

# Closures on CPOs Form Complete Lattices

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It is well known that closure operators on a complete lattice, ordered pointwise, give rise to a complete lattice, and this basic fact plays an important rôle in many fields of the semantics area, notably in domain theory and abstract interpretation. We strengthen that result by showing that closure operators on any directed-complete partial order (CPO) still form a complete lattice. An example of application in abstract interpretation theory is given. © 1999 Academic Press

*Key Words:* closure operator; complete lattice; CPO; domain theory; abstract interpretation.

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## 1. INTRODUCTION

Closure operators (closures for short) have been extensively investigated from an order-theoretic viewpoint since the 1940s, and they find relevant applications in many fields of mathematics and theoretical computer science. In mathematics, closures play an important rôle in algebra, logic, and topology (see, e.g., Gierz *et al.*'s (1980) “compendium” for a wide range of applications in topological algebra). In theoretical computer science, closures have been widely used in the semantics area, notably in domain theory (e.g., Abramsky and Jung, 1994; Sanchis, 1977), in program semantics (e.g., Falaschi *et al.*, 1997; Germano and Mazzanti, 1991) and in the theory of semantics approximation by abstract interpretation (Cousot and Cousot 1977, 1979b).

### *Motivations*

One key issue concerning closure operators has been the structure of posets of closures ordered by the standard pointwise relation between functions, here denoted by  $\sqsubseteq$ . If  $P$  is any poset and  $uco(P)$  denotes the set of all closure operators on  $P$ , then  $\langle uco(P), \sqsubseteq \rangle$  is a poset. It is worth recalling two basic peculiarities: (i) the image of a closure coincides with the set of its fixpoints (i.e., closed elements); (ii) the pointwise ordering coincides with the superset relation on the corresponding images: If  $\rho, \eta \in uco(P)$  then  $\rho \sqsubseteq \eta$  iff  $\eta(P) \subseteq \rho(P)$ . From these two observations, it

is not hard to derive Ward's (1942, Theorem 4.2) basic theorem stating that if  $C$  is a complete lattice then  $uco(C)$  is a complete lattice as well. This basic result turns out to be crucial especially in abstract interpretation, a well-known general theory widely used for approximating the semantics of discrete dynamic systems, e.g., for designing and proving correct static program analyses (see Cousot, 1996, for a short introduction). Within this framework, the approximated abstract semantics is obtained from the standard one by substituting the actual domain of computation (called concrete) and its basic operations with, respectively, an abstract domain encoding some approximated properties of the concrete domain and the corresponding abstract operations. As argued by Cousot and Cousot (1979b), abstract domains can be specified as closure operators on the concrete domain. Thus, for a given concrete domain  $D$ ,  $uco(D)$  turns out to be the collection of all possible abstract domains of  $D$  (up to domain isomorphism). Further, the pointwise ordering of  $uco(D)$  corresponds precisely to the standard relation used in abstract interpretation to compare abstract domains with regard to their precision:  $A_1$  is more precise than  $A_2$  iff  $A_1 \sqsubseteq A_2$  in  $uco(D)$ . Since the concrete domain  $D$  is generally assumed to be a complete lattice, by Ward's theorem,  $uco(D)$  turns out to be the so-called (complete) lattice of abstract interpretations of  $D$  (cf. Cousot and Cousot, 1977, Section 7, 1979b, Section 8; Mycroft, 1981, Section 2.4.2). In practical cases, designing the abstract domain is the most critical step. Therefore, research has been devoted to devise operators that systematically refine and simplify abstract domains (Filé *et al.*, 1996; Giacobazzi and Ranzato, 1998b), like the reduced product (Cousot and Cousot, 1979b), disjunctive completion (Cousot and Cousot, 1979b), complementation (Cortesi *et al.*, 1997), and least disjunctive basis (Giacobazzi and Ranzato, 1998a). Among the benefits, these operators provide high-level facilities to tune static program analyses in accuracy and cost. The definitions of most of these operators heavily rely on the hypothesis that  $uco(D)$  is a complete lattice, since they work by minimally transforming abstract domains w.r.t. a given domain property. However, in many common situations, e.g., in denotational semantics, often concrete domains are only required to be mere algebraic directed-complete partial orders (CPOs), or just something more, and, therefore, in these cases, one cannot resort to Ward's theorem. For example, the concrete domains in Jensen's (1997, Section 4) abstract interpretation framework for disjunctive functional program analysis are not complete lattices. Furthermore, as remarkable examples, Plotkin and Smyth powerdomain constructions, in general, do not give rise to complete lattices, and the Plotkin powerdomain is used by Mycroft (1981) for strictness and termination analysis of lazy functional programs, while a modified Plotkin powerdomain is used by Mycroft and Nielson (1983) (see also Nielson, 1984, Chapt. 5) for the so-called strong approach to abstract interpretation. These motivations stimulated us in trying to strengthen Ward's theorem, with the aim of proving that most posets ordinarily used in theoretical computer science give rise to complete lattices of closures.

### *Contributions of the Paper*

In general, it is not true that, for any poset  $P$ ,  $\langle uco(P), \sqsubseteq \rangle$  turns out to be always a complete lattice, as shown by Example 2.1 given later in Section 2. Morgado (1960, Theorem 28) gave a theorem characterizing all and only the posets  $P$  such

that  $uco(P)$  is a complete lattice, based on a notion of relative quasi-infimum in posets (Morgado, 1960, Definition 4). Unfortunately, Morgado's result is based on some erroneous lemmata (we refer to Ranzato, 1998, for details) and, therefore, it is unusable.

CPOs are largely used in various fields of theoretical computer science, and within the semantics area and in domain theory, being a CPO is definitely a very basic requirement for a poset. However, to the best of our knowledge, there are no available results concerning posets of closures on CPOs—we became aware of an unpublished related result by Abramsky (1999), who proved that continuous closure operators on CPOs form complete lattices, only when preparing the final version of this paper. The purpose of this work is to fill this gap by showing that closures on a CPO always give rise to a complete lattice.

Although Morgado's (1960) results contain a flaw, our approach has been loosely inspired by his work, in particular by an erroneous interpretation (Morgado, 1960, p. 120) of the notion of relative quasi-infimum in posets. By generalizing Hawrylycz and Reiner's (1993) use of maximal lower bounds for characterizing closure operators on finite (or satisfying the ascending chain condition, ACC) posets, we define a poset  $P$  to be relatively maximal lower bound complete (rmlb-complete for short) whenever for any  $Y \subseteq P$  and for any lower bound  $x$  of  $Y$ , the set of maximal lower bounds of  $Y$  which follow  $x$  is nonempty. Our proof consists in demonstrating that closures on rmlb-complete posets form complete lattices and then that CPOs are rmlb-complete. For any rmlb-complete poset  $P$ , the proof explicitly characterizes lub's, glb's, and the greatest element in  $uco(P)$ . It turns out that the lub coincides with the one of Ward's theorem—which is set-intersection of closure images—while the glb and the greatest element are characterized through a construction based on maximal lower bounds. We also give examples showing that if  $uco(P)$  is a complete lattice then  $P$  is not necessarily a rmlb-complete poset and that the class of CPOs is strictly contained in the class of rmlb-complete posets. It is worthwhile to mention that the proof showing that CPOs are rmlb-complete makes use of the axiom of choice by exploiting a formulation of Zorn's lemma involving CPOs. It turns out that Abramsky's (1999) proof for continuous closures on CPOs mentioned above can be adapted to our case of not-necessarily continuous closures. As we sketch at the end of Section 4, this provides an alternative proof which does not make use of the axiom of choice.

In order to show the usefulness of our main result, we exploit it in abstract interpretation theory for Cortesi *et al.*'s (1997) domain complementation operator. According to Cortesi *et al.* (1997), given a domain  $D$  and an abstract domain  $A \in uco(D)$ , the domain complement of  $A$  in  $D$  is defined as the lattice-theoretic pseudocomplement of  $A$  in  $uco(D)$ . Existence of such pseudocomplements is ensured by a Giacobazzi *et al.*'s (1996) result, showing that if  $D$  is a meet-continuous complete lattice then  $uco(D)$  is pseudocomplemented. We present here a novel theorem of pseudocomplementation, whose proof relies on our main result. If  $D$  is any poset satisfying the ACC then  $uco(D)$  is pseudocomplemented (actually, a slightly more general result is proved). This turns out to be a significant extension, since most practically used abstract domains satisfy the ACC in order to ensure that fixpoint computations are finitely convergent.

2. BASIC NOTATION AND CLOSURE OPERATORS

Basic Notation

We shall sometimes use Church's lambda notation for functions. The identity operator on any set will be denoted by *id*. If  $f: X \rightarrow Y$  then  $f(X) \stackrel{\text{def}}{=} \{f(x) \mid x \in X\}$ . Let  $\langle P, \leq \rangle$  be a poset. If  $x \in P$  and  $Y \subseteq P$  then we write  $x \leq Y$  when, for any  $y \in Y$ ,  $x \leq y$ , i.e., when  $x$  is a lower bound of  $Y$ . Let us remark that it follows that, for any  $x \in P$ ,  $x \leq \emptyset$ . If  $Y \subseteq P$  then  $Y$  has least element  $\text{lst}(Y)$  when  $\text{lst}(Y) \leq Y$  and  $\text{lst}(Y) \in Y$ . If  $Y \subseteq P$  then  $\text{max}(Y) \stackrel{\text{def}}{=} \{y \in Y \mid \forall z \in Y. (y \leq z) \Rightarrow (y = z)\}$ . If  $x \in P$  then  $\uparrow x \stackrel{\text{def}}{=} \{y \in P \mid x \leq y\}$ . Pointwise ordering between functions is denoted by the symbol  $\sqsubseteq$ : If  $f, g: Q \rightarrow P$ , then  $f \sqsubseteq g$  iff for any  $x \in Q$ ,  $f(x) \leq g(x)$ .

We will use the usual standard symbols for complete lattices:  $\vee$  and  $\wedge$  denote, respectively, the least upper bound (lub) and greatest lower bound (glb) operations, and  $\top$  and  $\perp$  denote, respectively, greatest and least elements. As usual, we assume that the glb (lub) of the empty set is the greatest (least) element.

A directed-complete partial order (CPO) here is not required to have necessarily the least element. We will use the following equivalent formulation of Zorn's lemma involving CPOs (Davey and Priestley, 1990, Theorem 4.22, p. 101).

ZORN'S LEMMA. *If  $P$  is a CPO then  $\text{max}(P) \neq \emptyset$ .*

Let us also recall the notion of pseudocomplement. Given a meet semilattice with least element  $\langle L, \leq, \wedge, \perp \rangle$ , the pseudocomplement of  $x \in L$ , if it exists, is the (unique) element  $x^* \in L$  such that  $x \wedge x^* = \perp$  and  $\forall y \in L. (x \wedge y = \perp) \Rightarrow (y \leq x^*)$ . When  $L$  is a complete lattice, if the pseudocomplement  $x^*$  exists then it is characterized as  $x^* = \vee \{y \in L \mid x \wedge y = \perp\}$ . If every element in  $L$  has the pseudocomplement, then  $L$  is called pseudocomplemented.

Closure Operators

An (upper) closure operator (briefly, closure) on a poset  $\langle P, \leq \rangle$  is an operator  $\rho: P \rightarrow P$  monotone, idempotent, and increasing (i.e.,  $\forall x \in P. x \leq \rho(x)$ ). Fixpoints of a closure are also called closed elements. Closures will be denoted by lowercase Greek letters  $\rho, \eta, \mu, \dots$ . Let  $\text{uco}(P)$  denote the set of all closure operators on the poset  $P$ . Closures on posets are partially ordered by pointwise ordering, i.e.,  $\langle \text{uco}(P), \sqsubseteq \rangle$  is a poset. Throughout the paper for any  $\rho \in \text{uco}(P)$ , we follow a standard notation by denoting the image  $\rho(P)$  simply by  $\rho$  itself: This does not give rise to ambiguity, since one can immediately distinguish the use as a function or set, according to the context. Let  $\rho, \eta \in \text{uco}(P)$ . The following are some known basic easy properties of closures on posets (cf. Hawrylycz and Reiner, 1993; Monteiro and Ribeiro, 1942; Morgado, 1960) that will be used throughout the paper:

- (1) the image of a closure coincides with its set of closed elements:  $\rho = \{x \in P \mid x = \rho(x)\}$ ;
- (2) the image of a closure is closed for maximal lower bounds: If  $S \subseteq \rho$  then  $\text{max}(\{x \in P \mid x \leq S\}) \subseteq \rho$ ;
- (3) as a consequence of (2), maximal elements are always closed:  $\text{max}(P) \subseteq \rho$ ;

(4) a subset  $Y \subseteq P$  is the set of fixpoints of a closure  $\rho \in uco(P)$  iff for all  $x \in P$ ,  $Y \cap \uparrow x$  has the least element; in such a case,  $\rho = \lambda x. lst(Y \cap \uparrow x)$ ;

(5) pointwise ordering coincides with the superset relation on the corresponding images:  $\rho \sqsubseteq \eta \Leftrightarrow \eta \subseteq \rho$ ;

(6) the identity is the least closure:  $id \in uco(P)$  and  $id \sqsubseteq \rho$ .

The following simple example shows that with no hypothesis on a poset  $P$ ,  $\langle uco(P), \sqsubseteq \rangle$  is not, in general, a complete lattice.

**EXAMPLE 2.1.** Consider the poset  $\mathbb{N}$  of nonnegative integers endowed with their standard ordering (i.e., the first infinite ordinal), and let us show that  $uco(\mathbb{N})$  does not have the greatest element. Consider, for any  $k \in \mathbb{N}$ ,  $\rho_k \stackrel{\text{def}}{=} \uparrow k$ . Clearly, any  $\rho_k$  is the set of fixpoints of a closure on  $\mathbb{N}$ . Hence, if the greatest element  $\tau \in uco(\mathbb{N})$  would exist, from  $\rho_k \sqsubseteq \tau$  for any  $k$ , by point (5) above, we would have that  $\tau \subseteq \bigcap_{k \in \mathbb{N}} \rho_k = \emptyset$ , which is a contradiction, since the set of fixpoints of a closure is never empty.

Ward's theorem (1942, Theorem 4.2) states that when  $C$  is a complete lattice,  $\langle uco(C), \sqsubseteq \rangle$  is a complete lattice, which is dually isomorphic to the complete lattice of all complete meet subsemilattices of  $C$ , ordered by set-inclusion. More in detail,  $\langle uco(C), \sqsubseteq, \bigsqcup, \bigsqcap, \lambda x. \top, id \rangle$  is a complete lattice, where for all  $\{\rho_i\}_{i \in I} \subseteq uco(C)$ :

- $(\bigsqcup_{i \in I} \rho_i)(C) = \bigcap_{i \in I} \rho_i$ ;
- $\bigsqcap_{i \in I} \rho_i = \lambda x. \bigwedge_{i \in I} \rho_i(x)$ ;
- $\lambda x. \top$  and  $id$  are, respectively, the greatest and least elements.

In particular, let us remark that:

(7) A subset  $Y \subseteq C$  is the set of fixpoints of a closure  $\rho \in uco(C)$  iff  $Y$  is meet-closed, i.e., for all  $S \subseteq Y$ ,  $\bigwedge S \in Y$ ; in such a case,  $\rho = \lambda x. \bigwedge (Y \cap \uparrow x)$ .

Moreover, let us also recall that  $uco(C)$  enjoys the property of being co-atomic (Ward, 1942, Theorem 5.1), namely co-atoms of  $uco(C)$  (i.e., closures covered by the greatest closure  $\lambda x. \top$ ) meet-generate all  $uco(C)$ .

### 3. RMLB-COMPLETE POSETS

Given a poset  $\langle P, \leq \rangle$  and any (possibly empty) subset  $Y \subseteq P$ ,  $Y^\downarrow$  will denote the set of lower bounds of  $Y$  in  $P$ , i.e.,  $Y^\downarrow \stackrel{\text{def}}{=} \{x \in P \mid x \leq Y\}$ . In particular,  $\emptyset^\downarrow = P$ .

**DEFINITION 3.1.** A poset  $P$  is *relatively maximal lower bound complete* (briefly, *rmlb-complete*) if for any  $Y \subseteq P$  and  $x \in P$ ,  $x \in Y^\downarrow \Rightarrow \max(Y^\downarrow) \cap \uparrow x = \emptyset$ .

Let us remark that a weakening to finite subsets only of the dual definition of rmlb-completeness (i.e., involving minimal upper bounds) yields one of the defining conditions (within one of the possible equivalent characterizations; cf. Abramsky and Jung, 1994, Section 4.2.1) of the so-called SFP or bifinite domains, introduced by Plotkin (1977, Section 4). Also, let us point out that if  $P$  is rmlb-complete then

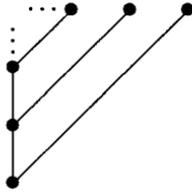


FIG. 1. The rmlb-complete poset  $R$ .

any  $x \in P$  is followed by some maximal element of  $P$ , because  $\max(P) \cap \uparrow x = \max(\emptyset^\downarrow) \cap \uparrow x \neq \emptyset$ . Thus, for instance, the poset  $\mathbb{N}$  of nonnegative integers considered in Example 2.1 is not rmlb-complete. The following result proves that any CPO is rmlb-complete.

LEMMA 3.2. *Any CPO is rmlb-complete.*

*Proof.* Let  $P$  be a CPO,  $Y \subseteq P$ , and  $x \in Y^\downarrow$ . Observe that  $Y^\downarrow \cap \uparrow x$  is a subCPO of  $P$ . In fact, since  $x \in Y^\downarrow \cap \uparrow x$ ,  $Y^\downarrow \cap \uparrow x$  is nonempty, and if  $Z \subseteq Y^\downarrow \cap \uparrow x$  is a directed subset, then  $x \leq \bigvee Z \leq Y$ . Thus, by Zorn's lemma in Section 2,  $\max(Y^\downarrow \cap \uparrow x) \neq \emptyset$ . Moreover, observe that  $\max(Y^\downarrow \cap \uparrow x) = \max(Y^\downarrow) \cap \uparrow x$ , and therefore, this concludes the proof. ■

The converse of the above result does not hold. In fact, it is easy to check that the poset  $R$  depicted in Fig. 1 is rmlb-complete, although  $R$  is not a CPO.

#### 4. CLOSURES ON CPOS

In this section, we will show that closures on rmlb-complete posets form complete lattices, thus obtaining, as a consequence of Section 3, that closures on CPOs form complete lattices as well. Let us first introduce a key definition of mlb-closedness for subsets of a poset.

DEFINITION 4.1. Let  $\langle P, \leq \rangle$  be a poset. A subset  $Y \subseteq P$  is *closed for maximal lower bounds* (mlb-closed for short) if

$$Y = M(Y) \stackrel{\text{def}}{=} \bigcup_{S \subseteq Y} \max(S^\downarrow).$$

Let  $Y \subseteq P$ . Let us remark that if  $Y$  is mlb-closed then  $Y$  contains the maximal elements of  $P$ , since  $\max(P) = \max(\emptyset^\downarrow) \subseteq Y$ . Thus, in particular, notice that  $M(\emptyset) = \max(P)$ . Moreover, since for any  $y \in Y$ ,  $\max(\{y\}^\downarrow) = \{y\}$ , notice that  $Y \subseteq M(Y)$  always holds.

LEMMA 4.2. *For any poset  $P$ , mlb-closed subsets of  $P$  are closed under arbitrary set-intersections.*

*Proof.* Let  $\{Y_i\}_{i \in I} \subseteq \wp(P)$  be a (possibly empty) family of mlb-closed subsets, and let us show that  $M(\bigcap_{i \in I} Y_i) \subseteq \bigcap_{i \in I} Y_i$ . If  $S \subseteq \bigcap_{i \in I} Y_i$  and  $x \in \max(S^\downarrow)$ , then, for any  $j \in I$ ,  $x \in M(Y_j) = Y_j$ , and therefore,  $x \in \bigcap_{i \in I} Y_i$ . ■

Since  $\langle \wp(P), \subseteq \rangle$  is a complete lattice, by (7) in Section 2, Lemma 4.2 means that mlb-closed subsets form the set of fixpoints of a closure operator  $\mathfrak{M}$  on the powerset of  $P$ . Hence, the closure  $\mathfrak{M} \in uco(\wp(P))$  is defined by

$$\mathfrak{M}(X) \stackrel{\text{def}}{=} \bigcap \{ Y \in \wp(P) \mid X \subseteq Y, M(Y) = Y \}. \quad (\dagger)$$

Thus,  $\mathfrak{M}(X)$ , called the mlb-closure of  $X$ , is the least (w.r.t. set-inclusion) mlb-closed subset containing  $X$ —in particular, let us note that the mlb-closure of the empty set coincides with the mlb-closure of the set of maximal elements of  $P$ . Equivalently, the mlb-closure of  $X$  is the least fixpoint of  $M$  containing  $X$ . As a consequence, let us remark that, by the transfinite formulation of Knaster–Tarski’s fixpoint theorem,  $\mathfrak{M}(X)$  can be obtained by applying transfinitely often  $M$  starting from  $X$ . In fact,  $\bigcup_{\alpha \in \mathbb{O}} M^\alpha(X)$  is the least fixpoint of  $M$  above  $X$ , where, for any ordinal  $\alpha \in \mathbb{O}$ , the ordinal (upper)  $\alpha$ -power  $M^\alpha(X)$  is defined by transfinite induction as:  $X$  if  $\alpha = 0$ ;  $M(M^{\alpha-1}(X))$  if  $\alpha$  is a successor ordinal;  $\bigcup_{\gamma < \alpha} M^\gamma(X)$  if  $\alpha$  is a limit ordinal.

Still concerning the aforementioned class of Plotkin’s bifinite domains, let us recall that one of the possible equivalent definitions for such class involves a dual mlb-closure, i.e., the minimal upper bound closure (cf. Abramsky and Jung, 1994, Section 4.2.1; Plotkin, 1977, Theorem 5).

In Section 5 we will need the following property of the mlb-closure.

LEMMA 4.3. *For all  $X \in \wp(P)$  and  $y \in P$ ,  $\mathfrak{M}(X) \cap \uparrow y \subseteq \mathfrak{M}(X \cap \uparrow y)$ .*

*Proof.* Let us first prove that  $M(X) \cap \uparrow y \subseteq M(X \cap \uparrow y)$  ( $\dagger$ ). Let  $z \in M(X) \cap \uparrow y$ , i.e.,  $z \in \max(S^\downarrow) \cap \uparrow y$  for some  $S \subseteq X$ . Then, since  $S \cap \uparrow y \subseteq X \cap \uparrow y$ , let us show that  $z \in \max((S \cap \uparrow y)^\downarrow)$ . Note that from  $y \leq z$  and  $z \in S^\downarrow$ , we get  $y \in S^\downarrow$ . Hence,  $S \cap \uparrow y = S$ , and therefore,  $(S \cap \uparrow y)^\downarrow = S^\downarrow$  and, in turn,  $\max((S \cap \uparrow y)^\downarrow) = \max(S^\downarrow)$ . Thus,  $z \in \max((S \cap \uparrow y)^\downarrow)$  and, therefore,  $z \in M(X \cap \uparrow y)$ .

Now, let us show by transfinite induction that for any ordinal  $\alpha \in \mathbb{O}$ , we have that  $M^\alpha(X) \cap \uparrow y \subseteq M^\alpha(X \cap \uparrow y)$  (\*). This trivially holds for  $\alpha = 0$ . For successor ordinals, we have that:

$$\begin{aligned} M^{\alpha+1}(X) \cap \uparrow y &= M(M^\alpha(X)) \cap \uparrow y && \text{(by } (\dagger) \text{)} \\ &\subseteq M(M^\alpha(X) \cap \uparrow y) && \text{(by induction and monotonicity of } M \text{)} \\ &\subseteq M(M^\alpha(X \cap \uparrow y)) \\ &= M^{\alpha+1}(X \cap \uparrow y). \end{aligned}$$

For a limit ordinal  $\alpha$  for any  $\gamma < \alpha$ , by induction,  $M^\gamma(X) \cap \uparrow y \subseteq M^\gamma(X \cap \uparrow y)$ , and therefore,

$$\begin{aligned} M^\alpha(X) \cap \uparrow y &= \left( \bigcup_{\gamma < \alpha} M^\gamma(X) \right) \cap \uparrow y \\ &= \bigcup_{\gamma < \alpha} (M^\gamma(X) \cap \uparrow y) \\ &\subseteq \bigcup_{\gamma < \alpha} M^\gamma(X \cap \uparrow y) \\ &= M^\alpha(X \cap \uparrow y). \end{aligned}$$

Thus, we conclude

$$\begin{aligned}
 \mathfrak{M}(X) \cap \uparrow y &= \left( \bigcup_{\alpha \in \mathbb{O}} M^\alpha(X) \right) \cap \uparrow y \\
 &= \bigcup_{\alpha \in \mathbb{O}} (M^\alpha(X) \cap \uparrow y) \quad (\text{by } (*)) \\
 &\subseteq \bigcup_{\alpha \in \mathbb{O}} M^\alpha(X \cap \uparrow y) \\
 &= \mathfrak{M}(X \cap \uparrow y). \quad \blacksquare
 \end{aligned}$$

The following result provides a crucial relationship between the notions of rmlb-complete poset and mlb-closure. This generalizes (Hawrylycz and Reiner, 1993, Proposition 3, p. 303) which was given under the strong hypothesis of finite (or satisfying the ACC) posets.

**THEOREM 4.4.** *Let  $P$  be a rmlb-complete poset and  $Y \subseteq P$ . Then,  $Y$  is the set of fixpoints of a closure operator on  $P$  iff  $Y = \mathfrak{M}(Y)$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\rho_Y \in \text{uco}(P)$  be the closure whose set of fixpoints is  $Y$ . Let us show that  $Y = M(Y)$ . So, let  $x \in M(Y)$ , i.e.,  $x \in \max(S^\downarrow)$  for some  $S \subseteq Y$ . Then, for any  $s \in S$ ,  $\rho_Y(x) \leq \rho_Y(s) = s$ , and therefore,  $\rho_Y(x) \in S^\downarrow$ . Thus, by maximality of  $x$ , from  $x \leq \rho_Y(x)$ , we get that  $x = \rho_Y(x)$  and, hence,  $x \in Y$ . Consequently,  $\mathfrak{M}(Y) = M(Y) = Y$ .

( $\Leftarrow$ ) By (4) in Section 2, we have to show that for any  $x \in P$ ,  $Y \cap \uparrow x$  has the least element. Let  $x \in P$ . Since  $x \leq Y \cap \uparrow x$ , by rmlb-completeness of  $P$ , we get that  $\max((Y \cap \uparrow x)^\downarrow) \cap \uparrow x \neq \emptyset$ . Moreover, since  $Y \cap \uparrow x \subseteq Y$  and  $M(Y) = Y$ , we also have that  $\max((Y \cap \uparrow x)^\downarrow) \subseteq Y$  ( $\dagger$ ). Let us show that  $\max((Y \cap \uparrow x)^\downarrow) \cap \uparrow x = \{m\}$  for some  $m \in P$ . If  $a, b \in \max((Y \cap \uparrow x)^\downarrow) \cap \uparrow x$  then  $x \leq a, b \leq (Y \cap \uparrow x)$ , and since, by ( $\dagger$ ),  $a, b \in Y$ , we therefore have that  $a \leq b$  and  $b \leq a$ , i.e.,  $a = b$ . Thus,  $\max((Y \cap \uparrow x)^\downarrow) \cap (Y \cap \uparrow x) = \{m\}$ , namely  $m = \text{lst}(Y \cap \uparrow x)$ .  $\blacksquare$

It is worth noting that if  $P$  is a complete lattice then, for any  $Y \subseteq P$ ,  $\mathfrak{M}(Y)$  results to be the meet-closure of  $Y$  in  $P$ . Thus, the above result can be viewed as a generalization of (7) in Section 2 from complete lattices to rmlb-complete posets. We exploit Theorem 4.4 in order to prove the following key result.

**THEOREM 4.5.** *If  $P$  is a rmlb-complete poset then  $\langle \text{uco}(P), \sqsubseteq \rangle$  is a complete lattice, where if  $\{\rho_i\}_{i \in I} \subseteq \text{uco}(P)$ , then  $\mathfrak{M}(\bigcup_{i \in I} \rho_i)$  and  $\bigcap_{i \in I} \rho_i$  are, respectively, the sets of fixpoints of the glb and lub in  $\text{uco}(P)$  of  $\{\rho_i\}_{i \in I}$ .*

*Proof.* By Theorem 4.4,  $\mathfrak{M}(\bigcup_{i \in I} \rho_i)$  is the least subset of  $P$  which contains  $\bigcup_{i \in I} \rho_i$  and is (the set of fixpoints of) a closure on  $P$ . Thus, by point (5) in Section 2,  $\mathfrak{M}(\bigcup_{i \in I} \rho_i)$  is the glb of  $\{\rho_i\}_{i \in I}$ . On the other hand, by Lemma 4.2 and Theorem 4.4,  $\mathfrak{M}(\bigcap_{i \in I} \rho_i) = \bigcap_{i \in I} \rho_i$ . Hence, by point (5) in Section 2,  $\bigcap_{i \in I} \rho_i$  is (the set of fixpoints of) the lub of  $\{\rho_i\}_{i \in I}$ .  $\blacksquare$

By Lemma 3.2 and Theorem 4.5, we therefore get the following relevant consequence.

**COROLLARY 4.6.** *If  $P$  is a CPO then  $\text{uco}(P)$  is a complete lattice.*

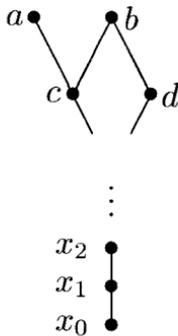


FIG. 2. The poset  $S$ .

It is worthwhile to read dually the above result for *lower* closure operators, also known as projections. Recall that, given a poset  $P$ ,  $\varphi: P \rightarrow P$  is a lower closure operator if  $\varphi$  is monotone, idempotent, and decreasing (i.e.,  $\forall x \in P. \varphi(x) \leq x$ ). Hence, if  $P^{op}$  denotes the dual poset of  $P$  and  $lco(P)$  denotes the set of all lower closure operators on  $P$ , then  $\varphi \in lco(P)$  iff  $\varphi \in uco(P^{op})$ . Then, it is easy to check that  $\langle lco(P), \sqsubseteq \rangle$  is isomorphic (via the identity mapping) to  $\langle uco(P^{op}), \supseteq \rangle$ . Thus, by Corollary 4.6, we have that if  $P$  is a co-CPO then  $\langle lco(P), \sqsubseteq \rangle$  is a complete lattice.

Rmlb-completeness does not characterize all and only the posets  $P$  such that  $uco(P)$  is a complete lattice. In fact, the following example, analogous to (Morgado, 1960, Example 7), shows that the converse of Theorem 4.5 does not hold.

EXAMPLE 4.7. Consider the poset  $S$  diagrammed in Fig. 2. Observe that  $S$  is not rmlb-complete: In fact, for any  $i \in \mathbb{N}$ ,  $\max(\{a, b, c, d\}^\perp) \cap \uparrow x_i = \emptyset$ . In spite of that,  $uco(S)$  is a complete lattice. The reason why  $uco(S)$  turns out to be a complete lattice should be clear. All glb's of  $uco(S)$  do exist, because the unique closure whose set of fixpoints is contained in  $\{a, b, c, d\}$  is the greatest closure, whose set of fixpoints is  $\{a, b, c\}$ : In fact, the only other possibility is given by  $\{a, b, c, d\}$ , which is not the set of fixpoints of a closure on  $S$ .

Let us remark that, given any family of closures  $\{\rho_i\}_{i \in I} \subseteq uco(P)$ , Theorem 4.5 characterizes explicitly its glb. In fact, by (4) in Section 2, the glb in  $uco(P)$  is  $\lambda x. lst(\mathfrak{M}(\bigcup_{i \in I} \rho_i) \cap \uparrow x)$ . This is a transfinite constructive characterization of the glb operation on closures, in the sense that each closed element  $lst(\mathfrak{M}(\bigcup_{i \in I} \rho_i) \cap \uparrow x)$  can be obtained from the possibly transfinite stationary limit of the iteration sequence for the operator  $M$  of Definition 4.1, starting from the set  $\bigcup_{i \in I} \rho_i$ . It is worth noticing that such approach is therefore reminiscent of the work by Cousot and Cousot (1979a), who gave a transfinite constructive characterization for the lub operation in complete lattices of closures defined on a complete lattice. As a particular relevant case, let us remark that Theorem 4.5 also characterizes the greatest closure of  $uco(P)$  as  $\lambda x. lst(\mathfrak{M}(\max(P)) \cap \uparrow x)$ , since, as noted above,  $\mathfrak{M}(\emptyset) = \mathfrak{M}(\max(P))$ . The least closure, of course, remains the identity *id*. Hence, complete lattices of closures on CPOs have been fully characterized.



FIG. 3. The poset  $T$  on the left, and the lattice  $uco(T)$  on the right.

We recalled in Section 2 that complete lattices of closures on complete lattices are co-atomic. The following finite example shows that this property does not hold anymore for closures on CPOs.

EXAMPLE 4.8. Consider the finite poset  $T$  depicted on the left of Fig. 3. It is a routine task to check that all the closure operators on  $T$  are

$$\begin{aligned} \rho_1 &= \{a, b, d\}, & \rho_2 &= \{a, b, d, f\}, & \rho_3 &= \{a, b, d, e\}, \\ \rho_4 &= \{a, b, d, e, f\}, & \rho_5 &= \{a, b, c, d, e\}, & \rho_6 &= T. \end{aligned}$$

By Corollary 4.6, they form the lattice  $uco(T)$  in Fig. 3, which obviously is not co-atomic.

*An Alternative Proof of Corollary 4.6*

The above proof of Corollary 4.6 makes use of the axiom of choice in Lemma 3.2, where Zorn’s lemma, as given in Section 2, is exploited. On the other hand, ordinals are not used, because the operator  $M$  of Definition 4.1 is monotone and increasing on a complete lattice (the powerset of the base poset), and hence, Knaster–Tarski’s fixpoint theorem on complete lattices allows us to define the operator  $\mathfrak{M}$  by the formula (‡). Samson Abramsky and Gordon Plotkin pointed out to the author that an alternative proof of Corollary 4.6 can be obtained by adapting a proof of an unpublished recent related result by Abramsky (1999). While working on a concurrent game semantics for linear logic, where strategies are represented by continuous closure operators (Abramsky and Melliès, 1999), Abramsky (1999) proved that *continuous* closure operators on CPOs form complete lattices. For the sake of comparison, let us sketch how Abramsky’s proof—some ideas of such a proof may well be folklore, e.g., (Borceux and Kelly, 1987, Sections 4 and 5; Escardó, 1998, Lemma 3.1.8)—can be adapted in order to prove Corollary 4.6.

Let  $P$  be a CPO. Given any monotone and increasing operator  $f: P \rightarrow P$ , for any  $x \in P$  there exists the least fixpoint of  $f$  which is greater than or equal to  $x$ , denoted by  $lfp_x(f)$ . It is easy to check that  $\lambda x.lfp_x(f) \in uco(P)$ . Moreover, it turns out that  $\lambda x.lfp_x(f)$  is the least (w.r.t.  $\sqsubseteq$ ) closure on  $P$  which is greater than or equal to  $f$ : If  $\rho \in uco(P)$  and  $f \sqsubseteq \rho$  then, for any  $x \in P$ ,  $\rho(x) \leq f(\rho(x)) \leq \rho(\rho(x)) = \rho(x)$ , i.e.,  $\rho(x)$  is a fixpoint of  $f$  such that  $x \leq \rho(x)$ , and therefore,  $lfp_x(f) \leq \rho(x)$ . In order to show that  $\langle uco(P), \sqsubseteq \rangle$  is a complete lattice, it is enough to check that it has

the least element, binary (i.e., finite) lub's, and directed lub's. By (6) in Section 2,  $id$  is the least element. If  $\rho, \eta \in uco(P)$ , then  $\rho \circ \eta$  is monotone and increasing on  $P$ , and  $\rho, \eta \sqsubseteq \rho \circ \eta$ . Then,  $\lambda x. lfp_x(\rho \circ \eta) \in uco(P)$  turns out to be the lub of  $\{\rho, \eta\}$ : It is an upper bound, and if  $\mu \in uco(P)$  is an upper bound of  $\{\rho, \eta\}$  then  $\rho \circ \eta \sqsubseteq \mu$  and, therefore, by what is shown above,  $\lambda x. lfp_x(\rho \circ \eta) \sqsubseteq \mu$ . Analogously, one shows that if  $R$  is a directed subset of  $uco(P)$ , then the pointwise directed lub  $\lambda y. \bigvee_{\rho \in R} \rho(y)$  is a monotone and increasing upper bound of  $R$ , and therefore,  $\lambda x. lfp_x(\lambda y. \bigvee_{\rho \in R} \rho(y))$  is the lub in  $uco(P)$  of  $R$ .

This proof does not need the axiom of choice. Furthermore, existence of the elements  $lfp_x(f)$  for any increasing and monotone operator  $f$  on a CPO can be proved without resorting to ordinals via Patarai's (1997) constructive proof of Knaster–Tarski's fixpoint theorem for monotone operators on CPOs. Patarai's proof is constructive and uses neither the axiom of choice nor ordinal theory (the proof could work in any topos), though it is impredicative (hence, it cannot be given within Martin–Löf's intuitionistic theory). On the other hand, we actually prove something more than Corollary 4.6. Without resorting to the axiom of choice, Theorem 4.5 shows that the result holds for the class of rmlb-complete posets, which, as shown in Lemma 3.2 by using the axiom of choice, properly includes the class of CPOs.

## 5. AN APPLICATION IN ABSTRACT INTERPRETATION

Complementation (Cortesi *et al.*, 1997) corresponds to the inverse operation (in the sense of Filé *et al.*, 1996; Giacobazzi and Ranzato, 1998b) of Cousot and Cousot's (1979b) reduced product of abstract domains. Given two domains  $A$  and  $D$  such that  $A$  is an abstraction of  $D$ , the complement of  $A$  in  $D$ , when it exists, is the most abstract domain  $D \sim A$  whose reduced product with  $A$  is exactly  $D$ . Complementation turns out to be a very useful tool for decomposing abstract domains in minimal components (Cortesi *et al.*, 1997; Filé and Ranzato, 1996), thus providing compact representations for complex domains and allowing modular verification. By the equivalence between closure operators and abstract domains,  $A$  is viewed as a closure operator on  $D$  and the glb operation of  $uco(D)$  coincides with the reduced product of abstract domains. Thus, Cortesi *et al.* (1997) argued that such a complement  $D \sim A$  exists precisely when the lattice-theoretic pseudo-complement of  $A$  in  $uco(D)$  exists.

The lattice-theoretic basis of domain complementation is provided by a Giacobazzi *et al.*'s (1996, Theorem 3.1) result. In order to recall it, we preliminarily need to introduce the following notions. Following the terminology used by Giacobazzi *et al.* (1996, Definition 2.1), given a meet semilattice  $L$  and  $x, y \in L$  such that  $y \leq x$ , we say that an element  $x \rightsquigarrow y \in L$  is the weak relative pseudocomplement of  $x$  with respect to  $y$ , if  $x \wedge x \rightsquigarrow y = y$  and  $\forall z \in L. (x \wedge z = y) \Rightarrow (z \leq x \rightsquigarrow y)$ . Clearly, if such  $x \rightsquigarrow y$  exists, then it is necessarily unique. Note that if  $L$  admits the least element  $\perp$ , then  $x \rightsquigarrow \perp$ , when it exists, is the pseudocomplement of  $x$ .  $L$  is weakly relatively pseudocomplemented if  $x \rightsquigarrow y$  exists for any  $x, y \in L$  such that  $y \leq x$ . Equivalently,  $L$  is weakly relatively pseudocomplemented if any principal filter

of  $L$  is pseudocomplemented. Giacobazzi *et al.* (1996, Examples 3.2 and 3.3) show that in general  $uco(C)$  is not weakly relatively pseudocomplemented. On the other hand, they give the following result (Giacobazzi *et al.*, 1996, Theorem 3.1). Recall (Gierz *et al.*, 1980) that a complete lattice  $C$  is meet-continuous if for any directed subset  $Y \subseteq C$  and  $x \in C$ ,  $x \wedge (\bigvee Y) = \bigvee_{y \in Y} (x \wedge y)$ .

**THEOREM 5.1** (Giacobazzi *et al.*, 1996). *If  $C$  is a meet-continuous complete lattice then for every  $\rho, \eta \in uco(C)$  such that  $\eta \sqsubseteq \rho$  and  $\eta$  is continuous, there exists  $\rho \rightsquigarrow \eta$ .*

As a consequence, one gets that if  $C$  is a meet-continuous complete lattice then  $uco(C)$  is pseudocomplemented (Giacobazzi *et al.*, 1996, Corollary 3.4).

By exploiting Theorem 4.5, as far as complete lattices of closures on CPOs are concerned, we are able to give the following result.

**THEOREM 5.2.** *If  $P$  is a poset satisfying the ACC then  $uco(P)$  is weakly relatively pseudocomplemented.*

*Proof.* Let  $\rho, \eta \in uco(P)$  such that  $\eta \sqsubseteq \rho$ . Since, by Corollary 4.6,  $uco(P)$  is a complete lattice, and its glb  $\sqcap$  and lub  $\sqcup$  operations are characterized by Theorem 4.5, the lub  $\varphi = \sqcup \{\mu \in uco(P) \mid \rho \sqcap \mu = \eta\}$  exists. Let us show that  $\rho \rightsquigarrow \eta = \varphi$ , i.e., that  $\eta = \rho \sqcap \varphi$ . Since  $\eta \sqsubseteq \rho$ , we have that  $\rho \sqcap \eta = \eta$ , from which we get  $\eta \sqsubseteq \varphi$ . Hence, from  $\eta \sqsubseteq \rho$ , we get that  $\eta \sqsubseteq \rho \sqcap \varphi$ . Let us show the other inequality by contradiction. Assume that  $\rho \sqcap \varphi \not\sqsubseteq \eta$ , i.e., that there exists some  $x_0 \in \eta$  such that  $x_0 \notin \rho \sqcap \varphi$ . Then, since  $\varphi \subseteq \rho \sqcap \varphi$ , we have that  $x_0 \notin \varphi$ . Hence, by Theorem 4.5,  $x_0 \notin \{\mu \in uco(P) \mid \rho \sqcap \mu = \eta\}$ . Therefore, there exists some  $\mu \in uco(P)$  such that  $\rho \sqcap \mu = \eta$  and  $x_0 \notin \mu$ . Since  $x_0 \in \rho \sqcap \mu$ , by Theorem 4.5,  $x_0 \in \mathfrak{M}(\rho \cup \mu)$ , and therefore,  $x_0 \in \mathfrak{M}(\rho \cup \mu) \cap \uparrow x_0$ . Since  $\rho \subseteq \rho \sqcap \varphi$ , note that  $\rho \cap \uparrow x_0 \subseteq \rho \sqcap \varphi$ . Moreover, we have that  $\mu \cap \uparrow x_0 \not\subseteq \rho \sqcap \varphi$ . Otherwise,  $(\rho \cup \mu) \cap \uparrow x_0 \subseteq \rho \sqcap \varphi$ , and hence, by monotonicity of  $\mathfrak{M}$  and Theorem 4.4, we would have that  $\mathfrak{M}((\rho \cup \mu) \cap \uparrow x_0) \subseteq \rho \sqcap \varphi$ . Since, by Lemma 4.3,  $\mathfrak{M}(\rho \cup \mu) \cap \uparrow x_0 \subseteq \mathfrak{M}((\rho \cup \mu) \cap \uparrow x_0)$ , we would therefore have that  $x_0 \in \rho \sqcap \varphi$ , which would be a contradiction. Thus, there exists some  $z \in \mu \cap \uparrow x_0$  such that  $z \notin \rho \sqcap \varphi$ . Define  $x_1 = z$ . By definition,  $x_0 \leq x_1$ . Also, since  $x_0 \notin \mu$  and  $x_1 \in \mu$ , we get that  $x_0 < x_1$ . Further,  $x_1 \in \mu \subseteq \rho \sqcap \mu = \eta$ . Hence,  $x_1$  is such that  $x_1 \in \eta$  and  $x_1 \notin \rho \sqcap \varphi$ . This means that by iterating this constructive process, we get an infinite strictly increasing chain of  $x_i$ 's. By the hypothesis on  $P$ , this is a contradiction, which closes the proof. ■

Hence, we get the following consequence.

**COROLLARY 5.3.** *If  $P$  is a poset satisfying the ACC then  $uco(P)$  is pseudocomplemented.*

Thus, the above result allows us to extend the range of application of the complementation operation to abstract domains which are mere posets satisfying the ACC. This is a significant extension, since most abstract domains used in practical abstract interpretation frameworks satisfy the ACC in order to ensure finite convergence of fixpoint computations. The problem of investigating whether the above result can be extended to the class of meet semilattices which are CPOs and meet-continuous (cf. Gierz *et al.*, 1980, Definition 4.6, p. 33) remains open.

As an example, consider the finite poset  $T$  in Fig. 3. Then, one can easily check that the corresponding lattice  $uco(T)$ , depicted in Fig. 3, actually is weakly relatively pseudocomplemented (accordingly with Theorem 5.2), and therefore, pseudocomplemented.

## 6. CONCLUSION

We have shown that closures on a CPO, ordered pointwise, form a complete lattice. The usefulness of this result stems from the relevant rôle played by closure operators in numerous fields of the semantics area. As an example, we have applied it in abstract interpretation by providing a significant extension of the operation of abstract domain complementation. As a further example, let us mention that it could be successfully applied to extend the constructive methodologies of Giacobazzi *et al.* (1998) for minimally making abstract interpretations complete. In fact, by exploiting our result, Giacobazzi *et al.*'s assumption of dealing with concrete and abstract domains that are complete lattices may be relaxed to CPOs, thus enabling a significantly wider range of application in denotational semantics.

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