Expected log-utility maximization under incomplete information and with Cox-process observations

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Abstract

We consider the portfolio optimization problem for the criterion of maximization of expected terminal log-utility. The underlying market model is a regime-switching diffusion model where the regime is determined by an unobservable factor process forming a finite state Markov process. The main novelty is due to the fact that prices are observed and the portfolio is rebalanced only at random times corresponding to a Cox process where the intensity is driven by the unobserved Markovian factor process as well. This leads to a more realistic modeling for many practical situations, like in markets with liquidity restrictions; on the other hand it considerably complicates the problem to the pointy that traditional methodologies cannot be directly applied. The approach presented here is specific to the log-utility. For power utilities a different approach is presented in the companion paper [8].

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1 Introduction

Among the optimization problems in finance, portfolio optimization is one of the first and most important problems. A classical formulation for this problem is the maximization of expected utility from terminal wealth. In the paper we consider this classical portfolio problem with the log-utility function (the power utility function is studied in the companion paper ([8]). What is novel in this paper is the market model, which implies that even for the classical portfolio optimization problem the standard approaches such as Dynamic Programming and convex duality cannot be directly applied and a novel approach is required. Our market model is first of all of the regime-switching type where the factor process that specifies the regime may not be fully observable. This is still a relatively classical situation, for which one may use techniques from stochastic control under incomplete information. In fact, various papers have appeared in such a context and here we just mention some of the most recent ones that also summarize previous work in the area, more precisely [2], [16], [20]. The main novelty of our model is however given by the fact that the prices S_t^i, or equivalently their logarithmic values $X_t^i := \log S_t^i$, of the risky assets in which one invests are supposed to be observable only at discrete random points in time $\tau_0, \tau_1, \tau_2, \cdots$, where the associated counting process is a Cox process (see e.g. [3], [11]) with intensity that depends on the same factor process that specifies the regime for the price evolution model. Such models are in fact relevant in financial applications since (see e.g. [7], [4], [17], [18]), especially on small time scales, prices do not vary continuously, but rather change and are observed only at discrete random points in time that correspond to the time instants when significant new information is absorbed by the market and/or market makers update their quotes. This setting leads to a stochastic control problem with incomplete information and observations given by a Cox process.

A classical approach to incomplete observation control problems is to first transform the problem into a so-called separated problem, where the unobservable part of the state is replaced by its conditional distribution. This requires to solve first the associated filtering problem, which already is non-standard and has been solved recently in [4] (see also [5]). Our major contribution here is on the control part of the separated problem that is approached in a non classical way. In particular we shall restrict the investment strategies to be rebalanced only at the random times τ_k where prices are observed. Although slightly less general from a theoretical point of view, restricting trading to discrete, in particular random times, is quite realistic in finance, where in practice one cannot rebalance a portfolio continuously: think of the case with transaction costs or with liquidity restrictions (in this latter context see e.g. [9], [10], [14], [17], [18], [19] where the authors consider illiquid markets, partly also with regime switching models as in this paper, but under complete information).

In the companion paper [8] we study the case of a power utility, for the solution of which one cannot simply carry over the approach that we develop here for the log-utility case, even if there are close analogies between the two approaches. In other words, for our nonclassical setup the approach that one has to use may depend on the specific case.

In section 2 we give a more precise definition of the model and of the investment strategy and specify the objective function. Section 3 deals with the associated filtering problem, while the specific control problem is then studied in section 4, where the first two subsections concern preliminary and auxiliary notions and results, while the main result is presented in subsection 4.3.

2 Market model and objective

2.1 Introductory remarks

As mentioned in the general Introduction, we consider here the problem of maximization of expected log-utility from terminal wealth, when the dynamics of the prices of the risky assets in which one invests are of the usual diffusion type but with the coefficients in the dynamics depending on an unobservable finite-state Markovian factor process (regime-switching model). In addition it is assumed that the risky asset prices S_t^i, or equivalently their logarithmic values $Z_t^i := \log S_t^i$, are observed only at random times $\tau_0, \tau_1, \cdots$ for which the associated counting process forms a Cox process with an intensity $n(\theta_t)$ that also depends on the unobservable factor process θ_t.

2.2 The market model and preliminary notations

Let θ_t be the hidden finite state Markovian factor process. With Q denoting its transition intensity matrix ($Q-$matrix) its dynamics are given by

$$d\theta_t = Q^* \theta_t dt + dM_t, \theta_0 = \xi. \tag{2.1}$$

where M_t is a jump-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. If N is the number of possible values of θ_t, we may without loss of generality take as its state space the set $E = \{e_1, \ldots, e_N\}$, where e_i is a unit vector for each $i = 1, \ldots, N$ (see [6]).

The evolution of θ_t may also be characterized by the process π_t given by the state probability vector that takes values in the set

$$\mathcal{S}_N := \{\pi \in \mathbb{R}^N \mid \sum_{i=1}^N \pi^i = 1, 0 \leq \pi^i \leq 1, i = 1, 2, \ldots, N\} \tag{2.2}$$

namely the set of all probability measures on E and we have $\pi_0^i = P(\xi = e_i)$. Denoting by $\mathcal{M}(E)$ the set of all finite nonnegative measures on E, it follows that $\mathcal{S}_N \subset \mathcal{M}(E)$. In our study it will be convenient to consider on $\mathcal{M}(E)$ the Hilbert metric $d_H(\pi, \bar{\pi})$ defined (see [1] [12] [13]) by

$$d_H(\pi, \bar{\pi}) := \log\Big(\sup_{\bar{\pi}(A)>0, A\subset E} \frac{\pi(A)}{\bar{\pi}(A)} \sup_{\pi(A)>0, A\subset E} \frac{\bar{\pi}(A)}{\pi(A)} \Big). \tag{2.3}$$

Notice that, while d_H is only a pseudo-metric on $\mathcal{M}(E)$, it is a metric on $\mathcal{S}_N$ ([1]).

In our market we consider m risky assets, for which the price processes $S^i = (S_t^i)_{t\geq 0}, i = 1, \ldots, m$ are supposed to satisfy

$$dS_t^i = S_t^i \{ r^i(\theta_t) dt + \sum_j \sigma_j^i(\theta_t) dB_t^j \}. \tag{2.4}$$

for given coefficients $r^i(\theta)$ and $\sigma_j^i(\theta)$ and with B_t^j $(j = 1, \cdots, m)$ independent $(\mathcal{F}_t, P)-$Wiener processes. Letting $X_t := \log S_t$, by Itô's formula we have

$$X_t = X_0 + \int_0^t r(\theta_s) - d(\sigma\sigma^*(\theta_s)) ds + \int_0^t \sigma(\theta_s) dB_s, \tag{2.5}$$

where by $d(\sigma\sigma^*(\theta))$ we denote the diagonal matrix $d(\sigma\sigma^*(\theta)) := (\frac{1}{2}(\sigma\sigma^*)^{11}(\theta), \ldots, \frac{1}{2}(\sigma\sigma^*)^{mm}(\theta))$. As usual there is also a locally non-risky asset (bond) with price S_t^0 satisfying

$$dS_t^0 = r_0 S_t^0 dt \tag{2.6}$$

where r_0 stands for the short rate of interest. In the interest of generality we shall also make use of discounted asset prices, namely

$$\tilde{S}_t^i := \frac{S_t^i}{S_t^0}, \quad \text{with} \quad \tilde{X}_t := \log \tilde{S}_t \tag{2.7}$$

for which, by Itô's formula

$$d\tilde{S}_t^i = \tilde{S}_t^i \{(r^i(\theta_t) - r_0)dt + \sum_j \sigma_j^i(\theta_t) dB_t^j\}, \tag{2.8}$$

$$d\tilde{X}_t^i = \{r^i(\theta_t) - r_0 - d(\sigma\sigma^*(\theta_t))^i\}dt + \sum_{j=1}^m \sigma_j^i(\theta_t) dB_t^j. \tag{2.9}$$

As already mentioned, the asset prices and thus also their logarithms are observed only at random times $\tau_0, \tau_1, \tau_2, \ldots$ and we shall put $X_k = (X_k^1, \ldots, X_k^m)$ with $X_k^i := X_{\tau_k}^i$, $(i = 1, \cdots, m;\ k \in \mathbb{N})$. The observations are thus given by the sequence $(\tau_k, X_k)_{k \in \mathbb{N}}$ that forms a multivariate marked point process with counting measure

$$\mu(dt, dx) = \sum_k \mathbf{1}_{\{\tau_k < \infty\}} \delta_{\{\tau_k, X_k\}}(t, x) dt dx. \tag{2.10}$$

The corresponding counting process $\Lambda_t := \int_0^t \int_{\mathbb{R}^m} \mu(dt, dx)$ is supposed to be a Cox process with intensity $n(\theta_t)$, i.e. $\Lambda_t - \int_0^t n(\theta_s) ds$ is an $(\mathcal{F}_t, P)-$ martingale. We consider two sub-filtrations related to $(\tau_k, X_k)_{k \in \mathbb{N}}$ namely

$$\begin{aligned} \mathcal{G}_t &:= \mathcal{F}_0 \vee \sigma\{\mu((0, s] \times B) : s \le t, B \in \mathcal{B}(\mathbb{R}^m)\}, \\ \mathcal{G}_k &:= \mathcal{F}_0 \vee \sigma\{\tau_0, X_0, \tau_1, X_1, \tau_2, X_2, \ldots, \tau_k, X_k\}. \end{aligned} \tag{2.11}$$

In our development below we shall often make use of the following notations. For the conditional (on $\mathcal{F}^\theta$) mean and variance of $\tilde{X}_t - \tilde{X}_k$ we set

$$\begin{aligned} m_k^\theta(t) &= \int_{\tau_k}^t [r(\theta_s) - r_0 \mathbf{1} - d(\sigma\sigma^*(\theta_s))] ds, \\ \sigma_k^\theta(t) &= \int_{\tau_k}^t \sigma\sigma^*(\theta_s) ds \end{aligned} \tag{2.12}$$

and, for $z \in \mathbb{R}^m$, we set

$$\rho_{\tau_k, t}^\theta(z) \sim N(z; m_k^\theta(t), \sigma_k^\theta(t)) \tag{2.13}$$

namely the joint conditional (on $\mathcal{F}^\theta$) density of $\tilde{X}_t - \tilde{X}_k$.

2.3 Investment strategies, portfolios, objective

As mentioned in the Introduction, since observations take place at random time points τ_k, we shall consider investment strategies that are rebalanced only at those same time points τ_k.

Let N_t^i be the number of assets of type i held in the portfolio at time t, $N_t^i = \sum_k \mathbf{1}_{[\tau_k,\tau_{k+1})}(t) N_{\tau_k}^i$. The wealth process is defined by

$$V_t := \sum_{i=0}^m N_t^i S_t^i.$$

Consider then the investment ratios

$$h_t^i := \frac{N_t^i S_t^i}{V_t},$$

and set $h_k^i := h_{\tau_k}^i$. The set of admissible investment ratios is given by

$$\bar{H}_m := \{(h^1, \ldots, h^m); h^1 + h^2 + \ldots + h^m \leq 1, 0 \leq h^i, i = 1, 2, \ldots, m\}, \tag{2.14}$$

i.e. no shortselling is allowed and notice that $\bar{H}_m$ is bounded and closed. Put $h = (h^1, \cdots, h^m)$. Analogously to [15] define next a function $\gamma : \mathbb{R}^m \times \bar{H}_m \to \bar{H}_m$ by

$$\gamma^i(z, h) := \frac{h^i \exp(z^i)}{1 + \sum\limits_{i=1}^m h^i(\exp(z^i) - 1)} \quad i = 1, , \ldots, m. \tag{2.15}$$

Noticing that N_t is constant on $[\tau_k, \tau_{k+1})$, for $i = 1, \ldots, m$, and $t \in [\tau_k, \tau_{k+1})$ let

$$\begin{aligned}
h_t^i &= \frac{N_t^i S_t^i}{\sum_{i=0}^m N_t^i S_t^i} = \frac{N_k^i S_t^i}{\sum_{i=0}^m N_k^i S_t^i} \\
&= \frac{N_k^i S_k^i S_t^i / S_k^i}{\sum_{i=0}^m N_k^i S_k^i S_t^i / S_k^i} = \frac{h_k^i S_t^i / S_k^i}{\sum_{i=0}^m h_k^i S_t^i / S_k^i} = \frac{h_k^i S_k^0 / S_t^0 S_t^i / S_k^i}{\sum_{i=0}^m h_k^i S_k^0 / S_t^0 S_t^i / S_k^i} \\
&= \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)}{h_k^0 + \sum\limits_{i=1}^m h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)} = \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)}{1 + \sum\limits_{i=1}^m h_k^i (\exp(\tilde{X}_t^i - \tilde{X}_k^i) - 1)} \\
&= \gamma^i(\tilde{X}_t - \tilde{X}_k, h_k).
\end{aligned} \tag{2.16}$$

The set of admissible strategies $\mathcal{A}$ is defined by

$$\mathcal{A} := \{\{h_k\}_{k=0}^\infty | h_k \in \bar{H}_m,\ \mathcal{G}_k \text{ m'ble for all } k \geq 0\}. \tag{2.17}$$

Furthermore, for $n > 0$, we let

$$\mathcal{A}^n := \{h \in \mathcal{A} | h_{n+i} = h_{\tau_{n+i}-} \text{ for all } i \geq 1\}. \tag{2.18}$$

Notice that, by the definition of $\mathcal{A}^n$, for all $k \geq 1$, $h \in \mathcal{A}^n$ we have

$$h_{n+k}^i = h_{\tau_{n+k}-}^i$$

$$\Leftrightarrow \frac{N_{n+k}^i S_{n+k}^i}{\sum_{i=0}^m N_{n+k}^i S_{n+k}^i} = \frac{N_{n+k-1}^i S_{n+k}^i}{\sum_{i=0}^m N_{n+k}^i S_{n+k}^i}$$

$$\Leftrightarrow N_{n+k} = N_{n+k-1}.$$

Therefore, for $k \geq 1$

$$N_{n+k} = N_n.$$

and

$$\mathcal{A}^0 \subset \mathcal{A}^1 \subset \cdots \mathcal{A}^n \subset \mathcal{A}^{n+1} \cdots \subset \mathcal{A}. \tag{2.19}$$

Remark 2.1. *Notice that, for a given finite sequence of investment ratios* $h_0, h_1, \cdots, h_n$ *such that* h_k *is an* $\mathcal{G}_k-$*measurable,* $\bar{H}_m-$*valued random variable for* $k \leq n$*, there exists* $h^{(n)} \in \mathcal{A}^n$ *such that* $h_k^{(n)} = h_k$, $k = 0, \cdots, n$*. Indeed, if* N_t *is constant on* $[\tau_n, T)$*, then for* h_t *we have* $h_t = \gamma(\tilde{X}_t - \tilde{X}_n, h_n)$, $\forall t \geq \tau_n$*. Therefore, by setting* $h_\ell^{(n)} = h_\ell$, $\ell = 0, \cdots, n$*, and* $h_{n+k}^{(n)} = h_{\tau_n + k}$, $k = 1, 2, \cdots$ *, since the vector process* S_t *and the vector function* $\gamma(\cdot, h_n)$ *are continuous, we see that* $h_{n+k}^{(n)} = h_{\tau_n + k-}$, $k = 1, 2, \cdots$ *.*

Finally, considering only self-financing portfolios, for their value process we have the dynamics

$$\frac{dV_t}{V_t} = [r_0 + h_t^* \{r(\theta_t) - r_0 \mathbf{1}\}]dt + h_t^* \sigma(\theta_t) dB_t. \tag{2.20}$$

Problem: Given a finite planning horizon $T > 0$, our problem of maximization of expected terminal log-utility consists in determining

$$\sup_{h \in \mathcal{A}} E[\log V_T | \tau_0 = 0, \pi_0 = \pi]$$

as well as an optimal maximizing strategy $\hat{h} \in \mathcal{A}$.

3 Filtering

As mentioned in the Introduction, the usual approach to stochastic control problems under incomplete information is to first transform them into a so-called separated problem, where the unobservable part of the state is replaced by its conditional (filter) distribution. This implies that we first have to study this conditional distribution and its (Markovian) dynamics, i.e. we have to study the associated filtering problem.

The filtering problem for our specific case, where the observations are given by a Cox process with intensity expressed as a function of the unobserved state, has been studied in [4] (see also [5]). In the first subsection 3.1 we therefore summarize the main results from [4] in view of their use in our control problem in section 4. In subsection 3.2 we present further results for our specific case.

3.1 General Filtering Equation

Recalling the definition of $\rho^\theta(z)$ in (2.13) and putting

$$\phi^\theta(\tau_k, t) = n(\theta_t) \exp(-\int_{\tau_k}^t n(\theta_s) ds), \tag{3.1}$$

for a given function $f(\theta)$ we let

$$\psi_k(f; t, x) := E[f(\theta_t) \rho^\theta_{\tau_k, t}(x - \tilde{X}_k) \phi^\theta(\tau_k, t) | \sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{3.2}$$

$$\bar{\psi}_k(f;t) := \int \psi_k(f;t,x)dx = E[f(\theta_t)\phi^\theta(\tau_k,t)|\sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \tag{3.3}$$

$$\pi_t(f) = E[f(\theta_t)|\mathcal{G}_t] \tag{3.4}$$

with ensuing obvious meanings of $\pi_{\tau_k}(\psi_k(f;t,y))$ and $\pi_{\tau_k}(\bar{\psi}_k(f;t))$ where we consider $\psi_k(f;t,y)$ and $\bar{\psi}_k(f;t)$ as functions of θ_{τ_k}. The process $\pi_t(f)$ is called the *filter process* for $f(\theta_t)$.

We have (see Lemma 4.1 in [4])

Lemma 3.1. *The compensator of the random measure $\mu(dt,dy)$ in (2.10) on the σ -algebra $\tilde{\mathcal{P}}(\mathcal{G}) = \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(R^m)$ with $\mathcal{P}(\mathcal{G})$ the predictable σ -algebra on $\Omega \times [0,\infty)$, is given by the following nonnegative random measure*

$$\nu(dt,dy) = \sum_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t) \frac{\pi_{\tau_k}(\psi_k(1,t,y))}{\int_t^\infty \pi_{\tau_k}(\bar{\psi}_k(1,s))ds} dtdy. \tag{3.5}$$

The main filtering result is the following (see Theorem 4.1 in [4]).

Theorem 3.1. *For any bounded function $f(\theta)$, the differential of the optimal filter $\pi_t(f)$ is given by*

$$\begin{aligned} d\pi_t(f) &= \pi_t(Lf)dt \\ &\quad + \int \sum_k \mathbf{1}_{(\tau_k,\tau_{k+1}]}(t)[\tfrac{\pi_{\tau_k}(\psi_k(f;t,y))}{\pi_{\tau_k}(\psi_k(1;t,y))} - \pi_{t-}(f)](\mu-\nu)(dt,dy), \end{aligned} \tag{3.6}$$

where L is the generator of the Markov process θ_t (namely $L = Q$).

Corollary 3.1. *We have*

$$\pi_{\tau_{k+1}}(f) = \frac{\pi_{\tau_k}(\psi_k(f;t,x))}{\pi_{\tau_k}(\psi_k(1;t,x))}\bigg|_{t=\tau_{k+1}, x=\tilde{X}_{k+1}}. \tag{3.7}$$

Recall that in our setting θ_t is an N-state Markov chain with state space $E = \{e_1,\ldots,e_N\}$, where e_i is a unit vector for each $i = 1,\ldots,N$. One may then write $f(\theta_t) = f(e_i)\mathbf{1}_{e_i}(\theta_t)$. For $i = 1,\ldots,N$ let $\pi_t^i = \pi_t(\mathbf{1}_{e_i}(\theta_t))$ and

$$r_{ji}(t,z) := E[\exp(\int_0^t -n(\theta_s)ds)\rho_{0,t}^\theta(z)|\theta_0 = e_j, \theta_t = e_i], \tag{3.8}$$

$$p_{ji}(t) := P(\theta_t = e_i|\theta_0 = e_j) \tag{3.9}$$

and, noticing that $\pi_t \in \mathcal{S}_N$, define the function $M : [0,\infty) \times \mathbb{R}^m \times \mathcal{S}_N \to \mathcal{S}_N$ by

$$M^i(t,x,\pi) \;:=\; \tfrac{\sum_j n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}{\sum_{ij} n(e_i)r_{ji}(t,x)p_{ji}(t)\pi^j}, \tag{3.10}$$

$$M(t,x,\pi) := (M^1(t,x,\pi), M^2(t,x,\pi),\ldots,M^N(t,x,\pi)). \tag{3.11}$$

For $A \subset E$

$$M(t,x,\pi)(A) := \sum_{i=1}^N M^i(t,x,\pi)1_{\{e_i\in A\}}. \tag{3.12}$$

The following corollary will be useful

Corollary 3.2. *For the generic i-th state one has*

$$\pi^i_{k+1} = M^i(\tau_{k+1} - \tau_k, \tilde{X}_{k+1} - \tilde{X}_k, \pi_k) \tag{3.13}$$

and the process $\{\tau_k, \pi_k, \tilde{X}_k\}_{k=1}^{\infty}$ *is a Markov process with respect to* $\mathcal{G}_k$*.*

Proof. The representation (3.13) and the fact that $\{\tau_k, \pi_k, \tilde{X}_k\}$ is a $\mathcal{G}_k-$adapted discrete stochastic processes on $[0,\infty) \times \mathcal{S}_N \times \mathbb{R}^m$ follow immediately from Corollary 3.1 and the preceding definitions. For the Markov property we calculate

$$\begin{aligned} P(\tau_{k+1} < t, \tilde{X}_{k+1} < x | \mathcal{G}_k) &= E[P(\tau_{k+1} < t, \tilde{X}_{k+1} < x | \mathcal{G}_k \vee \mathcal{F}^\theta) | \mathcal{G}_k] \\ &= E[\textstyle\int_{\tau_k}^t P(\tilde{X}_s < x | \mathcal{G}_k \vee \mathcal{F}^\theta) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds | \mathcal{G}_k] \\ &= E[\textstyle\int_{\tau_k}^t \int_{-\infty}^x \rho_{\tau_k, s}(z - \tilde{X}_k) n(\theta_s) \exp(-\int_{\tau_k}^s n(\theta_u) du) ds dz | \mathcal{G}_k] \\ &= \textstyle\int_{\tau_k}^t \int_{-\infty}^x \displaystyle\sum_{ij} n(e_i) r_{ji}(s - \tau_k, z - \tilde{X}_k) p_{ji}(s - \tau_k) \pi_k^j ds dz, \end{aligned}$$

and for any bounded measurable function g on $[0,\infty) \times \mathcal{S}_N \times \mathbb{R}^m$ it then follows that

$$\begin{aligned} & E[g(\tau_{k+1}, \pi_{k+1}, \tilde{X}_{k+1}) | \mathcal{G}_k] \\ &\quad = E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{k+1} - \tilde{X}_k, \pi_k), \tilde{X}_{k+1}) | \mathcal{G}_k] \\ &\quad = E[E[g(\tau_{k+1}, M(\tau_{k+1} - \tau_k, \tilde{X}_{k+1} - \tilde{X}_k, \pi_k), \tilde{X}_{k+1}) | \mathcal{G}(k) \vee \mathcal{F}^\theta] | \mathcal{G}_k] \\ &\quad = E[\textstyle\int_{\tau_k}^\infty E[g(t, M(t - \tau_k, \tilde{X}_t - \tilde{X}_k, \pi_k), \tilde{X}_t) n(\theta_t) \exp(-\int_{\tau_k}^t n(\theta_s) ds) | \mathcal{G}_k \vee \mathcal{F}^\theta] dt | \mathcal{G}_k] \\ &\quad = \textstyle\int_{\tau_k}^\infty \int_{\mathbb{R}^m} g(t, M(t - \tau_k, x - \tilde{X}_k, \pi_k), x) \sum_{ij} n(e_i) r_{ji}(t - \tau_k, x - \tilde{X}_k) p_{ji}(t - \tau_k) \pi_k^j dx dt. \end{aligned}$$

where the last equation depends only on $\{\tau_k, \pi_k, \tilde{X}_k\}$ thus implying the Markov property. $\square$

3.2 Further results

In view of the further results in this section we define an operator on $\mathcal{M}(E)$ as follows

$$K^i(t,x)\pi := \sum_j n(e_i) r_{ji}(t,x) p_{ji}(t) \pi^j. \tag{3.14}$$

$$K(t,x)\pi := (K^1(t,x)\pi, K^2(t,x)\pi, \ldots, K^N(t,x)\pi). \tag{3.15}$$

For $t \in [0,\infty),\ x \in \mathbb{R}^m,\ K^i(t,x)$ is a positive linear operator on $\mathcal{M}(E)$. For $A \subset E$ set

$$K(t,x)\pi(A) := \sum_{i=1}^N K^i(t,x)\pi 1_{\{e_i \in A\}}. \tag{3.16}$$

By the definition of $M^i(t,x,\pi)$ and $K^i(t,x)\pi$, setting $\kappa(t,x,\pi) := \sum_i K^i(t,x)\pi$,
for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi \in \mathcal{M}(E)$ we have

$$M^i(t,x,\pi) = \frac{1}{\kappa(t,x,\pi)} K^i(t,x)\pi\ . \tag{3.17}$$

By the definition of the Hilbert metric $d_H(\cdot,\cdot)$, for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi,\bar{\pi} \in \mathcal{M}(E)$ we then have

$$\begin{aligned} d_H(M(t,x,\pi), M(t,x,\bar{\pi})) &= \log(\sup \tfrac{M(t,x,\pi)(A)}{M(t,x,\bar{\pi})(A)} \sup \tfrac{M(t,x,\bar{\pi})(A)}{M(t,x,\pi)(A)}) \\ &= \log(\sup \tfrac{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)}{\frac{1}{\kappa(t,x,\bar{\pi})}K(t,x)\bar{\pi}(A)} \sup \tfrac{\frac{1}{\kappa(t,x,\bar{\pi})}K(t,x)\bar{\pi}(A)}{\frac{1}{\kappa(t,x,\pi)}K(t,x)\pi(A)}) \\ &= \log(\sup \tfrac{K(t,x)\pi(A)}{K(t,x)\bar{\pi}(A)} \sup \tfrac{K(t,x)\bar{\pi}(A)}{K(t,x)\pi(A)}) \\ &= d_H(K(t,x)\pi, K(t,x)\bar{\pi}). \end{aligned} \tag{3.18}$$

Applying [1], Lemma 3.8 in [12] and Theorem 1.1 in [13] , for the positive linear operator K on $\mathcal{M}(E)$ it then follows that

$$d_H(M(t,x,\pi), M(t,x,\bar{\pi})) = d_H(K(t,x)\pi, K(t,x)\bar{\pi}) \leq d_H(\pi,\bar{\pi}) \tag{3.19}$$

for $t \in [0,\infty),\ x \in \mathbb{R}^m,\ \pi,\bar{\pi} \in \mathcal{S}_N$. By Lemma 3.6 in [12] , for $\forall \pi,\bar{\pi} \in \mathcal{S}_N$ we also have

$$\|\pi - \bar{\pi}\|_{TV} \leq \frac{2}{\log 3} d_H(\pi,\bar{\pi}), \tag{3.20}$$

where $\|\cdot\|_{TV}$ is the total variation on $\mathcal{S}_N$.

We finally introduce a metric on $[0,\infty) \times \mathcal{S}_N \times \bar{H}_m$ by

$$|t-\bar{t}| + d_H(\pi,\bar{\pi}) + \sum_{i=1}^{m} |h^i - \bar{h}^i| \tag{3.21}$$

for $(t,\ \pi,\ h), (\bar{t},\ \bar{\pi},\ \bar{h}) \in [0,\infty) \times \mathcal{S}_N \times \bar{H}_m$ and considering the state space

$$\Sigma := [0,\infty) \times \mathcal{S}_N, \tag{3.22}$$

let $C_{b,lip}(\Sigma)$ be the set of bounded and Lipschitz continuous functions on Σ to $\mathbb{R}$ with norm

$$\|g\| := max_{x\in\Sigma} \mid g(x) \mid \tag{3.23}$$

We shall now show that, for $\forall g \in C_{b,lip}(\Sigma)$, the operator

$$\begin{aligned} &Jg(\tau,\pi) \\ :=\ & \textstyle\int_\tau^T \int_{\mathbb{R}^m} g(t, M(t-\tau,z,\pi)) \sum_{ij} n(e_i) r_{ji}(t-\tau,z) p_{ji}(t-\tau)\pi^j dz dt \\ =\ & E[g(\tau_1,\pi_1) 1_{\{\tau_1<T\}} | \tau_0 = \tau, \pi_0 = \pi], \end{aligned} \tag{3.24}$$

where M is defined in (3.10)-(3.11), takes values in $C_{b,lip}(\Sigma)$, namely $J : C_{b,lip}(\Sigma) \to C_{b,lip}(\Sigma)$.

First we have

Lemma 3.2. *J is a contraction operator on $C_{b,lip}(\Sigma)$ with contraction constant $c := 1 - e^{-\bar{n}T} < 1$, where $\bar{n} := \max n(\theta) = \max_i n(e_i)$.*

Proof. For $\forall g \in C_{b,lip}(\Sigma)$

$$\begin{aligned}
\mid Jg(t,\pi) \mid &= \mid E[g(\tau_1,\pi_1)1_{\{\tau_1<T\}}|\tau_0 = t, \pi_0 = \pi] \mid \\
&\leq E[\mid g(\tau_1,\pi_1) \mid 1_{\{\tau_1<T\}}|\tau_0 = t, \pi_0 = \pi] \\
&= \|g\|P(\tau_1 < T|\tau_0 = t) \\
&= \|g\|E[(1 - \exp(-\textstyle\int_t^T n(\theta_t)dt))] \\
&\leq \|g\|(1 - \exp(-\bar{n}(T - t)))
\end{aligned}$$

and so

$$\|Jg\| \leq c\|g\| \tag{3.25}$$

with c as specified in the statement. □

Proposition 3.1. *The operator J in (3.24) is an operator*

$$J : C_{b,lip}(\Sigma) \to C_{b,lip}(\Sigma)$$

Proof. Let us first prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to t. By assumption, for all $g \in C_{b,lip}(\Sigma)$,there exists a constant C_g s.t

$$|g(\tau,\pi) - g(\bar{\tau},\pi)| \leq C_g|\tau - \bar{\tau}|, \tag{3.26}$$

$$|g(\tau,\pi) - g(\tau,\bar{\pi})| \leq C_g d_H(\pi,\bar{\pi}). \tag{3.27}$$

We change variables from t to $t + \tau$,

$$Jg(\tau,\pi) = \int_0^{T-\tau} \int_{\mathbb{R}^m} g(t + \tau, M(t,z,\pi)) \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt. \tag{3.28}$$

We then have

$$\begin{aligned}
&|Jg(\tau,\pi) - Jg(\bar{\tau},\pi)| \\
&= \mid \textstyle\int_{T-\bar{\tau}}^{T-\tau} \int_{\mathbb{R}^m} g(t + \tau, M(t,z,\pi)) \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\quad +| \textstyle\int_0^{T-\tau} \int_{\mathbb{R}^m} \{g(t + \tau, M(t,z,\pi)) - g(t + \bar{\tau}, M(t,z,\pi))\} \\
&\qquad \cdot \textstyle\sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \qquad (3.29)\\
&\leq \bar{n}\|g\||\tau - \bar{\tau}| + \|C_g\||\tau - \bar{\tau}| \mid \textstyle\int_0^{T-\tau} \int_{\mathbb{R}^m} \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&= \bar{n}\|g\||\tau - \bar{\tau}| + \|C_g\||\tau - \bar{\tau}|P(\tau_1 < T|\tau_0 = \tau, \pi_0 = \pi) \\
&\leq \bar{n}\|g\||\tau - \bar{\tau}| + c\|C_g\||\tau - \bar{\tau}|.
\end{aligned}$$

Next, let us prove that $Jg(t,\pi)$ is Lipschitz continuous with respect to π.

$$\begin{aligned}
&|Jg(\tau,\pi) - Jg(\tau,\bar{\pi})| \\
&\leq \mid \textstyle\int_0^{T-\tau} \int_{\mathbb{R}^m} \{g(t, M(t,z,\pi)) - g(t, M(t,z,\bar{\pi}))\} \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \\
&\quad + \mid \textstyle\int_0^{T-\tau} \int_{\mathbb{R}^m} g(t, M(t,z,\bar{\pi})) \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)(\pi^j - \bar{\pi}^j)dzdt| \\
&\leq \mid \textstyle\int_0^{T-\tau} \int_{\mathbb{R}^m} \|C_g\| \frac{2}{\log 3} d_H(M(t,z,\pi), M(t,z,\bar{\pi})) \sum_{ij} n(e_i)r_{ji}(t,z)p_{ji}(t)\pi^j dzdt| \qquad (3.30)\\
&\quad + \|g\| \tfrac{2}{\log 3} d_H(\pi,\bar{\pi})P(\tau_1 < T|\tau_0 = \tau) \\
&\leq c\tfrac{2}{\log 3}\{\|C_g\|d_H(\pi,\bar{\pi}) + \|g\|d_H(\pi,\bar{\pi})\},
\end{aligned}$$

$\square$

4 The Control Problem/Log-utility

Recall from (2.20) that the value process of a self financing portfolio satisfies

$$\frac{dV_t}{V_t} = [r_0 + h_t^*\{r(\theta_t) - r_0\mathbf{1}\}]dt + h_t^*\sigma(\theta_t)dB_t. \tag{4.1}$$

We have by Itô's formula

$$\begin{aligned} \log V_T = \log v_0 \quad & + \quad \int_0^T h_t^*\sigma(\theta_t)dB_t \\ & + \quad \int_0^T [r_0 + h_t^*\{r(\theta_t) - r_0\mathbf{1}\} - \tfrac{1}{2}h_t^*\sigma\sigma^*(\theta_t)h_t]dt. \end{aligned} \tag{4.2}$$

Put

$$f(\theta, h) := r_0 + h^*\{r(\theta) - r_0\mathbf{1}\} - \frac{1}{2}h^*\sigma\sigma^*(\theta)h \tag{4.3}$$

and notice that this function $f(\cdot)$ is bounded under our assumptions. The expected log-utility of terminal wealth then becomes

$$E[\log V_T|\tau_0 = 0, \pi_0 = \pi] = \log v_0 + E[\int_0^T f(\theta_t, h_t)dt|\tau_0 = 0, \pi_0 = \pi] \tag{4.4}$$

and, as mentioned in section 2.3 we want to consider the problem of maximization of expected terminal log-utility, namely

$$\sup_{h\in\mathcal{A}} E[\log V_T|\tau_0 = 0, \pi_0 = \pi]$$

4.1 Preliminary definitions and properties

Definition 4.1. *Let $\hat{C}(\tau, \pi, h)$ be defined by*

$$\begin{aligned} \hat{C}(\tau, \pi, h) \quad & := \quad E[\int_t^{T\wedge\tau} f(\theta_s, h_s)ds|\tau_0 = t, \pi_0 = \pi] \\ & = \quad \int_\tau^T \int_{\mathbb{R}^m} \sum_{i,j} f(e_i, \gamma(x, h))r_{ji}(t - \tau, x)p_{ji}(t - \tau)\pi^j dxdt. \end{aligned} \tag{4.5}$$

where $\gamma(x, h) = [\gamma^1(x^1, h), \cdots, \gamma^m(x^m, h)]$.

Lemma 4.1.

(i) For the function defined by (4.3), we have the following equation

$$E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t, \pi_0 = \pi] = E[\sum_k \hat{C}(\tau_k, \pi_k, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_0 = \pi]. \tag{4.6}$$

(ii) $\hat{C}$ is a bounded and continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$.

For the proof see the Appendix.

Corollary 4.1.

(i) There exists a Borel function $\hat{h}(\tau,\pi)$ *such that* $\sup_{h\in\bar{H}_m}\hat{C}(\tau,\pi,h)=\hat{C}(\tau,\pi,\hat{h}(\tau,\pi))$.

(ii) the function

$$C(t,\pi) := \sup_{h\in\bar{H}_m} \hat{C}(t,\pi,h). \tag{4.7}$$

is Lipschitz continuous with respect to t,π *in the metric introduced in (3.21).*

Proof. $\bar{H}_m$ is compact and $\hat{C}(\tau,\pi,h)$ is a bounded continuous function on $[0,T]\times\mathcal{S}_N\times\bar{H}_m$; there exists then a Borel function $\hat{h}(\tau,\pi)$ such that (4.7) holds. Furthermore, $\hat{C}(t,\pi,h)$ is uniformly Lipschitz continuous with respect to t,π.

□

Definition 4.2. *For given initial data* $(\tau_0=t,\pi_0=\pi)$*, where we now start at a generic time* t*, consider the following value function for* $h\in\mathcal{A}$

$$\begin{aligned} W(t,\pi,h.) &:= E[\textstyle\int_t^T f(\theta_s,h_s)ds|\tau_0=t,\ \pi_0=\pi] \\ &= E[\sum_{k=0}^{\infty}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\ \pi_0=\pi]. \end{aligned} \tag{4.8}$$

and define

$$\begin{aligned} W(t,\pi) &:= \sup_{h\in\mathcal{A}} W(t,\pi,h.) \\ &= \sup_{h\in\mathcal{A}} E[\int_t^T f(\theta_s,h_s)ds|\tau_0=t,\ \pi_0=\pi] \\ &= \sup_{h\in\mathcal{A}} E[\sum_{k=0}^{\infty}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\ \pi_0=\pi], \end{aligned} \tag{4.9}$$

$$\begin{aligned} W^n(t,\pi) &:= \sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \\ &= \sup_{h\in\mathcal{A}^n} E[\int_t^T f(\theta_s,h_s)ds|\tau_0=t,\ \pi_0=\pi] \\ &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{\infty}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\ \pi_0=\pi]. \end{aligned} \tag{4.10}$$

where $\mathcal{A}^n$ *was defined in (2.18)*

Lemma 4.2. *For all* $n\geq 0$ *and* $h\in\mathcal{A}^n$*, we have the following equation*

$$\begin{aligned} W(t,\pi,h.) &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} \\ &\quad + \textstyle\int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\ \pi_0=\pi]. \end{aligned} \tag{4.11}$$

Proof. Fix $n \geq 0$. Recall the definition

$$h_n^i = \frac{N_n^i S_n^i}{\sum_{i=0}^m N_n^i S_n^i}.$$

Since S_t is continuous and V_t satisfies the self-financing condition, we obtain

$$h_{\tau_n -}^i = \frac{N_{n-1}^i S_{\tau_n -}^i}{V_{\tau_n -}} = \frac{N_{n-1}^i S_n^i}{V_n} = \frac{N_{n-1}^i S_n^i}{\sum_{i=0}^m N_n^i S_n^i}.$$

Using (2.16), (2.18), for *all* $k \geq 1, h \in \mathcal{A}^{n+k}, t \in [\tau_{n+k}, T]$, one furthermore has

$$\begin{aligned} h_t^i &= \gamma^i(\tilde{X}_t - \tilde{X}_{n+k}, h_{n+k}) = \tfrac{N_{n+k}^i S_t^i}{\sum_{i=0}^m N_{n+k}^i S_t^i} \\ &= \tfrac{N_n^i S_t^i}{\sum_{i=0}^m N_n^i S_t^i} = \tfrac{N_n^i S_n^i S_t^i / S_n^i}{\sum_{i=0}^m N_n^i S_n^i S_t^i / S_n^i} = \tfrac{h_n^i S_t^i / S_n^i}{\sum_{i=0}^m h_n^i S_t^i / S_n^i} \\ &= \gamma^i(\tilde{X}_t - \tilde{X}_n, h_n). \end{aligned}$$

Therefore, using lemma 4.1(i) for $h \in \mathcal{A}^n$

$$\begin{aligned} W(t, \pi, h.) &= E[\sum_{k=0}^{n-1} \int_{\tau_k}^{T \wedge \tau_{k+1}} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_k, h_k)) ds 1_{\{\tau_k < T\}} \\ &\quad + \sum_{k=n}^{\infty} \int_{\tau_k}^{T \wedge \tau_{k+1}} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_k, h_k)) ds 1_{\{\tau_k < T\}} | \tau_0 = t,\ \pi_0 = \pi] \\ &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_k, h_k) 1_{\{\tau_k < T\}} \\ &\quad + \sum_{k=n}^{\infty} \int_{\tau_k}^{T \wedge \tau_{k+1}} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_n, h_n)) ds 1_{\{\tau_k < T\}} | \tau_0 = t,\ \pi_0 = \pi] \\ &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_k, h_k) 1_{\{\tau_k < T\}} \\ &\quad + \textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_n, h_n)) ds 1_{\{\tau_n < T\}} | \tau_0 = t,\ \pi_0 = \pi]. \end{aligned} \tag{4.12}$$

□

Corollary 4.2. *For $n \geq 0,\ t \in [0, T],\ \pi \in \mathcal{S}_N$ we have the following equation*

$$\begin{aligned} W^n(t, \pi) &= \sup_{h \in \mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k, \pi_k, h_k) 1_{\{\tau_k < T\}} \\ &\quad + \textstyle\int_{\tau_n}^{T} f(\theta_s, \gamma(\tilde{X}_s - \tilde{X}_n, h_n)) ds 1_{\{\tau_n < T\}} | \tau_0 = t,\ \pi_0 = \pi] \end{aligned} \tag{4.13}$$

4.2 Further auxiliary notions and results

In view of computing the optimal value and the optimal controls we introduce now the following notions and results.

Lemma 4.3. *Set*

$$\bar{W} := \sum_{k=0}^{\infty} J^k C \tag{4.14}$$

where $C(t,\pi)$ is defined in Corollary 4.1 and J is defined in (3.24).

(i) We have the following equation

$$\bar{W}(t,\pi) = C(t,\pi) + J\bar{W}(t,\pi) \tag{4.15}$$

(ii) $\bar{W}(t,\pi)$ is bounded and Lipschitz continuous with respect to t,π.

Proof. Let us first prove (i). Because of Lemma 3.2, we see that (4.14) is a Neumann series and so $(I-J)^{-1}$ is defined and

$$(I-J)^{-1} = \sum_{k=0}^{\infty} J^k \tag{4.16}$$

Therefore

$$\bar{W} = (I-J)^{-1}C \tag{4.17}$$

$$\Leftrightarrow \bar{W} = C + J\bar{W} \tag{4.18}$$

Next, we prove (ii). By Proposition 3.1 and Corollary 4.1, $J^n C(t,\pi)$ is, $\forall n$, Lipschitz continuous with respect to t,π. By (i), $\lim_{n\to\infty} \| \sum_{k=0}^{n} J^k C - \bar{W} \| = 0$. Therefore $\bar{W}$ is Lipschitz continuous with respect to t,π. □

Definition 4.3. *We now define for $h \in \bar{H}_m$*

$$\bar{W}^0(t,\pi,h) := E[\int_t^T f(\theta_s, \gamma(Z_s,h))ds | \tau_0 = t,\ \pi_0 = \pi] \tag{4.19}$$

where $Z_s := \tilde{X}_s - \tilde{X}_t$. Furthermore, let

$$\bar{W}^0(t,\pi) := \max_{h\in\bar{H}_m} \bar{W}^0(t,\pi,h), \tag{4.20}$$

and, for $n \geq 1$

$$\begin{aligned} \bar{W}^n(t,\pi) &:= C(t,\pi) + J\bar{W}^{n-1}(t,\pi) \\ &= \sum_{k=0}^{n-1} J^k C(t,\pi) + J^n \bar{W}^0(t,\pi). \end{aligned} \tag{4.21}$$

Remark 4.1. *The function $\bar{W}^0(t,\pi,h)$ in (4.19) is bounded and continuous with respect to t,π,h. This follows by an analogous proof as in Lemma 4.1(ii).*

Lemma 4.4.

(i) We have the following equation

$$\bar{W}^n(t,\pi) = E[\sum_{k=0}^{n-1} C(\tau_k,\pi_k)1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\tau_0 = t,\ \pi_0 = \pi]. \tag{4.22}$$

(ii) For any $\epsilon > 0$, we set $n_\epsilon := (\log(1-c) + \log\varepsilon - \log\|\bar{W}^1 - \bar{W}^0\|)/\log c$, where c is the contraction constant defined in Lemma 3.2. For all $n > n_\epsilon$,

$$\|\bar{W} - \bar{W}^n\| < \epsilon. \tag{4.23}$$

Proof. We prove (i). For $n \geq 1$

$$\{\tau_{n-1} < T\} \supset \{\tau_n < T\}. \tag{4.24}$$

Therefore,

$$1_{\{\tau_{n-1}<T\}}1_{\{\tau_n<T\}} = 1_{\{\tau_n<T\}}. \tag{4.25}$$

For all $g \in C_b([0,T] \times \mathcal{S}_N)$ and $n \geq 0$, we have

$$\begin{aligned} &E[g(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\tau_0 = t, \pi_0 = \pi] \\ &= E[g(\tau_n,\pi_n)1_{\{\tau_{n-1}<T\}}1_{\{\tau_n<T\}}|\tau_0 = t, \pi_0 = \pi] \\ &= E[E[g(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\mathcal{G}_{n-1}]1_{\{\tau_{n-1}<T\}}|\tau_0 = t, \pi_0 = \pi]. \end{aligned} \tag{4.26}$$

Since (see (3.24))

$$\begin{aligned} &E[g(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\mathcal{G}_{n-1}] \\ &= \int_{\tau_{n-1}}^{T} \int_{\mathbb{R}^m} g(t, M(t-\tau_{n-1},z,\pi)) \sum_{ij} n(e_i) r_{ji}(t-\tau_{n-1},z) p_{ji}(t-\tau_{n-1}) \pi_{n-1}^j dz dt \\ &= Jg(\tau_{n-1},\pi_{n-1}), \end{aligned} \tag{4.27}$$

we have (see always (3.24))

$$\begin{aligned} E[g(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\tau_0 = t, \pi_0 = \pi] &= E[Jg(\tau_{n-1},\pi_{n-1})1_{\{\tau_{n-1}<T\}}|\tau_0 = t, \pi_0 = \pi] \\ &= J^n g(t,\pi). \end{aligned} \tag{4.28}$$

We then obtain

$$\begin{aligned} \bar{W}^n(t,\pi) &= \sum_{k=0}^{n-1} J^k C(t,\pi) + J^n \bar{W}^0(t,\pi) \\ &= E[\textstyle\sum_{k=0}^{n-1} C(\tau_k,\pi_k)1_{\{\tau_k<T\}} + \bar{W}^0(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\tau_0 = t,\ \pi_0 = \pi]. \end{aligned} \tag{4.29}$$

Next, we prove (ii). For any n,

$$\begin{aligned} \|\bar{W} - \bar{W}^n\| &= \|\lim_{k\to\infty} \bar{W}^{n+k} - \bar{W}^n\| = \lim_{k\to\infty} \|\bar{W}^{n+k} - \bar{W}^n\| \\ &\leq \lim_{k\to\infty} \sum_{i=0}^{k-1} \|\bar{W}^{n+i+1} - \bar{W}^{n+i}\| \leq \|\bar{W}^{n+1} - \bar{W}^n\| \sum_{i=0}^{\infty} c^i \\ &\leq \|\bar{W}^1 - \bar{W}^0\| c^n \sum_{i=0}^{\infty} c^i = \frac{c^n}{1-c}\|\bar{W}^1 - \bar{W}^0\|. \end{aligned} \tag{4.30}$$

□

Lemma 4.5. *For all $n \geq 0$, we have the equality*

$$W^n(t,\pi) = \bar{W}^n(t,\pi). \tag{4.31}$$

Proof. By Corollary 4.2, for all $n \geq 0$

$$\begin{aligned} W^n(t,\pi) &= \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} \\ &+ \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi]. \end{aligned} \tag{4.32}$$

Since $\bar{H}_m$ is compact and $\bar{W}^0(\tau,\pi,h)$ is a bounded continuous function on $[0,T]\times\mathcal{S}_N\times\bar{H}_m$, there exists a Borel function $w(\tau,\pi)$ such that $\sup_{h\in\bar{H}_m}\bar{W}^0(\tau,\pi,h)=\bar{W}^0(\tau,\pi,w(\tau,\pi))$. Furthermore, by Corollary 4.1(i) there exists a Borel function $\hat{h}(\tau,\pi)$ such that $\sup_{h\in\bar{H}_m}\hat{C}(\tau,\pi,h)=\hat{C}(\tau,\pi,\hat{h}(\tau,\pi))$ holds. For $n\geq 0$, we define the strategy

$$\begin{aligned} \tilde{h}_k &:= \hat{h}(\tau_k,\pi_k), && 0\leq k\leq n-1 \\ \tilde{h}_k &:= w(\tau_n,\pi_n), && k=n \\ \tilde{h}_k &:= \gamma(\tilde{X}_{\tau_n+(k-n)}-\tilde{X}_n,\tilde{h}_n), && k>n. \end{aligned} \tag{4.33}$$

By definition of $\{\tilde{h}_k\}_{k\in\mathbb{N}}$, we have $\{\tilde{h}_k\}_{k\in\mathbb{N}}\in\mathcal{A}^n$. Using Lemma 4.4(i) and Lemma 4.2, for $n\geq 0, t\in[0,T],\pi\in\mathcal{S}_N$

$$\begin{aligned} \bar{W}^n(t,\pi) &= E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_k,\tilde{h}_k)1_{\{\tau_k<T\}} \\ &\quad + \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,\tilde{h}_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\ &\leq \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} \\ &\quad + \int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\ &= W^n(t,\pi). \end{aligned} \tag{4.34}$$

Using again Lemma 4.2, (74) and Lemma 4.4(i), for all $n \geq 0, h \in \mathcal{A}^n, t \in [0,T], \pi \in \mathcal{S}_N$

$$\begin{aligned}
W(t,\pi,h.) &= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} \\
&\quad + \textstyle\int_{\tau_n}^T f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_0=\pi] \\
&\quad + E[\bar{W}^0(\tau_n,\pi_n,h_n)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&\leq E[\sum_{k=0}^{n-1} C(\tau_k,\pi_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_0=\pi] \\
&\quad + E[\bar{W}^0(\tau_n,\pi_n)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&= \bar{W}^n(t,\pi).
\end{aligned} \tag{4.35}$$

Therefore, we have

$$W^n(t,\pi) = \sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \leq \bar{W}^n(t,\pi). \tag{4.36}$$

and so we obtain for all $n \geq 0$

$$W^n(t,\pi) = \bar{W}^n(t,\pi). \tag{4.37}$$

□

Lemma 4.6. *For $n \geq 0$, we have the estimate*

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+1}(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{4.38}$$

Proof. By the definition of $\mathcal{A}^n$, for $n \geq 0, \mathcal{A}^n \subset \mathcal{A}^{n+1} \subset \mathcal{A}$, hence,

$$\sup_{h\in\mathcal{A}^n} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}^{n+1}} W(t,\pi,h.) \leq \sup_{h\in\mathcal{A}} W(t,\pi,h.). \tag{4.39}$$

By the definition of $W^n(t,\pi)$ and $W(t,\pi)$

$$W^n(t,\pi) \leq W^{n+1}(t,\pi) \leq W(t,\pi). \tag{4.40}$$

Using Lemma 4.5, for $n,m \geq 0$

$$\bar{W}^n(t,\pi) \leq \bar{W}^{n+m}(t,\pi) \leq W(t,\pi). \tag{4.41}$$

Letting $m \to \infty$

$$\bar{W}^n(t,\pi) \leq \bar{W}(t,\pi) \leq W(t,\pi). \tag{4.42}$$

□

Lemma 4.7. *The following estimate holds*

$$W(t,\pi) \leq \bar{W}(t,\pi) \tag{4.43}$$

for $t \in [0,T], \forall\pi \in \mathcal{S}_N$.

Proof. For $h \in \mathcal{A}$, we have

$$\begin{aligned}
W(t,\pi,h.) &= E[\sum_{k=0}^{\infty} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_0=\pi] \\
&= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_0=\pi] \\
&\quad + E[\sum_{k=n}^{\infty} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}|\tau_0=t,\pi_0=\pi] \\
&= E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}} \\
&\quad - \textstyle\int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&\quad + E[W(\tau_n,\pi_n,h.)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi].
\end{aligned} \tag{4.44}$$

We have the estimate

$$\begin{aligned}
&E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&\leq \sup_{h\in\mathcal{A}^n} E[\sum_{k=0}^{n-1} \hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}} + \int_{\tau_n}^{T} f(\theta_s,\gamma(\tilde{X}_s-\tilde{X}_n,h_n))ds1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&= W^n(t,\pi) = \bar{W}^n(t,\pi),
\end{aligned} \tag{4.45}$$

where we have used the representation of $W^n(t,\pi)$ in Corollary 4.2 (equation (4.13)) and Lemma 4.5. Furthermore

$$\begin{aligned}
&|E[\int_{\tau_n}^{T} f(\theta_t,\gamma(\tilde{X}_t-\tilde{X}_n,h_n))dt1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi]| \\
&\leq E[\textstyle\int_{\tau_n}^{T} |f(\theta_t,\gamma(\tilde{X}_t-\tilde{X}_n,h_n))|dt1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&\leq \|f\|E[(T-\tau_n)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \leq \|f\|TP(\tau_n<T|\tau_0=t).
\end{aligned} \tag{4.46}$$

Finally

$$\begin{aligned}
|E[W(\tau_n,\pi_n,h.)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi]| &\leq \|f\|E[(T-\tau_n)1_{\{\tau_n<T\}}|\tau_0=t,\pi_0=\pi] \\
&\leq \|f\|TP(\tau_n<T|\tau_0=t).
\end{aligned} \tag{4.47}$$

Therefore, we obtain

$$W(t,\pi,h.) \leq \bar{W}^n(t,\pi) + 2\|f\|TP(\tau_n<T|\tau_0=t) \tag{4.48}$$

for all $h \in \mathcal{A}$. Letting $n \to \infty$,

$$W(t,\pi,h.) \leq \bar{W}(t,\pi) \tag{4.49}$$

for all $h \in \mathcal{A}$. □

4.3 Main result

From the previous lemmas we obtain now the main result of this section, namely a Dynamic Programming-type approach to solve the log-utility maximization problem.

Theorem 4.1.

(i) Approximation theorem :
For any $\epsilon > 0, n > n_\epsilon$,

$$\|W - \bar{W}^n\| < \epsilon. \tag{4.50}$$

Where, n_ϵ *is the constant defined in Lemma 4.4(ii) and* $\bar{W}^n$ *are computed recursively according to (4.20) and (4.21).*

(ii) Dynamic programming principle : for any $n > 0$

$$\begin{aligned} W(t,\pi) = \sup_{h \in \mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_k, h_k) 1_{\{\tau_k < T\}} \\ + W(\tau_{n+1}, \pi_{n+1}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_0 = \pi] \end{aligned} \tag{4.51}$$

.

(iii) Optimal value and optimal strategy for the Log Utility Maximization Problem : for the utility maximization under the initial conditions $V_0 = v_0, \tau_0 = 0, \pi_0 = \pi$ *we have*

$$\begin{aligned} \sup_{h \in \mathcal{A}} E[\log V_T | \tau_0 = 0, \pi_0 = \pi] &= \log v_0 + \sup_{h \in \mathcal{A}} E[\int_0^T f(\theta_t, h_t) dt | \tau_0 = 0, \pi_0 = \pi] \\ &= \log v_0 + C(0,\pi) + \textstyle\sum_{k=1}^{\infty} E[\hat{C}(\tau_k, \pi_k, \hat{h}_k) 1_{\{\tau_k < T\}} | \tau_0 = 0, \pi_0 = \pi], \end{aligned} \tag{4.52}$$

where

$$\hat{h}^i_t = \gamma^i(\tilde{X}_t - \tilde{X}_k, \hat{h}_k), \quad \tau_k \le t < \tau_{k+1} \tag{4.53}$$

with $\hat{h}_k$ *defined in Corollary 4.1, namely* $\sup_{h \in \bar{H}_m} \hat{C}(\tau, \pi, h) = \hat{C}(\tau, \pi, \hat{h}(\tau, \pi))$ *and* $\hat{h}_k = \hat{h}(\tau_k, \pi_{\tau_k})$.

Proof. Let us first prove (i). By Lemma 4.6 and Lemma 4.7,

$$W(t,\pi) = \bar{W}(t,\pi). \tag{4.54}$$

Therefore, applying Lemma 4.4(ii) one obtains

$$\|W - \bar{W}^n\| < \epsilon. \tag{4.55}$$

Next, let us prove (ii). By (4.54), (4.15), (4.28) and by Corollary 4.1

$$\begin{aligned} W(t,\pi) &= \bar{W}(t,\pi) = \sum_{k=0}^{n} J^k C + J^{n+1} W(t,\pi) \\ &= E[\sum_{k=0}^{n} C(\tau_k, \pi_k) 1_{\{\tau_k < T\}} + W(\tau_{n+1}, \pi_{n+1}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_0 = \pi] \\ &= \sup_{h \in \mathcal{A}^n} E[\sum_{k=0}^{n} \hat{C}(\tau_k, \pi_k, h_k) 1_{\{\tau_k < T\}} + W(\tau_{n+1}, \pi_{n+1}) 1_{\{\tau_{n+1} < T\}} | \tau_0 = t, \pi_0 = \pi] \end{aligned} \tag{4.56}$$

Finally, (iii) is an immediate consequence of (4.4), Lemma 4.1 and Lemma 4.5 and its proof. $\square$

5 Appendix

Proof of Lemma 4.1.

Proof of statement (i). It follows from the two lemmas shown below.

Lemma 5.1. *We have the following representation,*

$$E[f(\theta_t, h_t)|\mathcal{G}_t] = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f_k(\theta_t, \tilde{X}_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}, \tag{5.1}$$

where,

$$f_k(\theta, x) := f(\theta, \gamma(x - \tilde{X}_k, h_k))$$

Proof. Any $\mathcal{G}_t-$adapted process Z_t has the following representation ([3])

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t) + Z_\infty 1_{]\tau_\infty,\infty[}(t), \tag{5.2}$$

where $Z_k(t)$ is $\mathcal{G}_k \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. Under our assumptions, for all $t > 0$, $\lim_{n\to\infty} 1_{\{\tau_n<t\}} = 0$.Therefore,

$$Z_t = \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t) \tag{5.3}$$

The result now follows by noticing that

$$\begin{aligned}
E[E[f(\theta_t, h_t)|\mathcal{G}_t]Z_t] &= E[f(\theta_t, h_t)Z_t] \\
&= E[f(\theta_t, h_t)\sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)] \\
&= \sum_{k\geq 0} E[E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]1_{\{\tau_k<t\}}Z_k(t)] \\
&= \sum_{k\geq 0} E[E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]\frac{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}1_{\{\tau_k<t\}}Z_k(t)] \\
&= \sum_{k\geq 0} E[\frac{E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}E[1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)|\mathcal{G}_k]] \\
&= \sum_{k\geq 0} E[E[\frac{E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}1_{]\tau_k,\tau_{k+1}]}(t)Z_k(t)|\mathcal{G}_k]] \\
&= E[\sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(t)\frac{E[f(\theta_t, h_t)1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{t\leq\tau_{k+1}\}}|\mathcal{G}_k]}Z_t]
\end{aligned}$$

Since from (2.16) it follows that

$$\begin{aligned}
f(\theta_t, h_t) &= \sum_{k=0}^{\infty} 1_{[\tau_k,\tau_{k+1})}(t)f(\theta_t, \gamma(\tilde{X}_t - \tilde{X}_k, h_k)) \\
&= \sum_{k=0}^{\infty} 1_{[\tau_k,\tau_{k+1})}(t)f_k(\theta_t, \tilde{X}_t).
\end{aligned}$$

we thus obtain (5.1). $\square$

Lemma 5.2. *We have the following equation*

$$E[\int_t^T f(\theta_s, h_s)ds|\tau_0 = t, \pi_0 = \pi] = E[\sum_{k\geq 0} \hat{C}(\tau_k, \pi_k, h_k)1_{\{\tau_k<T\}}|\tau_0 = t, \pi_0 = \pi]. \tag{5.4}$$

Where, $\hat{C}(t, \pi, h)$ is defined by (4.5) in Definition 4.1.

Proof. For simplicity, in the following formula we shall use the notation

$$E^{t,\pi}[\cdot] \equiv E[\cdot\,|\tau_0 = t, \pi_0 = \pi]$$

Using (5.1) we have

$$\begin{aligned}
&E^{t,\pi}[\int_t^T f(\theta_s, h_s)ds] = E^{t,\pi}[\int_t^T E[f(\theta_s, h_s)|\mathcal{G}_s]ds] \\
&= E^{t,\pi}[\int_t^T \sum_{k\geq 0} 1_{]\tau_k,\tau_{k+1}]}(s)\frac{E[f_k(\theta_s, \tilde{X}_s)1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{s\leq\tau_{k+1}\}}|\mathcal{G}_k]}ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[1_{\{s\leq\tau_{k+1}\}}|\mathcal{G}_k]\frac{E[f_k(\theta_s, \tilde{X}_s)1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]}{E[1_{\{s\leq\tau_{k+1}\}}|\mathcal{G}_k]}ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[f_k(\theta_s, \tilde{X}_s)1_{\{s<\tau_{k+1}\}}|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[e^{-\int_{\tau_k}^s n(\theta_u)du}f_k(\theta_s, \tilde{X}_s)|\mathcal{G}_k]ds] \\
&= E^{t,\pi}[\sum_{k\geq 0}\int_t^T 1_{]\tau_k,\infty)}(s)E[E[e^{-\int_{\tau_k}^s n(\theta_u)du}f_k(\theta_s, \tilde{X}_s)|\mathcal{G}_k \vee \sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds].
\end{aligned} \tag{5.5}$$

Since $(\theta_t, \tilde{X}_t)$ is a time homogeneous Markov process, we may write

$$E[e^{-\int_{\tau_k}^t n(\theta_s)ds)}f_k(\theta_t, \tilde{X}_t)|\mathcal{G}_k \vee \sigma\{\theta_{\tau_k}\}] = u_{\tilde{X}_k,h_k}(t-\tau_k, \theta_{\tau_k}, \tilde{X}_k), \tag{5.6}$$

where, recalling that $f_k(\theta, x) = f(\theta, \gamma(x - \tilde{X}_k, h_k)) := f_{\tilde{X}_k,h_k}(\theta, x)$, we have defined

$$u_{y,h}(t, \theta, x) := E[e^{-\int_0^t n(\theta_s)ds)}f_{y,h}(\theta_t, \tilde{X}_t)|\theta_0 = \theta, \tilde{X}_0 = x]. \tag{5.7}$$

We now have, recalling the definition of $r_{ji}(t,z)$ in (3.8),

$$
\begin{aligned}
&u_{y,h}(t,\theta,x)\\
&= E[e^{-\int_0^t n(\theta_s)ds} f_{y,h}(\theta_t,\tilde{X}_t)|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[e^{-\int_0^t n(\theta_s)ds} E[f_{y,h}(\theta_t,\tilde{X}_t-x+x)|\mathcal{F}_t^\theta,\theta_0=\theta,\tilde{X}_0=x]|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[e^{-\int_0^t n(\theta_s)ds} \textstyle\int_{\mathbb{R}^m} f_{y,h}(\theta_t,z+x)\rho_{0,t}^\theta(z)dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[\textstyle\int_{\mathbb{R}^m}\sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} e^{-\int_0^t n(\theta_s)ds} f_{y,h}(e_i,z+x)\rho_{0,t}^\theta(z)dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[\textstyle\int_{\mathbb{R}^m}\sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f_{y,h}(e_i,z+x)\\
&\qquad\qquad \times E[e^{-\int_0^t n(\theta_s)ds}\rho_{0,t}^\theta(z)|\theta_t=e_i,\theta_0=e_j]dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= E[\textstyle\int_{\mathbb{R}^m}\sum_{ij} 1_{\{\theta_t=e_i,\theta_0=e_j\}} f_{y,h}(e_i,z+x) r_{ji}(t,z)dz|\theta_0=\theta,\tilde{X}_0=x]\\
&= \textstyle\int_{\mathbb{R}^m}\sum_{ij} f_{y,h}(e_i,z+x) r_{ji}(t,z) p_{ji}(t) 1_{\{\theta=e_j\}}dz
\end{aligned}
\tag{5.8}
$$

On the other hand

$$
\begin{aligned}
&E[e^{-\int_{\tau_k}^t n(\theta_s)ds} f_k(\theta_t,\tilde{X}_t)|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}]\\
&= u_{\tilde{X}_k,h_k}(t-\tau_k,\theta_{\tau_k},\tilde{X}_k)\\
&= \textstyle\int_{\mathbb{R}^m}\sum_{ij} f_k(e_i,z+\tilde{X}_k) r_{ji}(t-\tau_k,z) p_{ji}(t-\tau_k) 1_{\{\theta_{\tau_k}=e_j\}}dz
\end{aligned}
\tag{5.9}
$$

Using (5.9) in (5.5) we finally have

$$
\begin{aligned}
&E^{t,\pi}[\int_t^T f(\theta_s,h_s)ds]\\
&= E^{t,\pi}[E[\sum_{k\geq 0}\int_t^T I_{]\tau_k,\infty)}(s) E[e^{-\int_{\tau_k}^s n(\theta_u)du} f_k(\theta_s,\tilde{X}_s)|\mathcal{G}_k\vee\sigma\{\theta_{\tau_k}\}]|\mathcal{G}_k]ds]\\
&= E^{t,\pi}[\sum_{k\geq 0} 1_{\{\tau_k<T\}}\int_{\tau_k}^T E[\int_{\mathbb{R}^m}\sum_{ij} f_k(e_i,z+\tilde{X}_k) r_{ji}(s-\tau_k,z) p_{ji}(s-\tau_k)\\
&\qquad\qquad \times 1_{\{\theta_{\tau_k}=e_j\}}dz|\mathcal{G}_k]ds]\\
&= E^{t,\pi}[\sum_{k\geq 0} 1_{\{\tau_k<T\}}\int_{\tau_k}^T \int_{\mathbb{R}^m}\sum_{ij} f_k(e_i,z+\tilde{X}_k) r_{ji}(s-\tau_k,z) p_{ji}(s-\tau_k)\\
&\qquad\qquad \times E[1_{\{\theta_{\tau_k}=e_j\}}|\mathcal{G}_k]dzds]\\
&= E^{t,\pi}[\sum_{k\geq 0} 1_{\{\tau_k<T\}}\int_{\tau_k}^T \int_{\mathbb{R}^m}\sum_{ij} f_k(e_i,z+\tilde{X}_k) r_{ji}(s-\tau_k,z) p_{ji}(s-\tau_k)\pi_k^j dzds]\\
&= E^{t,\pi}[\sum_{k\geq 0} 1_{\{\tau_k<T\}}\int_{\tau_k}^T \int_{\mathbb{R}^m}\sum_{ij} f(e_i,\gamma(z,h_k)) r_{ji}(s-\tau_k,z) p_{ji}(s-\tau_k)\pi_k^j dzds]\\
&= E^{t,\pi}[\sum_{k\geq 0}\hat{C}(\tau_k,\pi_k,h_k)1_{\{\tau_k<T\}}]
\end{aligned}
\tag{5.10}
$$

$\square$

Proof of statement (ii) of Lemma 4.1.

We start by proving that $\hat{C}(t,\pi,h)$ is Lipschitz continuous with respect to t.

$$\begin{aligned}\hat{C}(t,\pi,h) &= \int_t^T \int_{\mathbb{R}^m} \sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s-t,x)p_{ji}(s-t)\pi^j dxds \\ &= \int_0^{T-t} \int_{\mathbb{R}^m} \sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds.\end{aligned} \tag{5.11}$$

Thus

$$\begin{aligned}|\hat{C}(t,\pi,h)-\hat{C}(\bar{t},\pi,h)| &= |\int_{T-\bar{t}}^{T-t} \int_{\mathbb{R}^m} \sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds| \\ &\leq \|f\||t-\bar{t}|,\end{aligned} \tag{5.12}$$

where $\|f\| := \sup_{e\in E,h\in \bar{H}_m} \|f(e,h)\|$. Next, let us prove that $C(t,\pi,h)$ is Lipschitz continuous with respect to π (in the metric introduced in (3.21)).

$$\begin{aligned}|\hat{C}(t,\pi,h)-\hat{C}(t,\bar{\pi},h)| &= |\int_0^{T-t} \int_{\mathbb{R}^m} \sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)(\pi^j-\bar{\pi}^j)dxds| \\ &\leq \|f\|T|\pi-\bar{\pi}| = \|f\|T\textstyle\sum_{i=1}^N |\pi(e_i)-\bar{\pi}(e_i)| \\ &\leq \|f\|T\|\pi-\bar{\pi}\|_{TV} \leq \|f\|T\frac{2}{\log 3}d_H(\pi,\bar{\pi}),\end{aligned} \tag{5.13}$$

where we have used (3.20).

Next, let us prove that $C(t,\pi,h)$ is continuous with respect to h (always in the metric introduced in (3.21)). The function $f(e_i,h)$ is bounded and continuous with respect to h for all i. Furthermore, $\gamma(x,h)$ is continuous with respect to h for all $x\in\mathbb{R}^m$. Applying the dominated convergence theorem, for $h_n \subset \bar{H}_m$, s.t. $\lim\limits_{n\to\infty} h_n = h \in \bar{H}_m$

$$\begin{aligned}\lim_{n\to\infty}\hat{C}(t,\pi,h_n) &= \int_0^{T-t} \int_{\mathbb{R}^m} \sum_{i,j} \lim_{n\to\infty} f(e_i,\gamma(x,h_n))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\ &= \int_0^{T-t} \int_{\mathbb{R}^m} \sum_{i,j} f(e_i,\gamma(x,h))r_{ji}(s,x)p_{ji}(s)\pi^j dxds \\ &= \hat{C}(t,\pi,h).\end{aligned} \tag{5.14}$$

$\hat{C}(t,\pi,h)$ is thus continuous with respect to each of the variables t,π,h. However, continuity in t,π is independent of the other variable. Hence, $\hat{C}(t,\pi,h)$ is a continuous function on $[0,T]\times \mathcal{S}_N \times \bar{H}_m$.

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