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Some Global Features of Wave Propagation

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Abstract

After a discussion on two fundamental routes —weak discontinuity waves and high frequency asymptotic waves— both leading to Hamilton-Jacobi equation, we review two notions of weak solution for it, the minimax solution and the viscosity solution. We claim the coincidence of the two solutions for a general class of p-convex Hamiltonians of mechanical type and we sketch some technical details of the proof.

1 Introduction

The work reviewed in this paper is aimed at recollecting some fundamental routes to Hamilton-Jacobi equations, even outside the classical arena of analytical mechanics where this equation naturally arises.

We reconsider discontinuity and asymptotic wave propagation –see 1.1 and 1.2 below– leading us to Hamilton-Jacobi equation. For such an equation we recall two notions of weak solution: the *viscous* solution and the *minimax* solution. By concerning with the last one, a lot of examples can be found and produced, beyond classical mechanics, in this new topological background: e.g., in control theory [4], or in multi-time theory of H-J equations [8].

More precisely, we will investigate around H-J evolution equation:

$$\frac{\partial u}{\partial t} + H(t, q, \frac{\partial u}{\partial q}) = 0 \qquad u(0, q) = \sigma(q), \tag{1}$$

$t \in (0, T)$, $q \in N$, where N is a bounded domain of $\mathbb{R}^n$.
For T small enough, the Cauchy Problem (1) admits a unique classical solution and it is determined using characteristics method. However, even though the Hamiltonian H and the initial function σ are smooth, in general there exists a critical time in which the classical solution becomes multivalued, i.e. q-components of some characteristics cross each other. Hence, it comes out the requirement how to define, and then to construct weak solutions of (1), e.g. continuous and almost everywhere differentiable functions solving (1).

Before discussing the two announced types of weak solutions, we recall the concept of *geometric solution* of (1), which is a *Lagrangian* submanifold L –see 2– obtained by gluing together the characteristics of the Hamiltonian vector field $X_{\mathcal{H}}$, where $\mathcal{H}(t, q, \tau, p) = \tau + H(t, q, p)$. In general, Lagrangian submanifolds are described by their *generating functions*. The geometric solution L was intended to be a global object, showing, among other things, the multivalue features of the H-J problem. Multivalues, if any, produce in turn singularities, which were studied in past decades by theoretical physicists and, mainly, by mathematicians like Thom, Arnol'd and Mather, producing *singularity theory*[1], see [1].
From the one side, by means of several operations, recalled in Section 2, it was possible to make a theory of *local* classification of Lagrangian singularities.
From the other side, in the more recent *global* theory of generating functions – namely, *symplectic topology*– their use appears more extensive: up to these operations, uniqueness is reached for such mathematical tools, describing our involved Lagrangian submanifolds. Finally, for a general class of *p-convex Hamiltonians of mechanical type*: $H(q, p) = \frac{1}{2}|p|^2 + V(q)$, we claim the coincidence of the viscous and minimax solution and we sketch some technical details of the proof, see [3]. Viscous solution and minimax solution emerge from different, separate fields of mathematics: such a coincidence, which does work surely for physical Hamiltonians, seems to mark a sensible step towards the recognition of a robust good model of solution for wave propagation.

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[1]sometimes called catastrophe theory

1.1 Discontinuities

Let consider a general semi-linear evolution system of partial differential equations:

$$\frac{\partial u^i}{\partial t} + \sum_{L=1,\, j=1}^{d\, n} A_j^{i\,L}(t,q)\frac{\partial u^j}{\partial q^L} = b^i(t,q) \tag{2}$$

As is well known, it modelizes e.g. Maxwell equations or non-homogeneous linear elasticity. Weak discontinuities have support on propagating wave in the space-time $\mathbb{R}^{d+1}$ described by

$$\Phi(t, q^L) = 0 \qquad (\Phi : \mathbb{R}^{d+1} \to \mathbb{R}) \tag{3}$$

by denoting, as usual, $\left[\frac{\partial u^i}{\partial t}\right]$ and $\left[\frac{\partial u^i}{\partial q^L}\right]$ possible discontinuities of the derivatives of u^i through $\Phi = 0$, we obtain

$$\left[\frac{\partial u^i}{\partial t}\right] + \sum_{L=1,\, j=1}^{d\, n} A_j^{i\,L}(t,q)\left[\frac{\partial u^j}{\partial q^L}\right] = 0 \tag{4}$$

By recalling the Hugoniot-Hadamard compatibility conditions,

$$\left[\frac{\partial u^i}{\partial t}\right] = -\lambda^i v\,, \qquad \left[\frac{\partial u^i}{\partial q^L}\right] = -\lambda^i n_L\,, \tag{5}$$

where λ^i is the size of the jump, v the normal velocity of the wave, and n_L is the normal unit vector of $\Phi = 0$, in some more detail,

$$v = -\frac{\frac{\partial \Phi}{\partial t}}{\sqrt{\sum_{L=1}^{d}\left(\frac{\partial \Phi}{\partial q^L}\right)^2}}\,, \qquad n_L = \frac{\frac{\partial \Phi}{\partial q^L}}{\sqrt{\sum_{L=1}^{d}\left(\frac{\partial \Phi}{\partial q^L}\right)^2}}\,, \tag{6}$$

we write

$$\sum_{j=1}^{n}\left[\delta_j^i \frac{\partial \Phi}{\partial t} - \sum_{L=1}^{d} A_j^{i\,L}(t,q)\frac{\partial \Phi}{\partial q^L}\right]\lambda^j = 0\,, \tag{7}$$

hence non trivial solutions occur if

$$\det\left(\delta_j^i \frac{\partial \Phi}{\partial t} - \sum_{L=1}^{d} A_j^{i\,L}(t,q)\frac{\partial \Phi}{\partial q^L}\right) = 0 \tag{8}$$

A standard irreducible factorization of (8), like $\ldots \cdot \mathcal{F}_{\alpha-1} \cdot \mathcal{F}_\alpha \cdot \mathcal{F}_{\alpha+1} \cdot \ldots = 0$, produces Hamilton-Jacobi equations $\mathcal{F}_\alpha = 0$ for unknown functions Φ describing waves:

$$\mathcal{F}_\alpha\left(t, q, \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial q}\right) = 0. \tag{9}$$

For example, we ascertain in this framework *longitudinal* and *transversal* propagation waves of linear elasticity. And the universal suggestion for the reader is to make for the beautiful booklets by Levi Civita [14] and Boillat [5].

1.2 Asymptotics

We can discover another route to H-J equation belonging to the high frequency asymptotic approximation of semi-linear partial differential equations, like Schrödinger equation in quantum mechanics; for a particle of mass $m = 1$ in a field generated by the potential energy $V(t, q)$ (here $\varepsilon = h/2\pi$, the Planck constant, is the 'small' parameter):

$$i\varepsilon \frac{\partial \psi}{\partial t}(t, q) = -\frac{\varepsilon^2}{2} \Delta \psi(t, q) + V(t, q)\psi(t, q), \tag{10}$$

$t \in \mathbb{R},\ q \in \mathbb{R}^n$. Trying to solve (10) by a (highly) oscillating integral like

$$I(t, q; \varepsilon) = \int_{u \in U} b(t, q, u; \varepsilon) e^{\frac{i}{\varepsilon}\Phi(t,q,u)} du,$$

$(U \subset \mathbb{R}^k)$, which is a sort of superposition of oscillating functions, we produce, for some *amplitude* b and (real) *phase* Φ, independent of ε,

$$\int_{u \in U} \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla_q \Phi|^2 + V \right) b\, e^{\frac{i}{\varepsilon}\Phi(t,q,u)} du\; + O(\varepsilon) = 0$$

Non trivial amplitudes are admissible if the phase satisfies the H-J equation

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} |\nabla_q \Phi|^2 + V = 0 \tag{11}$$

which is exactly a H-J evolution equation like (1), related to the Hamiltonian of the connected classical model of the physical system: $H(t, q, p) = \frac{1}{2}|p|^2 + V$.

2 Geometric solutions and their generating functions

Let $\mathbb{R} \times N$ be the "space-time", $T^*(\mathbb{R} \times N) = \{(t, q, \tau, p)\}$ its cotangent bundle equipped with the standard symplectic 2-form $\omega = dp \wedge dq + d\tau \wedge dt$. Moreover, let $\mathcal{H}(t, q, \tau, p) = \tau + H(t, q, p)$ be the homogeneous Hamiltonian function related to the evolution type H-J problem (1):

$$\frac{\partial u}{\partial t} + H(t, q, \frac{\partial u}{\partial q}) = 0 \qquad u(0, q) = \sigma(q).$$

In other words, a function $u(t,q)$ is a classical solution if[2] $\mathrm{im}(du) \subset \mathcal{H}^{-1}(0)$, that is

$$(t, q, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial q}) \in \mathcal{H}^{-1}(0) \qquad \forall (t,q) \in (0,T) \times N.$$

We just recall that $\mathrm{im}(du)$ is an *exact* Lagrangian[3] submanifold of $T^*((0,T)\times N)$: a geometric solution L is nowadays defined as a Lagrangian submanifold –not required to be exact– which is still contained in the hypersurface $\mathcal{H}^{-1}(0)$:

$$L \subset \mathcal{H}^{-1}(0).$$

The following Theorem indicates how to construct such an L –see for example [6] and [9].

Theorem 2.1. *The geometric solution to (1) is the submanifold*

$$L := \bigcup_{0<t<T} \Phi^t(\Gamma_\sigma) \subset T^*(\mathbb{R} \times N), \tag{12}$$

where $\Phi^t : \mathbb{R} \times T^*(\mathbb{R}\times N) \to T^*(\mathbb{R}\times N)$ *is the flow generated by the Hamiltonian* $\mathcal{H}$ *and* Γ_σ *is the initial data submanifold*

$$\Gamma_\sigma := \{(0, q, -H(0, q, d\sigma(q)), d\sigma(q)) : \ q \in N\} \subset \mathcal{H}^{-1}(0) \subset T^*(\mathbb{R}\times N).$$

A classical Theorem by Maslov and Hörmander assures that, *at least locally*, Lagrangian submanifolds L are described by *generating functions* $f : (0,T) \times N \times \mathbb{R}^k \to \mathbb{R}$, $(t,q,\xi) \mapsto f(t,q,\xi)$, in the following way:

$$L := \left\{ \left(t, q, \frac{\partial f}{\partial t}(t,q,\xi), \frac{\partial f}{\partial q}(t,q,\xi) \right) : \ \frac{\partial f}{\partial \xi}(t,q,\xi) = 0 \right\},$$

where 0 is a regular value of the map

$$(t,q,\xi) \longmapsto \frac{\partial f}{\partial \xi}(t,q,\xi).$$

The direct use of generating functions in Calculus of Variations requires to search for conditions on L guaranteeing the existence and uniqueness of a *global* generating function for L. By concerning uniqueness, we remind that it can be stated up to three main operations on generating functions, which leave invariant the corresponding Lagrangian submanifolds:

[2] $\mathrm{im}(du)$ means the *image* of $du : (0,T) \times N \to T^*((0,T)\times N)$

[3] $L \subset T^*((0,T)\times N)$ is Lagrangian if (i) $\omega|_L = 0$, (ii) $\dim L = \dim((0,T)\times N)$

• *Fibered diffeomorphisms.* Let $f : (0,T) \times N \times \mathbb{R}^k \to \mathbb{R}$ be a generating function and $(0,T) \times N \times \mathbb{R}^k \ni (t,q,\eta) \mapsto (t,q,\xi(t,q,\eta)) \in N \times \mathbb{R}^k$ a map such that, $\forall (t,q) \in (0,T) \times N$,

$$\mathbb{R}^k \ni \eta \longmapsto \xi(t,q,\eta) \in \mathbb{R}^k$$

is a diffeomorphism. Then

$$f_1(t,q,\eta) := f(t,q,\xi(t,q,\eta))$$

generates the same Lagrangian submanifold of f.
• *Stabilization.* Let $f : (0,T) \times N \times \mathbb{R}^k \to \mathbb{R}$ be a generating function. Then

$$f_1(t,q,\xi,v) := f(t,q,\xi) + v^T B v,$$

where $v \in \mathbb{R}^l$ and $v^T B v$ is a nondegenerate quadratic form, generates the same Lagrangian submanifold of f.
• *Addition of a constant.* Finally, as a third –although trivial– invariant operation, we observe that by adding to a generating function f any arbitrary constant $c \in \mathbb{R}$ the described Lagrangian submanifold is invariant.

A fundamental step, which will allow us to construct the minimax solution of (1) is the following

Proposition 2.2. *The geometric solution $L \subset \mathcal{H}^{-1}(0)$ admits an unique global generating function $S : (0,T) \times N \times \mathbb{R}^k \to \mathbb{R}$, $(t,q,\xi) \mapsto S(t,q,\xi)$, which is quadratic at infinity (GFQI shortly): that is, for $|\xi| > C$*

$$S(t,q,\xi) = \xi^T Q \xi,$$

where $\xi^T Q \xi$ is a nondegenerate quadratic form.

This result, originally deduced for compact support Hamiltonians –see [17], [20], [18] and [19]–, has been extended to a more general class, like $H(q,p) = \frac{1}{2}|p|^2 + V(q)$, see [3].

3 Minimax solutions of H-J equations

Proposition 2.2 gives us a global object, the GFQI S, describing the geometric solution L. However, our aim is more ambitious: starting from S, the goal is the construction of weak solutions of (1). Note that, as a consequence of uniqueness[4] of S, this function results definitively linked *only* by the geometric solution L.

[4] i.e. up to the three operations

In 1991, Chaperon (see [9]) indicated how to utilize the existence and uniqueness result of Proposition 2.2 in order to construct a new type of weak solution for (1), called minimax solution. His definition is based on a special critical value (namely, the *minimax critical value*) of the function

$$\mathbb{R}^k \ni \xi \longmapsto S(t,q,\xi) \in \mathbb{R}, \tag{13}$$

$\forall (t,q) \in (0,T) \times N$ fixed.
In some more detail, let us consider the following sublevel sets related to S and Q ($S(t,q,\xi) = \xi^T Q\xi$, $|\xi| > C$):

$$S^c_{(t,q)} := \left\{ \xi \in \mathbb{R}^k : \ S(t,q;\xi) \leq c \right\}, \ (t,q) \in (0,T) \times N \text{ fixed},$$

$$Q^c := \left\{ \xi \in \mathbb{R}^k : \ Q(\xi) \leq c \right\}.$$

We observe that for $c > 0$ large enough, $S^c_{(t,q)}$ and Q^c are invariant from a homotopical point of view:

$$S^{\pm c}_{(t,q)} = Q^{\pm c},$$

and $S^{\pm \bar{c}}_{(t,q)}$ retracts on $S^{\pm c}_{(t,q)}$ for every $\bar{c} > c$. Let $A := Q^{(c-\epsilon)}$, $\epsilon > 0$ small. Then the isomorphisms below (the first one by excision and the second one by retraction) hold:

$$H^*\left(Q^c, Q^{-c}\right) \cong H^*\left(Q^c \setminus \mathring{A}, Q^{-c} \setminus \mathring{A}\right) \cong H^*\left(D^i, \partial D^i\right),$$

where i is the index of the quadratic form Q (that is, the number of negative eigenvalues of Q) and D^i denotes the disk (of radius $\sqrt{c}$) in $\mathbb{R}^i$. Hence $H^*\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ is 1-dimensional:

$$H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right) \cong H^h\left(D^i, \partial D^i\right) = \begin{cases} 0 & \text{if } h \neq i \\ \alpha \cdot \mathbb{R} & \text{if } h = i \end{cases} \tag{14}$$

where i is the Morse index of the quadratic form Q. Such a one-dimension cohomology forces us to select the unique critical value connected to α:

Definition 3.1. *(Minimax solution) Let $S(t,q,\xi)$ and $\xi^T Q\xi$ as above. For $c > 0$ large enough and for every $(t,q) \in (0,T) \times N$, let $0 \neq \alpha \in H^i\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ be the unique generator (up to a constant factor) as in (14) and*

$$i_\lambda : S^\lambda_{(t,q)} \hookrightarrow S^c_{(t,q)}.$$

The function

$$(t,q) \mapsto u(t,q) := \inf\left\{\lambda \in [-c,+c] : \ i^*_\lambda \alpha \neq 0\right\} \tag{15}$$

is the minimax solution of (1).

The following fundamental Theorem has been proved by Chaperon, see [9].

Theorem 3.2. *The minimax solution (15) is a weak solution to* (1)*, Lipschitz on finite times, which does not depend on the choice of the GFQI.*

4 Viscosity solutions of H-J equations

In this section we review some aspects of the basic theory of continuous viscosity solutions of the Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t} + H(t, q, \frac{\partial u}{\partial q}) = 0, \tag{16}$$

$t \in (0, T)$, $q \in N$. Special attention will be devoted later to the case where $H = H(p)$ and $p \mapsto H(p)$ is convex.

Definition 4.1. *A function $u \in C((0, T) \times N)$ is a viscosity subsolution [supersolution] of (16) if, for any $\phi \in C^1((0, T) \times N)$,*

$$\frac{\partial \phi}{\partial t}(\bar{t}, \bar{q}) + H(\bar{t}, \bar{q}, \frac{\partial \phi}{\partial q}(\bar{t}, \bar{q})) \leq 0 \qquad [\geq 0] \tag{17}$$

at any local maximum [minimum] point $(\bar{t}, \bar{q}) \in (0, T) \times N$ of $u - \phi$. Finally, u is a viscosity solution of (16) if it is simultaneously a viscosity sub- and supersolution.

The origin of the term "viscosity solution" is going back to the vanishing viscosity method:

$$-\varepsilon \Delta u_\varepsilon(q) + H(q, \frac{\partial u_\varepsilon}{\partial q}(q)) = 0, \qquad q \in N. \tag{18}$$

In this case, the Hamiltonian of the problem is given by

$$H_\varepsilon(q, p, M) = -\varepsilon \mathrm{tr}(M) + H(q, p),$$

converging in $C(N \times \mathbb{R}^n \times Sym_{n\times n})$ to $H(q, p)$. Giving a solution of (18), a natural question arises: if $\varepsilon \to 0$ does u_ε tends to a function u, solution (in some sense) of the limit equation $H(q, \frac{\partial u}{\partial q}(q)) = 0$?
The question is not so easy because the regularizing effect of the term $\varepsilon \Delta u_\varepsilon$ vanishes as $\varepsilon \to 0$ and we end up with an equation that has easily non regular solutions. The answer is that if $u_\varepsilon \to u$ uniformly on every compact sets, then u is a viscosity solution. This is actually the motivation for the terminology "viscosity solution", used in the original paper of Crandall and Lions [10].

Analogously for minimax solutions, existence and uniqueness theorems hold for viscous ones. Moreover, Bardi and Evans [2] directly constructed viscosity solutions for Liouville-integrable and convex Hamiltonians $H(p)$. Their representation of solutions is based on a Hopf's formula and on an inf-sup procedure on auxiliary parameters:

$$u_{visc}\left(t,q\right)=\inf_{\chi\in\mathbb{R}^n}\sup_{v\in\mathbb{R}^n}\left\{-H\left(v\right)t+\left(q-\chi\right)\cdot v+\sigma\left(\chi\right)\right\}. \tag{19}$$

The generating function involved in this representation formula,

$$S(t,q,\underbrace{(\chi,v)}_{\xi})=-H(v)t+\left(q-\chi\right)\cdot v+\sigma\left(\chi\right),$$

results quadratic at infinity under auxiliary hypothesis: for example, σ compact support and $H(p)=\frac{1}{2}|p|^2$.

The plan of construct viscosity solutions starting from generating functions has been rather fruitless; nevertheless, we can find similar representation formulas for state-dependent Hamiltonians, see [7] and [15], although they hold only under suitable restrictive assumptions.

5 Coincidence of minimax and viscosity solutions in the convex case

The two types of weak solutions for H-J equations, treated in previous Sections, result in general *different* –see [16]. However, in the p-convex case, the *coincidence* of viscous and minimax solutions has been guessed in [13] and largely considered to be true: in the paper [3], we have proved in detail this fact for *p-convex Hamiltonians of mechanical type*:

$$H(q,p)=\frac{1}{2}|p|^2+V(q)\in C^2(T^*\mathbb{R}^n;\mathbb{R}),$$

where V is compact support.

In our proof it is crucial the following representation of the weak solution $u:(0,T)\times\mathbb{R}^n\to\mathbb{R}$, $(t,q)\mapsto u(t,q)$, where we take $\left(\tilde{q}\left(\cdot\right),\tilde{p}\left(\cdot\right)\right)\in H^1\left(\left(0,T\right),T^*\mathbb{R}^n\right)$:

$$u\left(t,q\right):=\inf_{\substack{\tilde{q}\left(\cdot\right):\\ \tilde{q}:[0,t]\to\mathbb{R}^n\\ \tilde{q}\left(t\right)=q}}\ \sup_{\substack{\tilde{p}\left(\cdot\right):\\ \tilde{p}:[0,t]\to\mathbb{R}^n,\\ \tilde{p}(0)=\frac{\partial\sigma}{\partial q}(\tilde{q}(0))}}\left\{\sigma\left(\tilde{q}\left(0\right)\right)+\int_0^t\left(p\dot{q}-H\right)_{|_{(\tilde{q},\tilde{p})}}ds\right\}, \tag{20}$$

In fact the above formula (20) provides *both* the viscous *and* the minimax solution.

From the one hand, (20) is the Hamiltonian version of the Lax-Oleinik formula producing the viscosity solution à la Crandall-Evans-Lions, see [12], [11] and bibliography quoted therein.

From the other hand, Amann-Conley-Zehnder reduction does work for the Hamilton-Helmholtz functional involved in (20), producing a global generating function with a finite number of parameters. It turns out that such a function, under the p-convexity hypothesis, is quadratic at infinity with Morse index $i = 0$ (see (14): in other words, it is definitively positive defined for $|\xi| > C$, so that it admits global minimum. After the sup-procedure on the curves $\tilde{p}$ in (20) representing the Legendre transformation, the inf-procedure on the curves $\tilde{q}$ in (20) captures the above minimum, which is *exactly* the minimax critical value, proving that (20) is precisely the minimax solution proposed by Chaperon-Sikorav-Viterbo.

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