

# The experimental localization of Aubry–Mather sets using regularization techniques inspired by viscosity theory

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We provide a new method for the localization of Aubry–Mather sets in quasi-integrable two-dimensional twist maps. Inspired by viscosity theories, we introduce regularization techniques based on the new concept of “relative viscosity and friction,” which allows one to obtain regularized parametrizations of invariant sets with irrational rotation number. Such regularized parametrizations allow one to compute a curve in the phase-space that passes near the Aubry–Mather set, and an invariant measure whose density allows one to locate the gaps on the curve. We show applications to the “golden” cantorus of the standard map as well as to a more general case. © 2007 American Institute of Physics. [DOI: 10.1063/1.2756264]

## I. INTRODUCTION

Deterministic dynamical systems can have very different types of motions. In particular, in the quasi-integrable systems one can find invariant KAM (Kolmogorov–Arnold–Moser) tori supporting quasi-periodic motions related to strongly irrational frequencies,<sup>1–3</sup> as well as regions of chaotic motions related to the resonances of the system. The numerically computed phase portraits of symplectic maps reveal the presence of these structures. Another type of invariant sets with very peculiar topology are the so-called “cantori,” or Aubry–Mather sets,<sup>4,5</sup> which are not usually detected by standard numerical computations of phase portraits. Recently there has been a lot of research on the Aubry–Mather sets in connection with weak KAM theory,<sup>6,7</sup> which is based on the so-called viscosity solutions to Hamilton–Jacobi equations.<sup>8,9</sup> In this paper we define a new method for the localization of Aubry–Mather sets that is inspired by the techniques of viscosity theory. The novelty is the introduction of a sort of “relative friction” (the  $\nu_0$  term) and “relative viscosity” (the  $\nu_1$  term) in the Proposition 1 of Sec. II; then, by Proposition 2, we are able to perform choices  $(\nu_0, \nu_1)$  for the contraction term  $l(\nu_0, \nu_1) < 1$  involved in the iterative refinement procedure. It turns out that we obtain regularized parametrizations that allow one to compute a curve in the phase-space that passes near the Aubry–Mather set and an invariant measure whose density allows one to locate the gaps on the curve.

Since KAM theorem, a lot of work has been done to construct the invariant KAM tori of dynamical systems supporting quasi-periodic motions in quasi-integrable systems (see, for example, Refs. 10 and 11), with the aim to explain the stability properties of more complicated situations.<sup>12</sup> Another type of motion that is well characterized analytically is supported by the so-called Aubry–Mather sets, which support quasi-periodic motions related to a set of frequencies with irrational ratios. Of course, they include the KAM invariant tori, but also a more general type of invariant sets corresponding to those frequencies for which it does not exist a

KAM torus, such as non-diophantine frequencies or diophantine frequencies, but with a strong perturbation. The Aubry–Mather theory characterizes analytically these invariant sets. To fix ideas we consider quasi-integrable symplectic twist maps of the cylinder, i.e., maps  $\mathcal{C}(I, \varphi) = (I', \varphi')$  of the form

$$\varphi' = \text{mod}(\varphi + I, 2\pi), \quad I' = I + \varepsilon f(\varphi'), \quad (1)$$

where  $I$  is in an open interval  $B \subset \mathbb{R}$ ;  $\varphi \in \mathbb{S}^1$ ;  $f$  is analytic, periodic, and with zero average; and  $\varepsilon > 0$  is a parameter. We also denote by  $\tilde{\mathcal{C}}(I, \varphi) = (I', \varphi')$  the lift of the map  $\mathcal{C}$  to the set  $B \times \mathbb{R}$  defined by

$$\varphi' = \varphi + I, \quad I' = I + \varepsilon f(\varphi'). \quad (2)$$

A motion  $(I(t), \varphi(t)) = \mathcal{C}^t(I(0), \varphi(0))$ ,  $t \in \mathbb{Z}$  is characterized by the rotation number  $\alpha \in \mathbb{R}$  if, denoting with  $(I(t), \tilde{\varphi}(t)) = \tilde{\mathcal{C}}^t(I(0), \varphi(0))$ , it is

$$\lim_{t \rightarrow \infty} \frac{\tilde{\varphi}(t)}{t} = \alpha. \quad (3)$$

In the 1980s, it was proved<sup>4,13</sup> that for any  $\varepsilon$ , for any  $\alpha$  with  $\alpha/(2\pi)$  irrational, there exists either a curve or a Cantor set  $M_\alpha$ , which projects injectively on the circle  $I=0$ , which is invariant and supports only motions with rotation number  $\alpha$ . These sets are called in literature Aubry–Mather sets.

To continue our discussion, it is useful to recall that, for any irrational  $\alpha/(2\pi)$ , the proof of the existence of its Aubry–Mather set was presented in Ref. 4 by proving the existence of a monotone parametrization of  $\mathbb{S}^1$ :

$$U: \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad \vartheta \mapsto \varphi, \quad (4)$$

which conjugates the dynamics of (1) in  $M_\alpha \subseteq \mathbb{S}^1 \times \mathbb{R}$  to the constant twist of  $\alpha$ :

$$\vartheta \mapsto \vartheta + \alpha.$$

The equations defining (1) imply that such a conjugation  $U$ , when it exists, is the solution of the following equation:

$$U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ U = 0, \tag{5}$$

where  $U^+(\vartheta) = U(\vartheta + \alpha)$ ,  $U^-(\vartheta) = U(\vartheta - \alpha)$ , and the invariant set is:

$$M_\alpha = \{(I, \varphi) : \varphi = U(\vartheta), I = U(\vartheta + \alpha) - U(\vartheta), \vartheta \in S^1\}, \tag{6}$$

which defines the Aubry-Mather set related to the rotation number  $\alpha$ .

For any irrational  $\alpha/(2\pi)$ , Mather proved the existence of a parametrization  $U$ , having at most a countable set of discontinuities. If  $U$  turns out to be continuous, the set  $M_\alpha$  is an invariant curve, as it is the case of KAM curves. Instead, for any discontinuity of the function  $U$  there is a ‘‘hole’’ in the set  $M_\alpha$ . If  $U$  has indeed a countable set of discontinuities, the set  $M_\alpha$  is a Cantor set, and has been called ‘‘cantorus’’ (see Percival<sup>14</sup>).

This paper concerns the problem of the localization of these invariant sets, which are not found by standard computations of phase portraits of maps (1), mainly for two reasons. The first is related to the probability of choosing an initial condition near a cantorus: the relative measure of the rotation numbers related to cantori is 1, but the Hausdorff dimension of each cantorus is smaller than 1 [actually, MacKay proved that the Hausdorff dimension of hyperbolic cantori is zero (see Ref. 15)]. The second reason is related to the hyperbolicity of cantori: very small numerical errors likely push the numerically integrated orbit on the chaotic orbits related to the high order resonances whose rotation number approximate  $\alpha$ . Therefore, to localize the Aubry-Mather sets, one needs numerical methods that are specifically designed for this task.

Different successful methods in the localization of Aubry-Mather sets are based on the computation of the hyperbolic periodic orbits that approximate the cantorus. Among these, we quote the method used by MacKay<sup>16,17</sup> to compute with great accuracy the point of the hyperbolic periodic orbits approximating the cantorus that is on the so-called dominant symmetry line; the gradient method used by Aubry;<sup>5,18</sup> various versions of the Newtonian method (see, for example, Refs. 19 and 20).

Here we define a different approach that is based on a functional representation of the problem and on its regularization with friction and viscosity like terms. Percival<sup>14</sup> put the problem in a variational form, so that parametrization of cantori are the minima of a suitable functional. More precisely, as it was proved by Mather,<sup>4</sup> the minimum of:

$$I_\alpha[U] = \int_0^{2\pi} h(U(\vartheta), U^+(\vartheta)) d\vartheta, \tag{7}$$

where  $h(x_0, x_1) = (x_1 - x_0)^2/2 + \varepsilon F(x_1)$  ( $F' = f$ ), solves Eq. (5).

A variational approach to Aubry-Mather sets different from that of Mather is inspired by the so-called viscosity theory. It was Moser<sup>21,7,22</sup> who suggested to use viscosity methods in order to investigate the Aubry-Mather sets of symplectic twist maps. As is usual in viscosity theory, Moser suggested to introduce a term proportional to a viscosity constant  $\nu$  in the functional  $I_\alpha[U]$  to regularize the problem. Precisely, given the functional:<sup>22</sup>

$$I_\alpha^\nu[U] = \int_0^{2\pi} \left( -\nu \log_e \frac{\partial U}{\partial \vartheta} + h(U(\vartheta), U^+(\vartheta)) \right) d\vartheta.$$

Moser proved, by standard variational arguments, the existence of a regular solution  $U_\alpha^\nu$  of class  $C^2$  for any  $\alpha$ , and then proved that  $U_\alpha^\nu$  converged pointwise to the minima  $U_\alpha$  of  $I_\alpha$  at all points of continuity of  $U_\alpha$ .

Actually, the functional  $I_\alpha^\nu$  defined by Moser in Ref. 22 is not usual in viscosity theory, and the associated Euler-Lagrange equation is

$$-\nu \frac{U''}{U'^2} + U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ U = 0. \tag{8}$$

For the sake of simplicity, we refer to the more usual viscosity equation used in Ref. 21 (the special form of the viscosity term used in Ref. 22 does not play any role in this paper):

$$-\nu U'' + U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ U = 0. \tag{9}$$

In our case, with  $f$  analytic, it is possible to prove that the solutions to Eq. (9) are analytic as well.

We now analyze the problem from the point of view of dynamics, having in mind its application to the practical localization of Aubry-Mather sets. From a dynamical point of view the viscosity term smooths the small denominators. In fact, replacing  $U$  by  $\vartheta + u(\vartheta)$ , with  $u(\vartheta)$  periodic, we write Eq. (9) in the form

$$L_\nu u(\vartheta) = -\frac{\varepsilon}{2} f(\vartheta + u(\vartheta)),$$

where  $L_\nu$  is the linear operator defined by

$$L_\nu u = -\nu u'' + u - \frac{u^+ + u^-}{2}.$$

It turns out that  $L_\nu$  has the trigonometric exponentials  $e^{ik\vartheta}$  as a basis of eigenvectors in  $L^2[0, 2\pi]$  with eigenvalues  $(1 - \cos(k\alpha)) + \nu k^2$ , and therefore all eigenvalues are bounded from below by  $\nu > 0$ . In principle, this could help to define convergent practical methods to solve Eq. (9). However, the solutions to Eq. (9) are good approximations to solutions of Eq. (5) only if  $\nu$  is very small, so that the small denominators still complicate the numerical algorithms based on the inversion of operator  $L_\nu$ .

In this paper we define a new numerical procedure that produces good approximate localizations of the Aubry-Mather sets, with a number of points much bigger than the Lyapunov time.

The basis of the method is the refinement of an approximate solution  $U$  to Eq. (5) by means of corrections  $\delta u$ , which are approximate solutions of the equation

$$\begin{aligned} \nu_0 \delta u - \nu_1 \delta u'' + \delta u - \frac{\delta u^+ + \delta u^-}{2} \\ + U - \frac{U^+ + U^-}{2} + \frac{\varepsilon}{2} f \circ (U + \delta u) = 0. \end{aligned} \tag{10}$$

In such a way, the regularizing effect of ‘‘friction’’ (the  $\nu_0$  term) and ‘‘viscosity’’ (the  $\nu_1$  term) does not act on the com-

plete solution  $u$ , but only on the difference with respect to an approximate solution, so that we can approach the true solution with convenient friction and viscosity values,  $\nu_0$  and  $\nu_1$ , respectively, as will be established in Sec. II. This new idea of using relative friction and viscosity regularizing terms will turn out to be crucial in order to define good numerical algorithms, because it allows one to use regularization parameters  $\nu_0$  and  $\nu_1$  that are not very small, as in the usual viscosity theories.

We remark that in the usual viscosity approximations of Aubry-Mather sets, one first defines solutions of a modified equation such as (9), which is different from the true equation for a term proportional to the viscosity parameter  $\nu$ . One then approaches the true solution because the additional spurious term is reduced by taking the limit of  $\nu$  going to zero. With the relative viscosity procedure, Eq. (10) is still different from the true equation for terms proportional to the relative viscosity and friction parameters  $\nu_0$  and  $\nu_1$ , but the true solution is approached because the additional spurious term is reduced by the fact that the factors  $\delta u$  and  $\delta u''$  of  $\nu_0$  and  $\nu_1$  become very small as a result of an iterative procedure. It happens (see Proposition 2), that there are pairs of values  $\nu_0, \nu_1 > 0$  (depending on the rotation number  $\alpha$  and on  $\varepsilon$ ) that minimize the final error of this procedure. In this way, we will construct (see Secs. II and III) a function  $\tilde{U} = \vartheta + \tilde{u}(\vartheta)$  that is an approximate solution to Eq. (5).

How can such a function be used to represent the cantorus? Our answer is given by the effective construction of a *measure*. In some more detail, having used regularizing techniques, the function  $\tilde{U}$  is smooth, so that the true cantorus  $M_\alpha$  is located near the smooth curve:

$$\tilde{M}_\alpha = \{(I, \varphi) = (\tilde{U}(\vartheta + \alpha) - \tilde{U}(\vartheta), \tilde{U}(\vartheta)), \vartheta \in S^1\}.$$

Considering better approximations  $\tilde{U}$ , one can reduce the distance of the set  $M_\alpha$  from the set  $\tilde{M}_\alpha$ , but  $\tilde{M}_\alpha$  continues to be a smooth curve, so that it remains to consider the problem of the localization of the gaps of the cantorus on it. To overcome this problem, we compute a measure on  $S^1$ , which corresponds to the density of the points of the cantorus. Precisely, let us denote by  $U_*$  the exact solution of (5), and by  $B$  the set of points where  $U_*$  has finite derivative (which has measure 1 because  $U_*$  is monotone). We denote by  $\vartheta = \vartheta_*(\varphi)$  the inverse of  $U_*$  on  $B$  (which is well defined because  $U_*$  is strictly monotone). It is well defined the density function on  $S^1$ :

$$\rho_*(\varphi) = \begin{cases} \frac{1}{2\pi U'_*(\vartheta_*(\varphi))}, & \text{if } \varphi \in U_*(B), \\ 0, & \text{if } \varphi \notin U_*(B), \end{cases} \quad (11)$$

which represents the density of points of the cantorus. Let us remark that the measure  $\mu_*$  defined by the density  $\rho_*$  is an extension of the unique ergodic measure, which one can define on  $M_\alpha$ .

For any regularized approximation  $\tilde{U}$  to  $U_*$  we then define the density function on  $S^1$ :

$$\tilde{\rho}(\varphi) = \frac{1}{2\pi \tilde{U}'(\tilde{\vartheta}(\varphi))}, \quad \varphi \in S^1, \quad (12)$$

where  $\tilde{\vartheta}(\varphi)$  is the inverse of  $\tilde{U}(\vartheta)$ . It is easy to show that for any sequence of functions  $\tilde{U}$ , which converges pointwise almost everywhere to  $U_*$ , denoting by  $\tilde{\mu}$  the measure defined by  $\tilde{\rho}$ ,

$$\tilde{\mu} \rightarrow \mu_*$$

in the vague topology of measures. The approximate density function  $\tilde{\rho}(\varphi)$ , which can be computed from the knowledge of  $\tilde{U}$ , provides us the essential information about the approximate location of the gaps of the cantorus. The idea of describing the convergence to the Aubry-Mather set by means of a ‘regularized’ measure was suggested in Ref. 22. It is our contribution to use the information carried by the regularized measures  $\tilde{\mu}$  to localize the gaps on the cantorus.

The paper is organized as follows: Section II is dedicated to the set up of the problem and the discussion of the analytic basis of our method. Section III is devoted to the numerical implementation and its application to some significant symplectic maps.

## II. THE ANALYTIC BASIS OF THE METHOD

We will use the formalism of Fourier series. More precisely, we will look for solutions of (5) of the form

$$U(\vartheta) = \vartheta + u(\vartheta)$$

with  $u(\vartheta)$  periodic in  $[0, 2\pi]$ , so that it can be represented by its Fourier series:  $u = \sum_{k \in \mathbb{Z}} u_k e^{ik\vartheta}$ , for which we introduce the  $l^2$  norm:  $\|u\| = (\sum_{k \in \mathbb{Z}} |u_k|^2)^{1/2}$ .

Mather’s result guarantees that the solutions of (5) are functions with bounded variations, so that their  $l^2$  norm is convergent. We define also the operator  $F: l^2 \rightarrow l^2$ , such that

$$F[u](\vartheta) = u(\vartheta) - \frac{1}{2}(u^+(\vartheta) + u^-(\vartheta)) - \varepsilon f(\vartheta + u(\vartheta)).$$

Equation (5) can be written  $F[u]=0$ , and for any approximate solution  $u$  a convenient indicator of the distance from the true solution is  $\|F[u]\|$ . We remark that the error represented by  $\|F[u]\| > 0$  is sensitive to the error on all points of the cantorus.

Our goal is to define a sequence of functions  $u^j$  such that the quantities  $\|F[u^j]\|$  converge to a very small value. We will follow two strategies to be used in cascade: the first one is useful to pass from an initial guess  $u^0$ , which has a large error  $\|F[u^0]\|$ , to some  $u^j$ , which is near to the true solution, but the major refinement is then obtained by means of a second strategy.

**The preliminary steps.** The preliminary steps are useful to reduce the error of some initial guess  $u^0$ , and are inspired by the contraction lemma. We remark that the equation  $F[u]=0$  can be written as a fixed point equation:

$$u = \Gamma[u],$$

where  $\Gamma[u]$  is defined by

$$\Gamma[u](\vartheta) := \frac{1}{2}(u^+(\vartheta) + u^-(\vartheta) - \varepsilon f(\vartheta + u(\vartheta))). \tag{13}$$

If  $\Gamma$  were a contraction in  $l^2$ , then the sequence  $u^j = \Gamma^j[u^0]$  would converge to the fixed point. Unfortunately,  $\Gamma$  is not a contraction, so that we cannot find its fixed point by this strategy. Nevertheless, we expect that for a finite number of steps, it is

$$\|\Gamma^{j+1}[u^0]\| < \|\Gamma^j[u^0]\|.$$

In fact, using the Fourier representation of a function  $u$ , it is

$$\Gamma[u] = \sum_k u_k \cos(k\alpha) e^{ik\vartheta} + \mathcal{O}(\varepsilon),$$

and therefore the main effect of  $\Gamma$  is to reduce each Fourier harmonics of  $u$  by multiplying it by  $\cos(k\alpha)$ . As a matter of fact, by using any initial function  $u^{(0)}$ , the operator  $\Gamma$  reduces the Fourier components with  $\cos(k\alpha)$  very different from 1. After some applications of  $\Gamma$ , one obtains a function whose components are mainly the components such that  $\cos(k\alpha) \sim 1$ , and therefore successive applications of  $\Gamma$  will not reduce necessarily the error. This method is very effective when  $\alpha/(2\pi)$  is a strong irrational number and  $\varepsilon$  is small; i.e., for the KAM solutions. However, our main interest here is on the Aubry-Mather sets that are not KAM solutions, and in this case we can only use a small finite number of these steps.

**Iterative improvement.** The starting point of our iterative improvement is the approximate solution obtained by the previous method, which will be denoted from now on by  $u^{(0)}$ . By construction,  $u^{(0)}$  is analytic in some complex domain characterized by  $|\Im\varphi| \leq \sigma$  and is of order  $\varepsilon$ , so that we define  $\varepsilon\xi_0$  to be the initial error:

$$\max\{\|F[u^{(0)}]\|, \|F[u^{(0)}]\|_{\sigma}\} = \varepsilon\xi_0,$$

where we denote by  $|\cdot|_{\sigma}$  the sup-norm of a periodic function on the complex domain  $|\Im\varphi| \leq \sigma$ .

We now define a finite sequence of functions  $u^{(k)}$ ,  $k \leq K$ , which improve the solution to the equation  $F[u]=0$ , defined by  $u^{(k)} = u^{(k-1)} + \delta u^{(k)}$ , where  $\delta u$  satisfies ( $u$  denotes  $u^{(k-1)}$  and  $\delta u$  denotes  $\delta u^{(k)}$ ):

$$\begin{aligned} \nu_0 \delta u - \nu_1 \delta u'' + \delta u - \frac{\delta u^+ + \delta u^-}{2} \\ + u - \frac{u^+ + u^-}{2} + \frac{\varepsilon}{2} f(\vartheta + u) = 0. \end{aligned} \tag{14}$$

Let us present the following proposition.

**Proposition 1.** *Let  $u$  be an approximate solution to equation  $F[u]=0$  such that the average of  $F[u]$  is zero. For any  $\nu_0, \nu_1 \geq 0$ ,  $(\nu_0, \nu_1) \neq (0, 0)$ , let  $\delta u$  be the solution of:*

$$\begin{aligned} \nu_0 \delta u - \nu_1 \delta u'' + \delta u - \frac{\delta u^+ + \delta u^-}{2} + u - \frac{u^+ + u^-}{2} + \frac{\varepsilon}{2} f(\vartheta + u) \\ = 0. \end{aligned} \tag{15}$$

We can then say

$$\begin{aligned} F[u + \delta u] = \sum_{n \in \mathbb{Z} \setminus 0} \frac{(\nu_0 + \nu_1 n^2) F[u]_n e^{in\vartheta}}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} + \frac{\varepsilon}{2} \left( f(\vartheta + u(\vartheta)) \right. \\ \left. - \sum_{n \in \mathbb{Z} \setminus 0} \frac{F[u]_n e^{in\vartheta}}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} \right) - f(\vartheta + u(\vartheta)). \end{aligned} \tag{16}$$

Denote by  $\sigma$  the analyticity radius of  $u$  and let  $\xi$  be defined by:  $\max\{|u|_{\sigma}, \|F[u]\|_{\sigma}\} = \varepsilon\xi$ . Then, for any  $\delta \geq |u|_{\sigma} + |\delta u|_{\sigma}$  (which is in particular satisfied by  $\delta = 2\varepsilon\xi/\sigma$ ),

$$\frac{|F[u + \delta u]|_{\sigma-\delta}}{|F[u]|_{\sigma}} \leq \sum_{n \in \mathbb{Z} \setminus 0} \frac{\nu_0 + \nu_1 n^2 + \frac{\varepsilon}{2} \lambda(\sigma)}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta}, \tag{17}$$

where  $\lambda(\sigma)$  is a Lipschitz constant of  $f$  in the complex strip of radius  $\sigma$ . If  $f$  and  $u$  are odd with respect to some value  $\varphi_c$ , then  $\delta u$  is odd with respect to  $\varphi_c$  and the average of  $F[u + \delta u]$  is zero.

The proof of this proposition, based on standard tools, is given at the end of the section. Its importance is mainly how it can be used to improve approximations of cantori.

At first approximation, Eq. (16) can be written as

$$\begin{aligned} F[u + \delta u] = \sum_{n \in \mathbb{Z} \setminus 0} \frac{(\nu_0 + \nu_1 n^2) F[u]_n e^{in\vartheta}}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} \\ + \varepsilon \mathcal{O} \left( \|f'\| \left\| \sum_{n \in \mathbb{Z} \setminus 0} \frac{F[u]_n e^{in\vartheta}}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} \right\| \right). \end{aligned} \tag{18}$$

The second term is small (for example, in the  $l^2$  norm) if  $\nu_0$  and  $\nu_1$  are not too small, while the first term is small if  $\nu_0, \nu_1$  are sufficiently small. Therefore, one expects that there are values of  $\nu_0, \nu_1$  that produce optimal reduction of the norm of  $F[u + \delta u]$  with respect to the norm of  $F[u]$ . We still remark that, because of (18), one expects that if  $\nu_0, \nu_1$  are both very small, then the norm of  $F[u + \delta u]$  is not smaller than the norm of  $F[u]$ . In the following proposition, we provide precise estimates of this reduction, for which we use analytic norms.

**Proposition 2.** *Let  $\alpha/(2\pi) \in \mathbb{R} \setminus \mathbb{Q}$  and  $\delta > 0$  such that*

$$\sum_{n \in \mathbb{Z} \setminus 0} \frac{n^2}{1 - \cos n\alpha} e^{-|n|\delta} < \infty. \tag{19}$$

Let  $\lambda > 0$  and  $\tilde{l} \in (0, 1)$ . There exists a positive constant  $c(\alpha, \delta, \tilde{l})$  which is increasing with respect to  $\delta, \tilde{l}$  such that, for any  $\nu_0, \nu_1 \geq 0$  with  $\|(\nu_0, \nu_1)\| \leq c(\alpha, \delta, \tilde{l})$ ,

$$l(\nu_0, \nu_1, \delta) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{\nu_0 + \nu_1 n^2 + \frac{\varepsilon}{2} \lambda(\sigma)}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta} \leq \tilde{l} \tag{20}$$

provided that  $\varepsilon$  satisfies:

$$0 \leq \varepsilon \leq \frac{\tilde{l}}{2\lambda_{n \geq 1}} \inf\{\nu_0 + \nu_1 n^2 + (1 - \cos(n\alpha))\} (e^{\delta} - 1). \tag{21}$$

The proof of this proposition is given at the end of the section.

**Remark 1.** Proposition 2 provides estimates for the

right-hand sides of inequality (17), and therefore it provides sufficient conditions for  $|F[u + \delta u]|_{\sigma-\delta}/|F[u]|_{\sigma} \leq \tilde{l} < 1$ . However, as it frequently happens with upper bounds in perturbation theory, it is inefficient, so that the practical implementations of the method work much more efficiently than predicted by (17) and therefore (21).

**Remark 2.** Inequality (21) does not mean that  $\varepsilon$  is smaller than a quantity proportional to  $\nu_0$  and  $\nu_1$ , because the term

$$\inf_{n \geq 1} \{ \nu_0 + \nu_1 n^2 + (1 - \cos(n\alpha)) \}$$

depends on the distribution of the small divisors  $1 - \cos(n\alpha)$  with respect to  $\nu_0 + \nu_1 n^2$ , and can be definitely bigger than  $\nu_0, \nu_1$ . For example, if  $\alpha$  is the rotation number which we will use in Sec. III A,  $\inf\{10^{-2}n^2 + (1 - \cos(n\alpha))\} \sim 0.4$ .

**Remark 3.** The proposition is interesting even in the limit case of very small  $\varepsilon$ . In fact, condition (19) on the rotation number  $\alpha$  is weaker than the diophantine condition of the KAM theorem. Therefore, it is meaningful to look for cantori related to nondiophantine  $\alpha$  and small  $\varepsilon$  (example in Sec. III B).

**Remark 4.** In usual situations (see, for example, Fig. 2, left panel), if one sets both regularizing parameters  $\nu_0, \nu_1 = 0$ , the iterative procedure immediately fails because, very soon, one encounters values  $l(0, 0, \delta) > 1$ .

**Remark 5.** For fixed  $\varepsilon$ , the hypotheses of Proposition 2 are satisfied if  $\nu_0$  and  $\nu_1$  are not too big [in fact, one needs  $\|(\nu_0, \nu_1)\| \leq c(\alpha, \delta, \tilde{l})$ ], but also if  $\nu_0, \nu_1$ , and  $\delta$  are not too small, so that they satisfy (21).

The iterative procedure that we use to construct refined approximations of the cantorus works as follows. We construct a finite sequence of functions  $u^{(k)}, k \leq K$ , which improve the initial approximate solution, with  $u^{(k)} = u + \delta u$  and  $\delta u$  satisfies ( $u$  denotes  $u^{(k-1)}$ , and  $\delta u$  denotes  $\delta u^{(k)}$ ) Eq. (15). We also require that the average of  $F[u^{(k)}]$  is zero at any  $k$ , which, by Proposition 1, is satisfied in the case  $f$  is odd with respect to some point  $\varphi_c$  and  $u^0$  is odd with respect to the same point. Therefore, the procedure begins with a function  $u^{(0)}$ , which is odd with respect to  $\varphi_c$  and is analytic in a big complex strip  $\sigma$ . To start the iteration, we need to choose a reduction of the analyticity domain  $\delta$  such that  $\sigma > \delta > |u|_{\sigma} + |\delta u|_{\sigma}$ , and moreover,

$$\frac{|F[u^{(1)}]|_{\sigma-\delta}}{|F[u^{(0)}]|_{\sigma}} < 1.$$

Proposition 2 provides sufficient conditions for this inequality: for  $\sigma$  (and therefore also  $\delta$ ) of order 1 one can easily choose  $\nu_0$  and  $\nu_1$  that satisfy the hypotheses of Proposition 2.

The iteration of the procedure requires a reduction of the analyticity radius at each step, so that at a given point  $\delta$  is so small that we are no longer able to find  $\nu_0$  and  $\nu_1$  that allow one to continue the approximating procedure, which is stopped at this point.

In conclusion, Eq. (17) and Proposition 2 provide a framework to explain why regularization allows one to construct improved approximations of cantori, but, as it frequently happens with upper bounds in perturbation theory, it

is inefficient, so that the practical implementations of the method work much more efficiently. An example is reproduced in Fig. 2 (left panel).

**Numerical construction of  $u(\vartheta)$ .** To implement the method numerically, we use the discrete Fourier formalism. Precisely, we fix an integer  $N$  as large as possible (depending on the real machine possibilities) and we find an approximate solution to the equation  $F[u] = 0$  in the form

$$u = \sum_{|k| \leq N} u_k e^{ik\vartheta}.$$

Denoting by  $\Pi_N$  the projector defined by  $\Pi_N \sum_{k \in \mathbb{Z}} v_k e^{ik\vartheta} = \sum_{|k| \leq N} v_k e^{ik\vartheta}$ , we find approximate solutions to the equation

$$\sum_{|k| \leq N} u_k (1 - \cos(k\alpha)) e^{ik\vartheta} - \frac{\varepsilon}{2} \Pi_N f \left( \vartheta + \sum_{|k| \leq N} u_k e^{ik\vartheta} \right) = 0, \tag{22}$$

following the strategy described above. Precisely, we preliminary find a raw approximation for the solution  $u$  as

$$u = (\Pi_N \Gamma)^j [u_*],$$

where  $u_*(\vartheta) = 0$  for any  $\vartheta \in S^1$ , and  $j$  is such that  $\|\Pi_N F(\Pi_N \Gamma)^j [u_*]\| < \|\Pi_N F(\Pi_N \Gamma)^{j-1} [u_*]\| < \dots < \|\Pi_N F(\Pi_N \Gamma) \times [u_*]\|$ . We then construct the sequence of improved approximate solutions  $u^{(k)}$  with  $u^{(0)} = (\Pi_N \Gamma)^j [u_*]$  and  $u^{(k)} = u^{(k-1)} + \delta u^{(k)}$ , where  $\delta u$  satisfies the regularized equation ( $u$  denotes  $u^{(k-1)}$  and  $\delta u$  denotes  $\delta u^{(k)}$ ):

$$\begin{aligned} \nu_0 \delta u - \nu_1 \delta u'' + \delta u - \frac{\delta u^+ + \delta u^-}{2} + u - \frac{u^+ + u^-}{2} \\ + \frac{\varepsilon}{2} \Pi_N f(\vartheta + u) = 0, \end{aligned} \tag{23}$$

with  $\nu_0$  and  $\nu_1$  chosen in such a way to obtain the best result; i.e., the smaller possible  $\min_k \|\Pi_N F[u^{(k)}]\|$ . Denoting with  $\tilde{k}$  the integer on which such a minimum is obtained, we use  $\tilde{u} = u^{(\tilde{k})}$  as best approximation of the cantorus.

**Numerical construction of the cantorus.** The solution  $\tilde{u}$  provides the approximate location of the following  $N$  points of the cantorus:

$$\begin{aligned} \varphi_j = \text{mod}(\vartheta_j + \tilde{u}(\vartheta_j), 2\pi), \quad I_j = \alpha + \tilde{u}_+(\vartheta_j) - \tilde{u}(\vartheta_j), \\ \vartheta_j = \frac{j}{N} 2\pi, \quad j = 1, \dots, N. \end{aligned} \tag{24}$$

However, a small number of these points should not be considered. In fact, the true parametrization  $\vartheta + \tilde{u}(\vartheta)$  of a cantorus is a monotone discontinuous function, while any regularized approximation of  $u$  is smooth. Therefore, if  $\vartheta_d$  is a discontinuity point of the true solution  $\vartheta + \tilde{u}(\vartheta)$ , some of the points  $\varphi_j$  can be in the forbidden interval:

$$\varphi_j \in \left( \lim_{\vartheta \rightarrow \vartheta_d^-} (\vartheta + \tilde{u}(\vartheta)), \lim_{\vartheta \rightarrow \vartheta_d^+} (\vartheta + \tilde{u}(\vartheta)) \right).$$

These points can be recognized and discharged by computing the density function  $\tilde{\rho}$  defined in (12). More generally, as already explained in Sec. I, the localization of the gaps on

the cantorus can be done by computing  $\tilde{\rho}$ , which we approximate by

$$\tilde{\rho}(\varphi_j) = \frac{1}{N}(\min\{\varphi_j - \varphi_{j-1}, \varphi_{j+1} - \varphi_j\})^{-1}. \tag{25}$$

The procedure that we will adopt is in fact the following:

- (i) We fix a density limit  $\rho$  and we eliminate all points  $(I_j, \varphi_j)$  such that  $\tilde{\rho}(\varphi_j) < \rho$ .
- (ii) The set of  $K$  points  $(I_j, \varphi_j)$ , which remains after this eliminating procedure is iterated using the map  $\mathcal{C}$ .
- (iii) We apply another selection procedure on the set of  $2K$  points obtained at step (ii) by eliminating all points that are separated from their first neighbors of more than  $\rho/\lambda$ , where  $\lambda > 1$  depends on the hyperbolicity properties of the cantorus.
- (iv) We repeat steps (ii) and (iii) as much as possible. The choice of the free parameters  $\rho$  and  $\lambda$  must be tuned in order to obtain the best stability of the set of points with respect to the iteration of the map. Applications of this technique are reported in Sec. III.

*Proof of Proposition 1.* We define the Fourier harmonics of  $\delta u$  by

$$\delta u_n = \frac{-[F(u)]_n}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha}, \quad \text{if } n \neq 0, \tag{26}$$

and  $\delta u_0 = 0$ . If  $f$  and  $u$  are odd functions (if  $\varphi_c \neq 0$ , one preliminarily translates  $\varphi$  so that  $\varphi_c = 0$ ); also  $F[u]$  is odd, and therefore one obtains  $\delta u_{-n} = -\delta u_n$  for any  $n$ , so that  $\delta u$  is odd as well.

Using (26), we can compute

$$\begin{aligned} F[u + \delta u] &= u - \frac{u^+ + u^-}{2} + \delta u - \frac{\delta u^+ + \delta u^-}{2} + \frac{\varepsilon}{2} f(\vartheta + u + \delta u) \\ &= F[u] + \sum_n \delta u_n (1 - \cos n\alpha) e^{in\vartheta} + \frac{\varepsilon}{2} (f(\vartheta + u + \delta u) - f(\vartheta + u)) \\ &= \sum_n \frac{\nu_0 + \nu_1 n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos(n\alpha)} F[u]_n e^{in\vartheta} + \frac{\varepsilon}{2} (f(\vartheta + u + \delta u) - f(\vartheta + u)), \end{aligned}$$

which proves (16). Because  $F[u]$  is analytic in a complex domain of radius  $\sigma$ , its Fourier harmonics are estimated by

$$|F[u]_n| \leq |F[u]|_\sigma e^{-|n|\sigma},$$

so that, for any positive  $\delta$ , it is

$$\begin{aligned} &\left| \sum_n \frac{\nu_0 + \nu_1 n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos(n\alpha)} F[u]_n e^{in\vartheta} \right|_{\sigma-\delta} \\ &\leq \sum_n \frac{\nu_0 + \nu_1 n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos(n\alpha)} e^{-|n|\delta} |F[u]|_\sigma, \end{aligned} \tag{27}$$

as well as

$$|\delta u|_{\sigma-\delta} \leq \sum_{n \neq 0} \frac{1}{\nu_0 + \nu_1 n^2 + 1 - \cos(n\alpha)} e^{-|n|\delta} |F[u]|_\sigma.$$

Because  $\delta > |u|_\sigma + |\delta u|_\sigma$ ,

$$|f(\vartheta + u + \delta u) - f(\vartheta + u)|_{\sigma-\delta} \leq \lambda(\sigma) |\delta u|_{\sigma-\delta}.$$

All these inequalities prove (17). □

*Proof of Proposition 2.* Equation (19) guarantees  $l(\nu_0, \nu_1, \delta) < \infty$  for any positive  $\delta$  and for any  $\nu_0, \nu_1 \geq 0$ . The series

$$l_1(\nu_0, \nu_1, \delta) = \sum_{n \in \mathbb{Z} \setminus 0} \frac{\nu_0 + \nu_1 n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta}$$

is then continuous in the domain  $\nu_0, \nu_1 \geq 0$ , including the point  $(\nu_0, \nu_1) = (0, 0)$ . In fact,  $l_1(0, 0, \delta) = 0$ , and because

$$\sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta},$$

$$\sum_{n \in \mathbb{Z} \setminus 0} \frac{n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta}$$

are convergent, the limit of

$$\begin{aligned} l_1 &= \nu_0 \sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta} \\ &\quad + \nu_1 \sum_{n \in \mathbb{Z} \setminus 0} \frac{n^2}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta}, \end{aligned} \tag{28}$$

is zero for  $\nu_0, \nu_1$ , which tends to  $(0, 0)$  in the domain of  $l$ . Continuity of  $l_1$  implies that there exists  $c(\alpha, \delta, \tilde{l})$  such that for any  $\|(\nu_0, \nu_1)\| \leq c(\alpha, \delta, \tilde{l})$ ,  $l_1(\nu_0, \nu_1, \delta) \leq \tilde{l}/2$ . Inequality (21) implies

$$\frac{\lambda}{2} \sum_{n \in \mathbb{Z} \setminus 0} \frac{1}{\nu_0 + \nu_1 n^2 + 1 - \cos n\alpha} e^{-|n|\delta} \leq \frac{\tilde{l}}{2},$$

so that the proof of the proposition is complete. □

### III. APPLICATIONS

#### A. The ‘‘golden’’ cantorus of the standard map

We consider the standard map defined by  $f(\varphi) = -\sin(\varphi)$ , and as it was done in Refs. 16 and 17, we consider the rotation number

$$\alpha = \frac{2\pi}{\omega^2}, \quad \omega = \frac{1 + \sqrt{5}}{2},$$

characterized by the fact that  $\alpha/(2\pi)$  is a diophantine number with the worst possible rational approximations, and the perturbation  $\varepsilon = \varepsilon_c + 0.01 \sim 0.981\,635\,406$ , where  $\varepsilon_c \sim 0.971\,635\,406$  is the critical value at which the Aubry-Mather set  $M_\alpha$  breaks into a discontinuous cantorus. To compute an approximation of the Aubry-Mather set, we use the method described in Sec. II. We first fix  $N = 2^8$  and look for the preliminary approximation. As explained in Sec. II, we set  $u^0 = 0$ , and then construct  $u^{(j)} = (\Pi_N \Gamma)^j u^{(0)}$ , where the operator  $\Gamma$  is defined in (13). In Fig. 1 we report the numerical computation of  $\|F[u^{(j)}]\|$ , using both the  $l^1$  norm and the  $l^2$

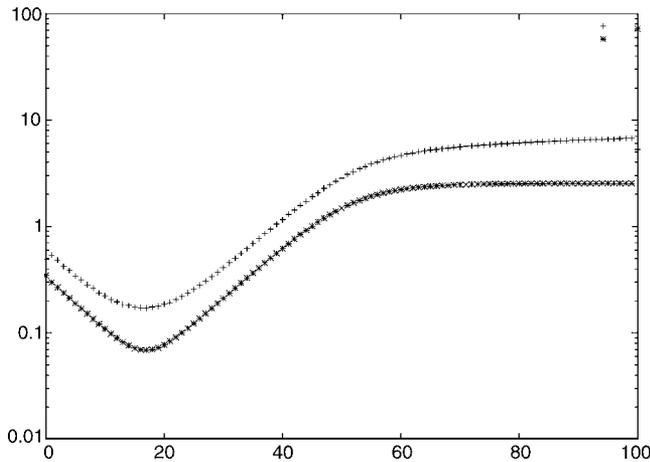


FIG. 1. Numerical computation of  $\|F[u^{(j)}]\|$  for the preliminary steps, using both the  $l^1$  norm (upper curve) and the  $l^2$  norm (lower curve). In both cases, the minimum value of  $\|F[u^{(j)}]\|$  is obtained for  $j=16$ .

norms. In both cases, the minimum value of  $\|F[u^{(j)}]\|$  is obtained for  $j=16$ , so that we choose  $u=u^{(16)}$  as the starting function for the iterative improvement.

The iterative improvement is then applied by using a set of values for  $\nu_0$  and  $\nu_1$ , from which we select the values which produce the best approximations of the cantor. Precisely, we applied  $k=1000$  iterative steps with a value of  $\nu_0 \in [0, 1]$  and  $\nu_1=0$  and represent the norm of  $\|\Pi_N F[u^{(k)}]\|$  versus  $\nu$ . The result is reported in Fig. 2 on the left: for values of  $\nu_0 < 0.065$  the iterative procedure fails, because it does not produce a reduction of  $\|\Pi_N F[u^{(k)}]\|$  with respect to  $\|\Pi_N F[u^{(0)}]\|$ . We remark that this range of values includes  $\nu_0=0$ , which corresponds to the case for which no regularizations are applied.

For  $\nu_0 \sim 0.068$ , there is a consistent reduction of  $\|\Pi_N F[u^{(k)}]\|$  with respect to  $\|\Pi_N F[u^{(0)}]\|$  of about seven orders of magnitude; for  $\nu_0 > 0.068$  there is still a reduction, but is less effective. Therefore, we selected  $\nu_0=0.068$  and continued the iteration until it was possible. In Fig. 2 on the right, we report the value of  $\|\Pi_N F[u^{(j)}]\|$  versus the number of iteration steps  $j$ . The norm is reduced up to order  $10^{-25}$  for  $j \sim 16\,000$ , and after this value the precision of our numerical computation evidently affects the result. Let us remark that the norm reported in Fig. 2 is a functional norm that measures the error of the parametrizations  $u^{(j)}$ , considering the Fourier harmonics up to order  $N$  (the error introduced by neglecting the Fourier harmonics with order larger than  $N$  is not estimated by this indicator, and in fact, in the applications, we find that a more suitable approximation requires  $N=2^{16}$ ). Therefore, we decided to take  $\tilde{u}=u^{(16\,000)}$  as the best approximation to the solution of equation  $\Pi_N F[u]=0$ , which allowed us to obtain 256 approximate points of the cantor. However, this cannot be considered a good representation of the cantor for two reasons: first of all, the 256 points are not enough to appreciate the presence of holes and accumulations, which are typical of the cantor structure; second, we need to repeat the computations with bigger values of  $N$  to test the stability of the result taking into account higher order Fourier representations.

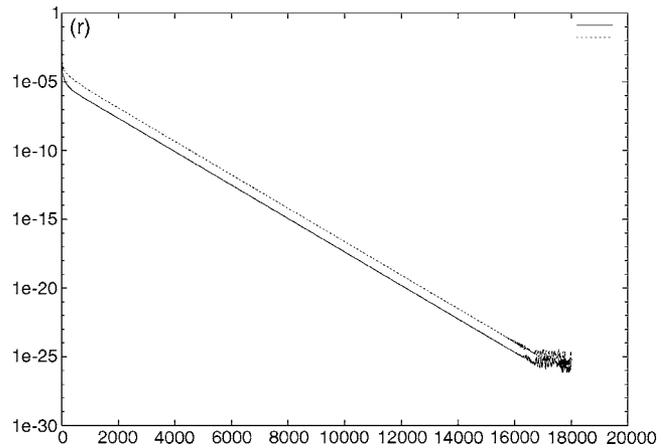
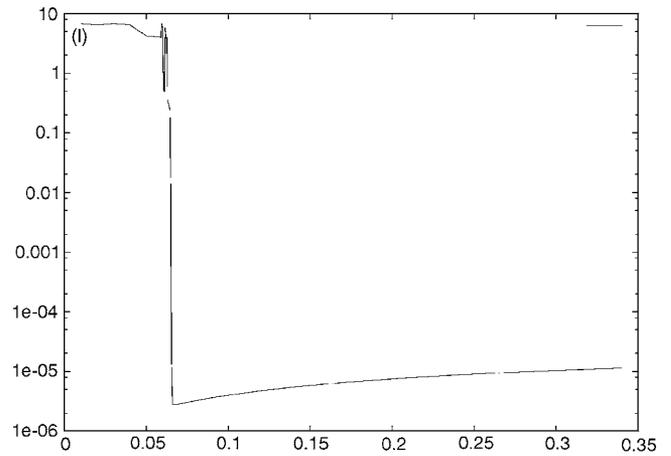


FIG. 2. On the left: we report the numerical computation of  $\|\Pi_N F[u^{(1000)}]\|$  for different values of  $\nu_0 \in [0, 1]$ . The minimum is found for  $\nu_0=0.068$ . On the right: numerical computation of  $\|\Pi_N F[u^{(k)}]\|$  (continuous line) and  $r$  (dashed line) versus  $k$ , obtained for  $\nu_0 \sim 0.068$ . The norm is reduced up to order  $10^{-25}$  for  $j \sim 16\,000$ , and after this value the precision of the numerical computation evidently affects the computations.

Therefore, we repeated the computations with larger values of  $N$ . In general, increasing the parameter  $N$  can make worse the convergence properties of the method, because on the one hand one introduces new small denominators, and on the other the errors are computed on a larger set of points  $\varphi_j$ . For the case under consideration, the convergence of the method is already worse at  $N=2^9$ , but then stabilizes up to the much larger  $N=2^{16}$ , so that, if one wishes to represent more points with respect to the  $N=2^8$  case, it is convenient to use  $N=2^{16}$ . After some trials, we found that one of the best choices for the parameters in the case  $N=2^{16}$  was  $\nu_0=0.1$  and  $\nu_1=10^{-8}$ . We will consider as an indicator of the largest individual error on the parametrization the quantity

$$r = \max_j \left| \tilde{u}(\vartheta_j) - \frac{\tilde{u}_+(\vartheta_j) + \tilde{u}_-(\vartheta_j)}{2} + \frac{\varepsilon}{2} f(\vartheta_j + \tilde{u}(\vartheta_j)) \right|, \quad (29)$$

which, in this case, is  $r \sim 5 \times 10^{-8}$  after 20 000 iterations of the method (the functional error  $\|\Pi_N F[u^{20\,000}]\|$  is approximately the same). On the set of  $2^{16}$  points  $(\varphi_j, I_j)$  we then applied the procedure described in Sec. II, points (i) to (iv). The density function  $\tilde{\rho}$  is represented in Fig. 3 on the top, and then we realized that a good first choice for the limit

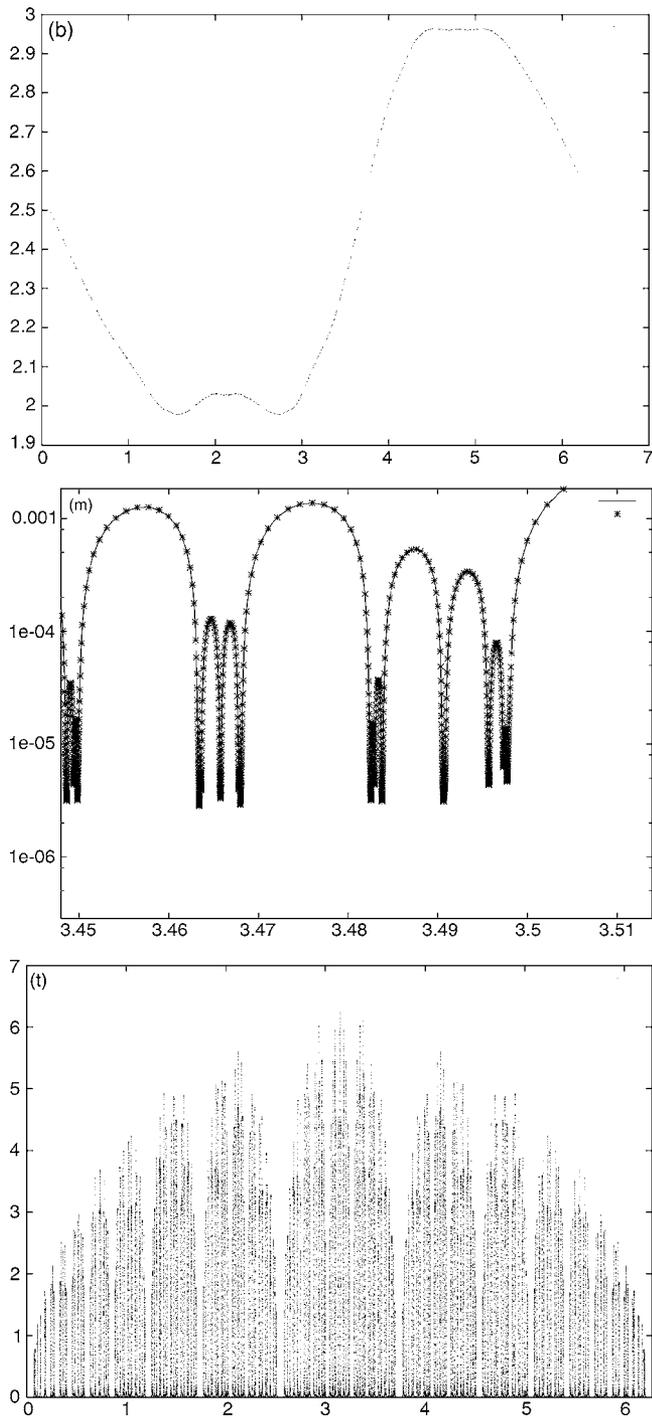


FIG. 3. On the top we represent  $\tilde{\rho}(\varphi_j)$  vs  $\varphi_j$ ; on the middle we represent (part of the plot of)  $\varphi_j - \varphi_{j-1}$  vs  $\varphi_j$ . The density of the points indicate that a good value for  $\rho$  is  $0.7629$ , which corresponds to the limit distance  $\varphi_j - \varphi_{j-1}$  of  $2 \times 10^{-5}$ . On the bottom: representation in the phase-space  $(\varphi, I)$  of the 42 204 approximate points of the cantor that satisfy (30).

density parameter  $\rho$  is  $2 \times 10^{-5}$  (see the middle of Fig. 3). Therefore, we removed from the set of points  $(\varphi_j, I_j)$  those with

$$\min\{\varphi_j - \varphi_{j-1}, \varphi_{j+1} - \varphi_j\} > \rho, \tag{30}$$

obtaining a set of 42 204 points represented in Fig. 3 bottom. To test the stability of this new set of points under the iterations of the map we apply the steps (ii), (iii), and (iv) 100

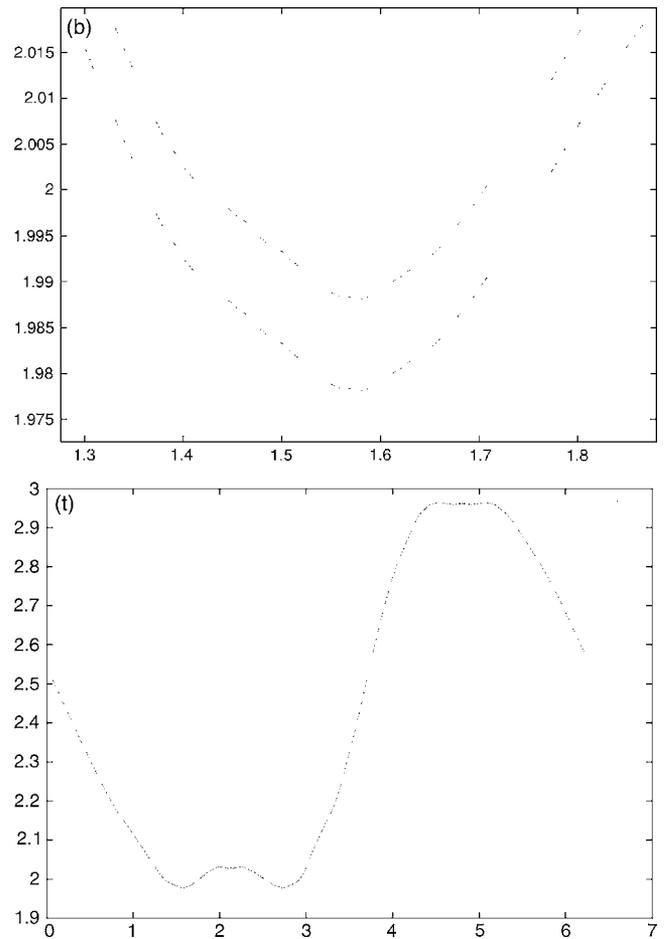


FIG. 4. On the top: representation in the phase-space  $(\varphi, I)$  of 1 947 069 approximate points of the cantor. On the bottom: zoom of the 1 947 069 points of the cantor and a zoom of the 42 204 points shifted of 0.02 in the action coordinate (to compare the two sets). It is evident a very good correspondence of the two sets, with no significant enlargements of the part of the cantor passing from the original set to the one obtained with 100 iterations.

times, with  $\lambda=k$ ,  $k$  denoting the  $k$ th step of the procedure, thus obtaining a set of 1 947 069 approximate points of the cantor, showing very good stability properties: in Fig. 4 we report a zoom of the 1 947 069 points of the cantor and a zoom of the 42 204 points shifted of 0.02 in the action coordinate (to compare the two sets). It is evident a very good correspondence of the two sets, with no significant enlargements of the part of the cantor passing from the original set to the one obtained with 100 iterations.

### B. Rotation number near the Liouville number

The situation described in the previous subsection is quite peculiar for two reasons: the standard map, characterized by  $f(\varphi) = -\sin \varphi$ , has only one Fourier harmonic. Moreover, the small divisors associated to that rotation number have very good asymptotic properties because the continued fraction expansion of  $\alpha/(2\pi)$  is  $[0, 2, 1, 1, 1, \dots]$ . In this subsection we change these two conditions to more unfavorable ones: on the one hand, we use as perturbation the function

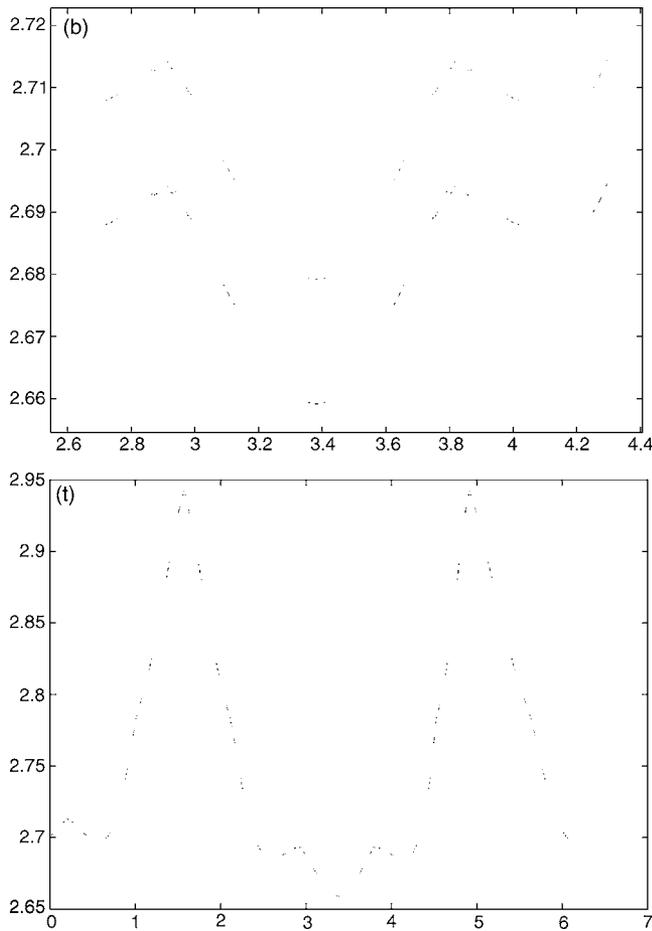


FIG. 5. On the top: representation in the phase-space  $(\varphi, I)$  of a set of 1 622 521 approximate points of the cantorus. On the bottom: zoom of the 1 622 521 points of the cantorus and a zoom of the 54 305 points shifted of 0.02 in the action coordinate (to compare the two sets). It is evident a very good correspondence of the two sets, with no significant enlargements of the part of the cantorus passing from the original set to the one obtained with 100 iterations.

$$f(\varphi) = \frac{\cos \varphi}{(\sin \varphi + 1.2)^2}, \quad (31)$$

which has infinite Fourier harmonics, with very slow exponential decay of their amplitudes because the analyticity radius is  $\sigma \sim 0.6$ ; on the other, we choose a rotation number that is near the Liouville number, whose small divisors have very bad asymptotic properties. Precisely, we choose  $\alpha = 4\pi(0.110\,001 + 0.0075/2\pi)$ , because the rational 0.110 001 is close to the Liouville number, while  $0.0075/2\pi$  is a small irrational correction (to the third digit). We then selected  $\varepsilon = 0.04$ , for which the cantorus is in a quite large chaotic region, but near its border, and repeated the computations of Sec. III A with the following choice of the parameters:  $N = 2^{16}$ ,  $\nu_0 = 0.1$ , and  $\nu_1 = 10^{-9}$ , ten preliminary steps, and 6000 further steps (with a final error of  $3 \times 10^{-5}$ ). We then selected the initial conditions to iterate as explained in Sec. II. Figure 5 reports the result of such a computation.

## IV. CONCLUSIONS

In this paper we have defined a new procedure for the localization of cantori of quasi-integrable twist maps of the cylinder, such as the standard map, based on regularization techniques. The construction of a parameterization of the cantorus is obtained through an iterative procedure: Proposition 1 describes how to construct improvements of approximate parameterizations, which depend on two parameters which we call “relative friction” and “relative viscosity.” These parameters act on the difference among the approximate parameterization and the true one. Proposition 2 describes also how to choose optimal values of these parameters in order to minimize the contraction term  $l(\nu_0, \nu_1) < 1$  involved in the iterated refinement procedure. In this regard, our method is different from usual applications of viscosity to Aubry-Mather theory, which work in the limit of very small viscosity parameter (instead of a positive optimal ones). Once an approximate parameterization of the cantorus is obtained, we use it to construct a measure that allowed us to localize the point of the cantorus in the cylinder. We report the numerical application of the method to two symplectic twist maps and two rotation numbers.

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