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Pseudocomplements of closure operators on posets

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Abstract

Some recent results provide sufficient conditions for complete lattices of closure operators on complete lattices, ordered pointwise, to be pseudocomplemented. This paper gives results of pseudocomplementation in the more general setting of closure operators on mere posets. The following result is first proved: closure operators on a meet-continuous meet-semilattice form a pseudocomplemented complete lattice. Furthermore, the following orthogonal result (actually, a slightly more general result) is proved: Closure operators on a directed-complete poset which is transfinitely generated by maximal lower bounds from its set of completely meet-irreducible elements—any poset satisfying the ascending chain condition belongs to this class—form a pseudocomplemented complete lattice. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The structure of posets of closure operators (closures for short), ordered by the standard pointwise relation between functions, is an important and well studied topic. Given any poset P, $\langle \mathrm{uco}(P), \sqsubseteq \rangle$ will denote the poset of all closures on P. In this paper, we focus on the properties of pseudocomplementation of $\mathrm{uco}(P)$.

It has been shown recently by Giacobazzi et al. [7] that if C is a meet-continuous complete lattice then $\mathrm{uco}(C)$ is a pseudocomplemented complete lattice. Also, Filé and Ranzato [5] proved that if C is a complete lattice meet-generated by its set of meet-irreducible elements then $\mathrm{uco}(C)$ is a pseudocomplemented complete lattice. Such results heavily exploit the well-known basic theorem by Ward [15], stating that if C is a complete lattice then $\mathrm{uco}(C)$ is a complete lattice dually isomorphic to the

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complete lattice of complete meet-subsemilattices of C, ordered by set-inclusion. To the best of our knowledge, no result of pseudocomplementation for closures on mere posets is available. In this more general setting, we provide two novel theorems of pseudocomplementation.

It turns out that the converse of Ward's theorem does not hold, i.e., if uco(P) is a complete lattice then P need not be a complete lattice. Recently, Ranzato [13] strengthened Ward's theorem as follows. Define a poset P to be relatively maximal lower bound complete (rmlb-complete for short) whenever for any $Y \subseteq P$ and for any lower bound $x \in P$ of Y, the set of maximal lower bounds of Y which are above x is nonempty. Then, Ranzato [13] proves that if P is rmlb-complete then uco(P) is a complete lattice. Moreover, in such a case, greatest lower bound and least upper bound operations in uco(P) are explicitly characterized. It is worth remarking that rmlb-completeness results in a fairly weak condition: Ranzato [13] shows that any directed-complete poset is rmlb-complete. The results of this paper are in turn based on Ranzato's [13] constructions, since we will establish some sufficient conditions on a rmlb-complete poset P guaranteeing that uco(P) is a pseudocomplemented complete lattice.

Our first result subsumes the aforementioned theorem by [7]. In fact, we show that if P is a meet-continuous meet-semilattice then uco(P) is a pseudocomplemented complete lattice.

The second result is orthogonal to the first one and more complex to state. Following [13], given a poset P and $Y \subseteq P$, if mlb(Y) denotes the set of maximal lower bounds of Y in P, we define the operator $M(Y) \stackrel{\text{def}}{=} \bigcup_{S \subseteq Y} \text{mlb}(S)$. We then say that the poset P is generated by maximal lower bounds from Y if P can be obtained by applying M transfinitely starting from Y. Let us also recall that an element $x \in P$ is completely meet-irreducible whenever $\{y \in P \mid y > x\}$ is a principal filter of P. Then, our second theorem is as follows: if P is rmlb-complete and generated by maximal lower bounds from the set of completely meet-irreducibles of P then uco(P) is a pseudocomplemented complete lattice. This second result enjoys the advantage of providing an explicit characterization for pseudocomplements of closures. We show that any poset satisfying the ascending chain condition satisfies the hypotheses of this second result.

Let us mention that our results may find applications in theoretical computer science, notably in abstract interpretation [3], a well-known technique widely used for approximating the semantics of discrete dynamic systems, e.g. for designing and proving correct static program analyses (cf. [12]). Abstract interpretation theory heavily uses closure operators as a way to represent the so-called abstract description domains. Cortesi et al. [2] introduced a useful complementation operation between abstract domains, based on the aforementioned pseudocomplementation theorem by [7]. Thus, in the approach of Cortesi et al. [2], abstract domains were required to be complete lattices. The results of this paper can therefore be used in order to widen the range of abstract domains which can be complemented. This may be useful in practice, since in many common situations, e.g. in denotational program semantics, abstract domains are not complete lattices.

2. Preliminaries

Let us introduce the basic notations and notions that will be used throughout the paper. The identity operator on any set will be denoted by id. If X and Y are sets then $X \setminus Y$ denotes their set-difference and if $f : X \to Y$ then $f(X) \stackrel{\text{def}}{=} \{f(x) \mid x \in X\}$. Let $\langle P, \leqslant \rangle$ be a poset. If $x, y \in P$ then $x \prec y$ and $y \succ x$ denote that y covers x, i.e., $x < y$ and for any $z \in P$, $x < z \leqslant y$ implies $z = y$. If $x \in P$ then $\uparrow x \stackrel{\text{def}}{=} \{y \in P \mid x \leqslant y\}$ and $\Uparrow x \stackrel{\text{def}}{=} \{y \in P \mid x < y\}$. If $x \in P$ and $Y \subseteq P$ then we write $x \leqslant Y$ when x is a lower bound of Y, i.e., when for any $y \in Y$, $x \leqslant y$. $Y^{\downarrow}$ will denote the set of lower bounds of Y in P, i.e., $Y^{\downarrow} \stackrel{\text{def}}{=} \{x \in P \mid x \leqslant Y\}$. The notation $x < Y$ means that $x \in Y^{\downarrow} \setminus Y$. Let us remark that it follows that, for any $x \in P$, $x \leqslant \emptyset$, i.e., $\emptyset^{\downarrow} = P$. Throughout, the notations $x \leqslant Y$ and $x \in Y^{\downarrow}$ will be used interchangeably. If $Y \subseteq P$ then Y has (necessarily unique) least element $\mathrm{lst}(Y)$ whenever $\mathrm{lst}(Y) \in Y^{\downarrow} \cap Y$. If $Y \subseteq P$ then $\max(Y) \stackrel{\text{def}}{=} \{y \in Y \mid \forall z \in Y.\ (y \leqslant z) \Rightarrow (y = z)\}$. If $Y \subseteq P$ then $\mathrm{mlb}(Y)$ denotes the (possibly empty) set of maximal lower bounds of Y, i.e., $\mathrm{mlb}(Y) \stackrel{\text{def}}{=} \max(Y^{\downarrow})$. In particular, let us point out that $\mathrm{mlb}(\emptyset) = \max(P)$, and that, for any $Y \subseteq P$, $\mathrm{mlb}(Y) = \{x\}$, for some $x \in P$, iff x is the greatest lower bound (glb) of Y. We will use the following easy property of maximal lower bounds.

Lemma 2.1. *If* $Y, Z \subseteq P$, $y \in Y$ *and* $y \in Z^{\downarrow}$ *then* $\mathrm{mlb}(Y) = \mathrm{mlb}(Y \cup Z)$.

If P is a poset and $Y \subseteq P$ then Y is directed if $Y \neq \emptyset$ and any finite subset of Y has an upper bound in Y. A poset P is directed-complete (dcpo for short) if any directed subset of P has least upper bound (lub) in P. We will exploit a well-known equivalent formulation of dcpo's involving chains [10]: a poset P is directed-complete iff any (nonempty) chain in P has lub. A function between dcpo's is said to be continuous if it preserves lub's of directed subsets (equivalently, lub's of chains [10]).

Let $\langle P, \leqslant, \wedge \rangle$ be a meet-semilattice. Let us recall from [8, Definition 4.6, p. 33] that P is meet-continuous if P is directed-complete (existing lub's are denoted by $\bigvee$) and for any $x \in P$ and directed subset $Y \subseteq P$, $x \wedge \bigvee Y = \bigvee_{y \in Y} x \wedge y$. Thus, meet-continuous complete lattices form a proper subclass of meet-continuous meet-semilattices. Again, in the previous definition, directed subsets can be equivalently replaced by chains [1,10].

Given a dcpo P, an element $x \in P$ is compact when for any directed subset $Y \subseteq P$, $x \leqslant \bigvee Y$ implies that there exists some $y \in Y$ such that $x \leqslant y$. The set of compact elements of P is denoted by $\mathbb{K}_P$. Let us recall [8, Exercise 4.28, p. 94] that a poset P is algebraic if P is a dcpo and for any $x \in P$, $\{y \in \mathbb{K}_P \mid y \leqslant x\}$ is directed and $x = \bigvee \{y \in \mathbb{K}_P \mid y \leqslant x\}$.

Let $\langle P, \leqslant, \wedge, \bot \rangle$ be a meet-semilattice with least element $\bot$. Let us recall that, given $x \in P$, $x^* \in P$ is the (necessarily unique) pseudocomplement of x if $x \wedge x^* = \bot$ and for any $y \in P$, $x \wedge y = \bot \Rightarrow y \leqslant x^*$. P is pseudocomplemented if all its elements have pseudocomplements.

Pointwise ordering between functions is denoted by the symbol $\sqsubseteq$: if P is a poset and $f, g : X \to P$, then $f \sqsubseteq g$ iff for any $x \in X$, $f(x) \leqslant g(x)$.

We will make use of the following equivalent formulation of the Axiom of Choice (see e.g. [4]).

Hausdorff's Maximal Principle. *Every chain in a poset P can be extended to a maximal chain in P.*

3. Closure operators

A closure operator (shortly, closure) on a poset $\langle P, \leqslant\rangle$ is an operator $\rho : P \to P$ monotone, idempotent and extensive (i.e., $\forall x \in P.\ x \leqslant \rho(x)$). Fixpoints of a closure are also called closed elements. Closures will be denoted by lowercase Greek letters $\rho, \eta, \mu, \ldots$. Let $\mathrm{uco}(P)$ denote the set of all closure operators on the poset P. Closures on posets are partially ordered by pointwise ordering, i.e. $\langle \mathrm{uco}(P), \sqsubseteq\rangle$ is a poset. Throughout the paper, for any $\rho \in \mathrm{uco}(P)$, we will follow a standard notation for closure operators by denoting the image $\rho(P)$ simply by ρ itself. This does not give rise to ambiguity, since one can immediately distinguish the use as function or set according to the context. Let us recall that the image of a closure $\rho \in \mathrm{uco}(P)$ coincides with its set of closed elements: $\rho = \{x \in P \mid x = \rho(x)\}$. Let $\rho, \eta \in \mathrm{uco}(P)$. The following are some basic easy properties of closures on posets (cf. [9,15]).

(1) The image of a closure is closed for maximal lower bounds: If $Y \subseteq \rho$ then $\mathrm{mlb}(Y) \subseteq \rho$; in particular, $\max(P) \subseteq \rho$.
(2) Pointwise ordering coincides with the superset relation on the corresponding images: $\rho \sqsubseteq \eta \Leftrightarrow \eta \subseteq \rho$.
(3) A subset $Y \subseteq P$ is the set of fixpoints of a closure $\rho \in \mathrm{uco}(P)$ iff for all $x \in P$, $Y \cap \uparrow x$ has least element; in such a case, $\rho = \lambda x.\ \mathrm{lst}(Y \cap \uparrow x)$.
(4) The identity is the least closure: $\mathrm{id} \in \mathrm{uco}(P)$ and $\mathrm{id} \sqsubseteq \rho$.
(5) If P is a meet-semilattice then $\mathrm{uco}(P)$ is a meet-semilattice, where meets of closures are defined pointwise: For any $x \in P$, $(\rho \sqcap \eta)(x) = \rho(x) \wedge \eta(x)$.

Thus, by (3), closures are univocally determined by their sets of fixpoints. Whenever P is a mere poset, in general, $\mathrm{uco}(P)$ is not a complete lattice. For example, if ω is the first infinite ordinal, then $\mathrm{uco}(\omega)$ is not a complete lattice. On the other hand, it is not too hard to prove Ward's theorem [15, Theorem 4.2] stating that if C is a complete lattice then $\langle \mathrm{uco}(C), \sqsubseteq\rangle$ is a complete lattice. In particular, the set of fixpoints of the lub of a subset $\{\rho_i\}_{i\in I} \subseteq \mathrm{uco}(C)$ is the set-intersection $\bigcap_{i\in I} \rho_i$.

Recently, Ranzato [13] strengthened Ward's theorem. Let us recall this result. A poset P is called relatively maximal lower bound complete (rmlb-complete for short) if for any $Y \subseteq P$ and $x \in P$, $x \in Y^{\downarrow}$ implies $\mathrm{mlb}(Y) \cap \uparrow x \neq \emptyset$. A subset $Y \subseteq P$ is closed for maximal lower bounds (mlb-closed for short) if

$$Y = M(Y) \stackrel{\mathrm{def}}{=} \bigcup_{S \subseteq Y} \mathrm{mlb}(S).$$

Notice that if Y is mlb-closed then Y contains the maximal elements of P, since $\max(P)=\mathrm{mlb}(\emptyset)\subseteq Y$. [13] observes that mlb-closed subsets are closed under arbitrary intersections. Then, one can define the mlb-closure $\mathfrak{M}\in \mathrm{uco}(\langle\wp(P),\subseteq\rangle)$ on the powerset of P as follows:

$$\mathfrak{M}(X)\stackrel{\text{def}}{=}\bigcap\{Y\in\wp(P)\mid X\subseteq Y,\ M(Y)=Y\}.$$

Thus, $\mathfrak{M}(X)$ is the least (w.r.t. set-inclusion) mlb-closed subset of P containing X—in particular, let us note that the mlb-closure of the empty set coincides with the mlb-closure of the set of maximal elements of P. Equivalently, the mlb-closure of X is the least fixpoint of M containing X. As a consequence, let us remark that, by the transfinite formulation of Knaster–Tarski's fixpoint theorem and since M is a monotone and extensive operator on the complete lattice $\langle\wp(P),\subseteq\rangle$, $\mathfrak{M}(X)$ can be obtained by applying M transfinitely often starting from X: In fact, $\bigcup_{\alpha\in\mathbb{O}} M^{\alpha}(X)$ is the least fixpoint of M above X, where, for any ordinal $\alpha\in\mathbb{O}$, the ordinal (upper) α-power $M^{\alpha}(X)$ is defined, by transfinite induction, as: X if $\alpha=0$; $M(M^{\alpha-1}(X))$ if α is a successor ordinal; $\bigcup_{\gamma<\alpha} M^{\gamma}(X)$ if α is a limit ordinal.

Since $\mathfrak{M}$ is a closure operator on the complete lattice $\langle\wp(P),\subseteq\rangle$, it satisfies the following property, which will be useful later on:

$$\text{For all } \{X_i\}_{i\in I}\subseteq\wp(P),\quad \mathfrak{M}\left(\bigcup_{i\in I} X_i\right)=\mathfrak{M}\left(\bigcup_{i\in I}\mathfrak{M}(X_i)\right). \tag{‡}$$

Moreover, the following property also holds.

Lemma 3.1. *If P is a poset, $\rho\in\mathrm{uco}(P)$ and $X\subseteq\rho$, then $\mathfrak{M}(X)\subseteq\rho$.*

Proof. Because, by point (1) above, ρ is closed for maximal lower bounds. $\square$

We are now in position to recall Ranzato's [13, Theorem 4.5] result.

Theorem 3.2 (Ranzato [13]). *If P is a rmlb-complete poset then $\langle\mathrm{uco}(P),\sqsubseteq\rangle$ is a complete lattice, where if $\{\rho_i\}_{i\in I}\subseteq\mathrm{uco}(P)$, then $\mathfrak{M}(\bigcup_{i\in I}\rho_i)$ and $\bigcap_{i\in I}\rho_i$ are, respectively, the sets of fixpoints of the glb and lub in $\mathrm{uco}(P)$ of $\{\rho_i\}_{i\in I}$.*

As far as rmlb-completeness is concerned, let us remark that if P is rmlb-complete, then any $x\in P$ is below some maximal element of P. Thus, for instance, the first infinite ordinal ω is not rmlb-complete. [13] shows that any dcpo is rmlb-complete, while the converse does not hold, as one can easily check by means of the poset R depicted in Fig. 1.

Finally, let us mention that Morgado [11, Theorem 28] gave a theorem characterizing all and only the posets P such that $\mathrm{uco}(P)$ is a complete lattice, based on a notion

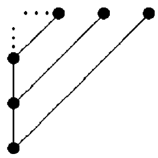

Fig. 1. The rmlb-complete poset R.

of relative quasi-infimum in posets [11, Definition 4]. Unfortunately, Morgado's result is based on some erroneous lemmata (we refer to [14] for details), and therefore it is unusable.

4. On pseudocomplements of closures

Following the terminology introduced in [7, Definition 2.1], a meet-semilattice $\langle P, \leqslant, \wedge \rangle$ is weakly relatively pseudocomplemented if any principal filter of P is pseudocomplemented. Given $x, y \in P$ such that $x \in \uparrow y$, the pseudocomplement of x in $\uparrow y$ is called the weak relative pseudocomplement of x with respect to y, and it is denoted by $x \rightsquigarrow y$. Note that $x \rightsquigarrow \bot$, when it exists, is the pseudocomplement of x (in P)—thus, a weakly relatively pseudocomplemented meet-semilattice with least element is pseudocomplemented. We refer to the discussions in [6,7] for the relationship with the well-known notion of relative pseudocomplementation.

If P is a poset and $\eta \in \mathrm{uco}(P)$ then one can consider η itself as a poset endowed with the order inherited from P, and therefore one can also consider the poset $\langle \mathrm{uco}(\eta), \sqsubseteq \rangle$ of closures on η. In this context, the following observation holds.

Lemma 4.1. $\langle \mathrm{uco}(\eta), \sqsubseteq \rangle \cong \langle \uparrow \eta, \sqsubseteq \rangle$.

Proof. Consider the map $\Phi : \uparrow \eta \to \mathrm{uco}(\eta)$ defined as follows: For any $\rho \in \mathrm{uco}(P)$ such that $\eta \sqsubseteq \rho$, for any $x \in \eta$, $\Phi(\rho)(x) \stackrel{\text{def}}{=} \rho(x)$. In other words, Φ is the identity on the sets of fixpoints of the closures. It is then a routine task to check whether Φ is a well-defined isomorphism. $\square$

Let us consider some interesting observations derived from the above isomorphism. Given $\rho, \eta \in \mathrm{uco}(P)$ such that $\eta \sqsubseteq \rho$, the weak relative pseudocomplement $\rho \rightsquigarrow \eta$ exists iff the pseudocomplement of ρ in $\mathrm{uco}(\eta)$ exists, and in such a case, the corresponding sets of fixpoints coincide. Consequently, for closure operators, weak relative pseudocomplementation reduces, up to the obvious isomorphism of Lemma 4.1, to pseudocomplementation. Giacobazzi et al. [7] first gave a result of weak relative pseudocomplementation for closures, and as a corollary obtained a result of pseudocomplementation for closures. By what has been observed above, instead, we will first concentrate on the results of pseudocomplementation, which are both conceptually and notationally simpler, and then the results of weak relative pseudocomplementation will be derived as consequences.

5. The first result

Given a complete lattice C, Giacobazzi et al. [7, Examples 3.2, 3.3] show that in general $\langle \mathrm{uco}(C), \sqsubseteq \rangle$ is not pseudocomplemented. On the other hand, Giacobazzi et al. [7, Corollary 3.4] give the following result.

Theorem 5.1 (Giacobazzi et al. [7]). *If C is a meet-continuous complete lattice, then* $\mathrm{uco}(C)$ *is pseudocomplemented.*

We strengthen this result by showing that it holds for the class of meet-continuous meet-semilattices. The proof follows and generalizes the one given in [7, Theorem 3.1], where maximal lower bounds somehow play the role of infinite glb's in complete lattices.

Theorem 5.2. *If P is a meet-continuous meet-semilattice, then* $\mathrm{uco}(P)$ *is a pseudo-complemented complete lattice.*

Proof. By Theorem 3.2, $\mathrm{uco}(P)$ is a complete lattice. Let $\sqcup$ and $\sqcap$ denote the lub and glb operations of this complete lattice—by Theorem 3.2, $\sqcup$ is characterized as set-intersection of images. Let $\rho \in \mathrm{uco}(P)$. Thus, the lub $\varphi \stackrel{\mathrm{def}}{=} \sqcup\{\mu \in \mathrm{uco}(P) \mid \rho \sqcap \mu = \mathrm{id}\}$ exists, and it is characterized as $\varphi = \bigcap\{\mu \in \mathrm{uco}(P) \mid \rho \sqcap \mu = \mathrm{id}\}$. We show that $\rho^* = \varphi$, i.e. that $\rho \sqcap \varphi = \mathrm{id}$. By contradiction, assume that there exists some $\hat{x} \in P$ such that $\hat{x} \notin \rho \sqcap \varphi$. We show that for each ordinal $\alpha \in \mathbb{O}$ we are able to construct a strictly increasing chain $\{x_\beta\}_{\beta \leqslant \alpha}$ in P. Hence, this leads to a contradiction for ordinals α whose cardinality is strictly greater than the cardinality of P. In detail, we prove by transfinite induction on $\alpha \in \mathbb{O}$ that it is possible to construct in P two sets $\{x_\beta\}_{\beta \leqslant \alpha}$, with $x_0 = \hat{x}$, and $\{y_\beta\}_{\beta < \alpha}$, such that:

(i) $\{x_\beta\}_{\beta \leqslant \alpha}$ is a strictly increasing chain in $P \setminus (\rho \sqcap \varphi)$;
(ii) $\forall \beta < \alpha.\ y_\beta \in \rho$;
(iii) $x_0 \in \mathrm{mlb}(\{x_\alpha\} \cup \{y_\beta\}_{\beta < \alpha})$;
(iv) $\forall \beta, \gamma \in \mathbb{O}.\ \beta \leqslant \gamma < \alpha \Rightarrow x_\beta < y_\gamma$.

Let us consider the three cases $\alpha = 0$, α successor ordinal, and α limit ordinal.

($\alpha = 0$): Define $x_0 = \hat{x}$. Then, $x_0 \in P \setminus (\rho \sqcap \varphi)$ by definition, and therefore (i) is satisfied. Conditions (ii) and (iv) are vacuously satisfied, while condition (iii) is trivially true, since $\mathrm{mlb}(\{x_0\}) = \{x_0\}$.

($\alpha+1$): By induction hypothesis (i), $x_\alpha \notin \rho \sqcap \varphi$, and therefore $x_\alpha \notin \rho \cup \varphi$. Hence, there exists a closure $\mu \in \mathrm{uco}(P)$ such that $\rho \sqcap \mu = P$ and $x_\alpha \notin \mu$. Since $\rho \sqcap \mu = P$, we have that $x_\alpha \in \rho \sqcap \mu$. Therefore, by (5) in Section 3, $x_\alpha = \rho(x_\alpha) \wedge \mu(x_\alpha)$. Since $x_\alpha \notin \rho \cup \mu$, we have that $\mu(x_\alpha) \in \mu \backslash \rho$ and $\rho(x_\alpha) \in \rho \backslash \mu$ (otherwise, we would have $x_\alpha \in \rho \cup \mu$). Define $x_{\alpha+1} = \mu(x_\alpha)$ and $y_\alpha = \rho(x_\alpha)$, and observe that (ii) and (iv) are satisfied.

Let us check (iii). By induction hypothesis (iii), we have $x_0 \in \mathrm{mlb}(\{x_\alpha\} \cup \{y_\beta\}_{\beta<\alpha})$. Since $\mathrm{mlb}(\{x_\alpha\} \cup \{y_\beta\}_{\beta<\alpha}) = \mathrm{mlb}(\{y_\alpha \wedge x_{\alpha+1}\} \cup \{y_\beta\}_{\beta<\alpha}) = \mathrm{mlb}(\{x_{\alpha+1}\} \cup \{y_\beta\}_{\beta<\alpha+1})$, we have that $x_0 \in \mathrm{mlb}(\{x_{\alpha+1}\} \cup \{y_\beta\}_{\beta<\alpha+1})$. With respect to (i), observe that $x_\alpha \leqslant x_{\alpha+1}$, and $x_\alpha \neq x_{\alpha+1}$, since $x_\alpha \notin \mu$ whilst $x_{\alpha+1} \in \mu$. Since $\{y_\beta\}_{\beta<\alpha+1} \subseteq \rho \subseteq \rho \sqcap \varphi$, we deduce that $x_{\alpha+1}$ cannot belong to $\rho \sqcap \varphi$, otherwise, from $x_0 \in \mathrm{mlb}(\{x_{\alpha+1}\} \cup \{y_\beta\}_{\beta<\alpha+1})$, by (1) in Section 3, we would get the contradiction $x_0 \in \rho \sqcap \varphi$.

(α limit ordinal): Define $x_\alpha = \bigvee_{\beta<\alpha} x_\beta$. Note that we do not need to define y_α for a limit ordinal α. Conditions (ii) and (iv) are trivially satisfied by the corresponding induction hypotheses.

Let us consider (iii). By induction hypothesis (iii), we have that, for any $\beta < \alpha$, $x_0 \in \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\beta})$. By induction hypothesis (iv), for $\beta < \alpha$, $x_\beta \leqslant \{y_\gamma\}_{\beta \leqslant \gamma < \alpha}$ holds. Therefore, by Lemma 2.1, $\mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\beta}) = \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\beta\leqslant\gamma<\alpha} \cup \{y_\gamma\}_{\gamma<\beta}) = \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$. Hence, we derive $x_0 \in \bigcap_{\beta<\alpha} \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$. Thus, in order to conclude (iii), we show that $\bigcap_{\beta<\alpha} \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha}) \subseteq \mathrm{mlb}(\{\bigvee_{\beta<\alpha} x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$. Let $x \in \bigcap_{\beta<\alpha} \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$. Then, $x \in (\{\bigvee_{\beta<\alpha} x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})^{\downarrow}$. Let $z \in (\{\bigvee_{\beta<\alpha} x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})^{\downarrow}$ such that $x \leqslant z$. Since $z \leqslant \{y_\gamma\}_{\gamma<\alpha}$, and since, as recalled in Section 3, any dcpo is rmlb-complete, there exists some $m \in \mathrm{mlb}(\{y_\gamma\}_{\gamma<\alpha}) \cap \uparrow z$. Then, $z \leqslant m \wedge \bigvee_{\beta<\alpha} x_\beta$, and therefore, by meet-continuity, $z \leqslant \bigvee_{\beta<\alpha} m \wedge x_\beta$. Consider now any $\beta < \alpha$. Notice that $m \wedge x_\beta \leqslant \{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha}$. Moreover, since $z \leqslant m$, we have that $x \leqslant m$, hence $x \leqslant \{m, x_\beta\}$, and therefore $x \leqslant m \wedge x_\beta$. Thus, since $x \in \mathrm{mlb}(\{x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$, by maximality of x, we get $x = m \wedge x_\beta$ (for any $\beta < \alpha$). Consequently, $x = \bigvee_{\beta<\alpha} m \wedge x_\beta$, and therefore $z \leqslant x$, i.e., $x = z$. As desired, this means that $x \in \mathrm{mlb}(\{\bigvee_{\beta<\alpha} x_\beta\} \cup \{y_\gamma\}_{\gamma<\alpha})$, and therefore $x_0 \in \mathrm{mlb}(\{x_\alpha\} \cup \{y_\gamma\}_{\gamma<\alpha})$. Finally, we show (i). Since all the elements of $\{x_\beta\}_{\beta<\alpha}$ are distinct, also x_α is strictly greater than x_β, for any $\beta < \alpha$. Analogously to the successor case, since $\{y_\beta\}_{\beta<\alpha} \subseteq \rho \sqcap \varphi$, it turns out that x_α cannot belong to $\rho \sqcap \varphi$, otherwise, from $x_0 \in \mathrm{mlb}(\{x_\alpha\} \cup \{y_\gamma\}_{\gamma<\alpha})$, by (1) in Section 3, we would get the contradiction $x_0 \in \rho \sqcap \varphi$. $\square$

By Section 4, we get the following consequence subsuming the result of Giacobazzi et al.'s [7, Theorem 3.1].

Corollary 5.3. *Let P be a meet-continuous meet-semilattice. Then, for every $\rho, \eta \in \mathrm{uco}(P)$ such that $\eta \sqsubseteq \rho$ and η is continuous, there exists $\rho \rightsquigarrow \eta$.*

Proof. Let us show that η is a meet-continuous meet-semilattice. Firstly, η is trivially a meet-subsemilattice of P. Also, η is directed-complete: If $Y \subseteq \eta$ is directed, then, by continuity of η, $\eta(\bigvee Y) = \bigvee \eta(Y) = \bigvee Y$, and therefore $\bigvee Y \in \eta$. Thus, since η is a meet-subsemilattice of P and a sub-dcpo of P, we get that η inherits from P the property of meet-continuity. Hence, by Theorem 5.2, $\mathrm{uco}(\eta)$ is pseudocomplemented, and therefore, by Section 4, $\rho \rightsquigarrow \eta$ exists. $\square$

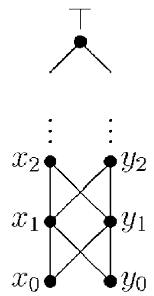

Fig. 2. The poset S.

We may note in the proof of Theorem 5.2 that the hypothesis of meet-continuity is exploited in order to prove that the containment

$$\bigcap_{\beta<\alpha} \mathrm{mlb}(\{x_\beta\} \cup Y) \subseteq \mathrm{mlb}\left(\left\{ \bigvee_{\beta<\alpha} x_\beta \right\} \cup Y\right) \qquad (*)$$

holds for any chain $\{x_\beta\}_{\beta<\alpha} \subseteq P$ and subset $Y \subseteq P$. Thus, one might wonder whether it is possible to widen the class of meet-continuous meet-semilattices of Theorem 5.2 to the class of dcpo's satisfying the condition $(*)$. The following example shows that this is not the case.

Example 5.4. Consider the dcpo S depicted in Fig. 2. It is clear from the Hasse diagram that S satisfies the condition $(*)$. In fact, it is enough to observe that if $\{x_\beta\}_{\beta<\alpha}$ is an infinite chain in S then, for any $Y \subseteq S$, $\bigcap_{\beta<\alpha} \mathrm{mlb}(\{x_\beta\} \cup Y) = \emptyset$. Nevertheless, $\mathrm{uco}(S)$ is not pseudocomplemented. Let us consider the closure $\rho = \{y_i\}_{i\in\mathbb{N}} \cup \{\top\} \in \mathrm{uco}(S)$. It turns out that the pseudocomplement of ρ does not exist. In fact, consider the family of closures $\{\eta_k\}_{k\in\mathbb{N}} \subseteq \mathrm{uco}(S)$ defined as follows: For any $k \in \mathbb{N}$, $\eta_k = \{x_j \mid j \geqslant k\} \cup \{\top\}$. Then, notice that for any $k \in \mathbb{N}$, $\mathfrak{M}(\rho \cup \eta_k) = S$, and therefore, by Theorem 3.2, $\rho \sqcap \eta_k = \mathrm{id}$. Consequently, the pseudocomplement of ρ should be greater than every η_k, and therefore it should be contained in $\bigcap_{k\in\mathbb{N}} \eta_k = \{\top\}$. But this is evidently a contradiction.

6. The second result

Let $\langle P, \leqslant \rangle$ be a poset. Let us recall [8, Definition 4.19, p. 92] that an element $x \in P$ is completely meet-irreducible in P when $\mathrm{lst}(\Uparrow x)$ exists. Thus, equivalently, x is completely meet-irreducible iff $\Uparrow x$ is a principal filter. The set of completely meet-irreducibles of a poset P is denoted by MI_P. Observe that maximal elements are not completely meet-irreducibles, i.e., $\max(P) \cap MI_P = \emptyset$. For complete lattices, the following standard definition is equivalent: If C is a complete lattice then $x \in MI_C$ iff for any $Y \subseteq C$, $x = \bigwedge Y$ implies that $x \in Y$. The following result provides an interesting generalization for posets of this latter alternative definition, which involves maximal lower bounds.

Theorem 6.1. *Let P be a poset and $x \in P$. The following statements are equivalent*:

(1) $x \in MI_P$;
(2) $\forall Y \subseteq P.\ x \in \text{mlb}(Y) \Rightarrow x \in Y$;
(3) $\forall Y \subseteq P.\ x \in \mathfrak{M}(Y) \Rightarrow x \in Y$.

Proof. (1) $\Rightarrow$ (2): Let $x \in \text{mlb}(Y)$. If $x \notin Y$ then $Y \subseteq\ \Uparrow x = \uparrow z$, for some $z \in P$. Hence, $x \leqslant z \in Y^{\downarrow}$, and therefore, by maximality of x, we get the contradiction $x = z$.

(2) $\Rightarrow$ (1): Let us first notice that $\Uparrow x \neq \emptyset$: Otherwise, we would have $x \in \max(P) = \text{mlb}(\emptyset)$, which, by hypothesis, is a contradiction. Thus, there exists some $z \in\ \Uparrow x$. By Hausdorff's maximal principle, there exists a maximal chain $Z \subseteq\ \Uparrow x$ containing z. Thus, $x < Z$ but $x \notin \text{mlb}(Z)$: Otherwise, by hypothesis, we would get the contradiction $x \in Z$. Hence, there exists some $y \in P$ such that $x < y \leqslant Z$. Since Z is a maximal chain in $\Uparrow x$, this implies that $y \in Z$, i.e., $y = \text{lst}(Z)$. Moreover, it turns out that $x \prec y$: In fact, if, for some w, $x < w \leqslant y$, then, still by maximality of the chain Z, $w \in Z$, and hence $w = y$. Let us show that $\Uparrow x = \uparrow y$. Assume by contradiction that there exists some $v > x$ such that $v \ngeqslant y$. Then, since $x < \{y, v\}$, by exploiting the hypothesis, $x \notin \text{mlb}(\{y, v\})$, and this implies that there exists some u such that $x < u \leqslant \{y, v\}$. It must be $u < y$, otherwise from $u = y$ we would get the contradiction $y \leqslant v$. But $x < u < y$ is a contradiction, since y is a cover of x.

(2) $\Rightarrow$ (3): Let $Y \subseteq P$. Let us first prove by transfinite induction that for any ordinal $\alpha \in \mathbb{O}$, if $x \in M^{\alpha}(Y)$ then $x \in Y$. The case $\alpha = 0$ is trivial because $Y = M^0(Y)$. If α is a successor ordinal then let $x \in M(M^{\alpha-1}(Y))$. Thus, there is some $S \subseteq M^{\alpha-1}(Y)$ such that $x \in \text{mlb}(S)$. Hence, by hypothesis, $x \in S$, and therefore $x \in M^{\alpha-1}(Y)$. Thus, by induction, $x \in Y$. If α is a limit ordinal then $x \in \bigcup_{\beta<\alpha} M^{\beta}(Y)$. Thus, there is some ordinal $\beta < \alpha$ such that $x \in M^{\beta}(Y)$, and therefore, by induction, $x \in Y$. To conclude, note that if $x \in \mathfrak{M}(Y)$ then there exists some ordinal α such that $x \in M^{\alpha}(Y)$, and therefore $x \in Y$.

(3) $\Rightarrow$ (2): If for some $Y \subseteq P$, $x \in \text{mlb}(Y)$, then $x \in M(Y) \subseteq \mathfrak{M}(Y)$, and therefore, by hypothesis, $x \in Y$. □

It is then natural to introduce the following definition of maximal lower bound generation.

Definition 6.2. Let P be a poset and $S \subseteq P$. Then, P is *mlb-generated* by S if $P = \mathfrak{M}(S)$. In particular, P is mlb-generated by completely meet-irreducibles if $P = \mathfrak{M}(MI_P)$.

We have therefore the following result of pseudocomplementation.

Theorem 6.3. *If P is a rmlb-complete poset which is mlb-generated by completely meet-irreducibles then* uco(P) *is a pseudocomplemented complete lattice*, *where*, *for any* $\rho \in \text{uco}(P)$, $\rho^* = \mathfrak{M}(MI_P \setminus \rho)$.

Proof. Let $\rho \in \mathrm{uco}(P)$. Then,

$$\begin{aligned}
&\rho \sqcap \mathfrak{M}(MI_P \setminus \rho) && = \text{(by Theorem 3.2)}\\
&\mathfrak{M}(\rho \cup \mathfrak{M}(MI_P \setminus \rho)) && = \text{(by (‡) in Section 3)}\\
&\mathfrak{M}(\mathfrak{M}(\rho) \cup \mathfrak{M}(\mathfrak{M}(MI_P \setminus \rho))) && = \text{(by idempotency of } \mathfrak{M})\\
&\mathfrak{M}(\mathfrak{M}(\rho) \cup \mathfrak{M}(MI_P \setminus \rho)) && = \text{(by (‡) in Section 3)}\\
&\mathfrak{M}(\rho \cup (MI_P \setminus \rho)) && \supseteq \text{(by monotonicity of } \mathfrak{M})\\
&\mathfrak{M}(MI_P) && = \text{(by hypothesis on } P)\\
&P.
\end{aligned}$$

Thus, $\rho \sqcap \mathfrak{M}(MI_P \setminus \rho) = \mathrm{id}$. Let $\psi \in \mathrm{uco}(P)$ such that $\rho \sqcap \psi = \mathrm{id}$. Let us show that $\psi \sqsubseteq \mathfrak{M}(MI_P \setminus \rho)$. By Theorem 3.2, $P = \rho \sqcap \psi = \mathfrak{M}(\rho \cup \psi)$. If $x \in MI_P$ then $x \in \mathfrak{M}(\rho \cup \psi) \cap MI_P$, and therefore, by Theorem 6.1, $x \in \rho \cup \psi$. Thus, $MI_P \subseteq \rho \cup \psi$. Hence, $MI_P \setminus \rho \subseteq \psi$, and therefore, by applying the monotone operator $\mathfrak{M}$, we get $\mathfrak{M}(MI_P \setminus \rho) \subseteq \psi$, as desired. □

By Section 4, we get the following straight consequence.

Corollary 6.4. *Let P be a rmlb-complete poset and $\rho, \eta \in \mathrm{uco}(P)$ such that $\eta \sqsubseteq \rho$. If η is mlb-generated by its set of completely meet-irreducibles then $\rho \rightsquigarrow \eta$ exists, and*[2] $\rho \rightsquigarrow \eta = \mathfrak{M}_\eta(MI_\eta \setminus \rho)$.

It is important to remark that Theorems 5.2 and 6.3 are orthogonal to each other, as the following example shows.

Example 6.5. Consider the unit interval of real numbers $U = \langle [0,1], \leqslant \rangle$, where $\leqslant$ is the standard ordering on reals. Observe that, being a complete chain, U is a meet-continuous complete lattice, and therefore Theorem 5.2 applies to U. By contrast, it turns out that U is not mlb-generated by completely meet-irreducibles: In fact, since for any $x \in [0,1]$, $\Uparrow x$ does not have the least element, we have that $MI_U = \emptyset$, and therefore $\mathfrak{M}(MI_U) = \{1\}$. Hence, Theorem 6.3 cannot be applied to U.

On the other hand, even for meet-semilattices, Theorem 6.3 is not subsumed by Theorem 5.2. In fact, consider the complete lattice C depicted in Fig. 3. Then, observe that $MI_C = \{y_i\}_{i \in \mathbb{N}} \cup \{z_i\}_{i \in \mathbb{N}}$ and $\mathfrak{M}(MI_C) = C$, and therefore Theorem 6.3 can be applied to C. Instead, observe that C is not meet-continuous: E.g., for any $k \in \mathbb{N}$, $y_k \wedge \bigvee_{i \in \mathbb{N}} x_i = y_k$, while $\bigvee_{i \in \mathbb{N}} y_k \wedge x_i = x_k$. Thus, Theorem 5.2 does not apply to C.

On mlb-generation by completely meet-irreducibles. It is well known that algebraic complete lattices are meet-generated by completely meet-irreducibles [8, Theorem 4.23, p. 93]. Given a poset P, by Theorem 6.1, one could arguably term an element $x \in MI_P$ as maximal lower bound irreducible. Thus, one could hope to generalize the above re-

[2] MI_η denotes the set of completely meet-irreducibles of the poset $\langle \eta, \leqslant \rangle$ (notice that this is not $MI_P \cap \eta$), and $\mathfrak{M}_\eta$ denotes the mlb-closure operator w.r.t. the poset $\langle \eta, \leqslant \rangle$.

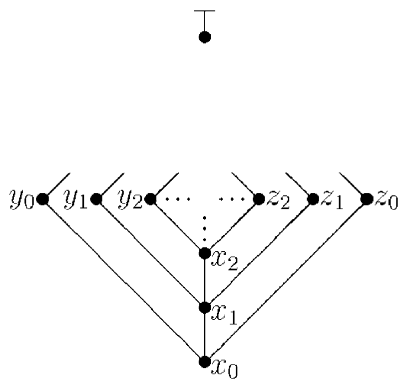

Fig. 3. The complete lattice C.

sult from algebraic complete lattices to algebraic posets as follows: If P is an algebraic poset then $P = \mathfrak{M}(MI_P)$. The next example shows that this is not the case.

Example 6.6. Consider the dcpo S in Fig. 2. Since $\mathbb{K}_S = S\setminus\{\top\}$, S turns out be an algebraic dcpo. On the other hand, we have that $MI_S = \emptyset$, and therefore $\mathfrak{M}(MI_S) = \{\top\}$.

On the other hand, just by adding a strong additional hypothesis to algebraicity, we observe the following fact.

Remark 6.7. If P is an algebraic join-semilattice then P is mlb-generated by completely meet-irreducibles.

Proof. It is well known that any dcpo with least element which is a join-semilattice actually is a complete lattice. Thus, if P has least element then P is an algebraic complete lattice, and therefore P is meet-generated, and hence mlb-generated, by completely meet-irreducibles. Otherwise, if P does not have least element, we consider the algebraic complete lattice $P_\perp$ obtained from P by adding a least element. Then, $P_\perp$ is still mlb-generated by completely meet-irreducibles, and hence this clearly holds for P as well. □

Thus, the above remark holds because an algebraic join-semilattice is "almost" an algebraic complete lattice, in the sense that only a least element may be missing.

On the other hand, any poset satisfying the ascending chain condition (ACC) is mlb-generated by completely meet-irreducibles.

Theorem 6.8. *Any ACC poset is mlb-generated by completely meet-irreducibles.*

Proof. Let us assume that $\{x \in P \mid x \notin \mathfrak{M}(MI_P)\} \neq \emptyset$. Thus, since P is ACC, there exists $x \in \max(\{x \in P \mid x \notin \mathfrak{M}(MI_P)\})$. Hence, $x \notin MI_P$, and therefore, by Theorem 6.1, there exists some $S \subseteq P$ such that $x \in \mathrm{mlb}(S)\setminus S$. Thus, if $s \in S$ then $x < s$, and therefore, by maximality of x, $s \in \mathfrak{M}(MI_P)$. Hence, from $S \subseteq \mathfrak{M}(MI_P)$ we obtain $\mathrm{mlb}(S) \subseteq M(\mathfrak{M}(MI_P)) = \mathfrak{M}(MI_P)$, and consequently, we get the contradiction $x \in \mathfrak{M}(MI_P)$. $\square$

Analogous to algebraic complete lattices [8, Remark 4.20, p. 92], we can still make the following observation.

Remark 6.9. If P is mlb-generated by S then $MI_P \subseteq S$.

Proof. Let $x \in MI_P$. By hypothesis, $x \in \mathfrak{M}(S)$. Thus, by Theorem 6.1, $x \in S$. $\square$

Determination of relevant classes of posets which are mlb-generated by completely meet-irreducibles appears to be an interesting problem—the problem of strengthening Theorem 6.8 is left open.

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References

[1] G. Bruns, A lemma on directed sets and chains, Arch. Math. 18 (1967) 561–563.
[2] A. Cortesi, G. Filé, R. Giacobazzi, C. Palamidessi, F. Ranzato, Complementation in abstract interpretation, ACM Trans. Program. Lang. Syst. 19 (1) (1997) 7–47.
[3] P. Cousot, R. Cousot, Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints, Conference Record of the Fourth ACM Symposium on Principles of Programming Languages (POPL'77), ACM Press, New York, 1977, pp. 238–252.
[4] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, Cambridge, UK, 1990.
[5] G. Filé, F. Ranzato, Complementation of abstract domains made easy, in: M. Maher (Ed.), Proceedings of the 1996 Joint International Conference and Symposium on Logic Programming, The MIT Press, Cambridge, MA, 1996, pp. 348–362.
[6] R. Freese, J.B. Nation, Clarification to 'Congruence lattices of semilattices', unpublished manuscript, Electronically available from Ralph Freese's Home Page: http://www.math.hawaii.edu/∼ralph, 1995.
[7] R. Giacobazzi, C. Palamidessi, F. Ranzato, Weak relative pseudo-complements of closure operators, Algebra Universalis 36 (3) (1996) 405–412.
[8] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, 1980.
[9] M. Hawrylycz, V. Reiner, The lattice of closure relations on a poset, Algebra Universalis 30 (3) (1993) 301–310.
[10] G. Markowsky, Chain-complete posets and directed sets with applications, Algebra Universalis 6 (1) (1976) 53–68.
[11] J. Morgado, Some results on the closure operators of partially ordered sets, Portugal. Math. 19 (2) (1960) 101–139.
[12] F. Nielson, H.R. Nielson, C. Hankin, Principles of Program Analysis, Springer, Berlin, 1999.
[13] F. Ranzato, Closures on CPOs form complete lattices, Inform. Comput. 152 (2) (1999) 236–249.
[14] F. Ranzato, A counterexample to a result concerning closure operators, Portugal. Math. 58 (1) (2001) 121–125.
[15] M. Ward, The closure operators of a lattice, Ann. Math. 43 (2) (1942) 191–196.