c})$. As a consequence, and since $f : \mathbb{T}^n \times \mathbb{R}^{3n+k} \to \mathbb{R}$ is a GFQI, Corollary 2.11 does work, so that we obtain the well-known estimate:

$$\#\ \mathrm{fix}(\phi_H^1) = \#\ \mathrm{crit}(f) \geq \mathrm{cl}(\mathbb{T}^n) = n + 1. \ \square$$

5 Fixed points: Nondegenerate case

Whenever all the fixed points of ϕ_H^1 are *a priori* nondegenerate, so that the corresponding critical points of f are, it happens that the GFQI f becomes also a so-called *Morse function*, and in this case we caught a rather better estimate.

Definition 5.1 *Let N be a smooth manifold. A fixed point $x \in N$ of a diffeomorphism $\Phi : N \to N$ is said nondegenerate if the graph of Φ intersects the diagonal of $N \times N$ transversally at (x, x), that is,*

$$\det\big(d\Phi(x) - \mathbb{I}\big) \neq 0.$$

The notion of nondegeneracy for fixed points of diffeomorphisms corresponds to the notion of nondegeneracy for critical points of functions, originally due to Morse.

Definition 5.2 *Let N be a smooth manifold and $f : N \to \mathbb{R}$ be a C^2 function. A critical point x for f, $\nabla f(x) = 0$, is said nondegenerate if the Hessian $\frac{\partial^2 f}{\partial x^i \partial x^j}(x)$ of f at x is nondegenerate.*

(Recall that the Hessian of a scalar function f at its critical points is a well-defined tensorial object.) Starting from the study of the sublevel sets N^ν (see (2)), where ν is not a critical value of f, Morse proved the following famous lower bound on the number of critical points of f.

Theorem 5.3 *(Morse inequality) Let N be a compact manifold and $f : N \to \mathbb{R}$ be a Morse function. Then*

$$\operatorname{crit}(f) \geq \sum_{k=0}^{dimN} H^k(N) =: \sum_{k=0}^{dimN} b_k(N),$$

where the values $b_k(N)$ are called the Betti numbers of N.

As in the degenerate case, the preceding estimate still holds when $f : N \times \mathbb{R}^n \to \mathbb{R}$ is a GFQI (see for example [8]):

Theorem 5.4 *Let N be a compact manifold and $f : N \times \mathbb{R}^n \to \mathbb{R}$ be a GFQI. If all the critical points of f are nondegenerate, then*

$$\operatorname{crit}(f) \geq \sum_{k=0}^{dimN} b_k(N).$$

The expected estimate on the number of nondegenerate fixed points for the Hamiltonian flow ϕ_H^1 is a straight consequence of the above Theorem 5.4.

Theorem 5.5 *Same hypothesis of the Theorem 4.5. Then ϕ_H^1 has at least 2^n nondegenerate fixed points and they correspond to homotopically trivial closed orbits of the Hamiltonian flow.*

Proof. Nondegenerate fixed points of ϕ_H^1 correspond (via the diffeomorphisms h and ψ) to transversal intersections between $h(\Gamma_K)$ and $0_{\mathbb{R}^{2n}}$. We observe now that the Lagrangian submanifold $h(\Gamma_K)$ intersects transversally $0_{\mathbb{R}^{2n}}$ in the point $(\bar{q}, \bar{p}, \bar{u}) := (\bar{x}, \bar{u}) \in h(\Gamma_K)$ if

$$\det\Big(\frac{\partial^2 S}{\partial x^i \partial x^j}\Big)(\bar{x}, \bar{u}) \neq 0. \tag{17}$$

Moreover, since the point $(\bar{x}, \bar{u}) \in h(\Gamma_K)$, the transversality condition guarantees that

$$\mathrm{rk}\Big(\frac{\partial^2 S}{\partial x^i \partial u^j}, \frac{\partial^2 S}{\partial u^i \partial u^j}\Big)(\bar{x}, \bar{u}) = \max. \tag{18}$$

Then, from the conditions (17) and (18), we conclude that the nondegenerate fixed points of ϕ_H^1 correspond exactly to the nondegenerate critical points of S, which are essentially (that is up to periodicity) the nondegenerate critical points of f. Now $f : \mathbb{T}^n \times \mathbb{R}^{3n+k} \longrightarrow \mathbb{R}$ is a GFQI, then, as a consequence of Theorem 5.4, we obtain

$$\#\ \text{nondeg-fix}(\phi_H^1) = \#\ \text{nondeg-crit}(f) \geq 2^n.\ \square$$

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