

Expected power-utility maximization under incomplete information and with Cox-process observations

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Abstract

We consider the problem of maximization of expected terminal power utility (risk sensitive criterion). The underlying market model is a regime-switching diffusion model where the regime is determined by an unobservable factor process forming a finite state Markov process. The main novelty is due to the fact that prices are observed and the portfolio is rebalanced only at random times corresponding to a Cox process where the intensity is driven by the unobserved Markovian factor process as well. This leads to a more realistic modeling for many practical situations, like in markets with liquidity restrictions; on the other hand it considerably complicates the problem to the point that traditional methodologies cannot be directly applied. The approach presented here is specific to the power-utility. For log-utilities a different approach is presented in the companion paper [8].

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1 Introduction

In this paper we consider a classical portfolio optimization problem, namely the maximization of expected utility from terminal wealth. As criterion we take a variant of expected power utility maximization (the log utility function is studied in the companion paper [8]). More precisely we consider a risk-sensitive type criterion of the form

$$\sup_{\mu} \frac{1}{\mu} E \{V_T^{\mu}\}$$

where we restrict ourselves to the case of $\mu < 0$ (concave utility). While the problem as such is a classical one, the novelty in this paper is given by a more realistic underlying market model, which implies that standard approaches such as Dynamic Programming and convex duality cannot be directly applied and a novel approach is required. Our market model is first of all of the regime-switching type where the factor process that specifies the regime may not be fully observable. This is still a relatively classical situation, for which one may use techniques from stochastic control under incomplete information. In fact, various papers have appeared in such a context and here we just mention some of the most recent ones that also summarize previous work in the area, more precisely [2], [16], [20]. The main novelty of our model is however given by the fact that the prices S_t^i , or equivalently their logarithmic values $X_t^i := \log S_t^i$, of the risky assets in which one invests are supposed to be observable only at discrete random points in time $\tau_0, \tau_1, \tau_2, \dots$, where the associated counting process is a Cox process (see e.g. [3], [11]) with intensity that depends on the same factor process that specifies the regime for the price evolution model. Such models are in fact relevant in financial applications since (see e.g. [7], [4], [17], [18]), especially on small time scales, prices do not vary continuously, but rather change and are observed only at discrete random points in time that correspond to the time instants when significant new information is absorbed by the market and/or market makers update their quotes. This setting leads to a stochastic control problem with incomplete information and observations given by a Cox process.

A classical approach to incomplete observation control problems is to first transform the problem into a so-called separated problem, where the unobservable part of the state is replaced by its conditional distribution. This requires to solve first the associated filtering problem, which already is non-standard and has been solved recently in [4] (see also [5]). Our major contribution here is on the control part of the separated problem that is approached in a non classical way. In particular we shall restrict the investment strategies to be rebalanced only at the random times τ_k where prices change. Although slightly less general from a theoretical point of view, restricting trading to discrete, in particular random times, is quite realistic in finance, where in practice one cannot rebalance a portfolio continuously: think of the case with transaction costs or with liquidity restrictions (in this latter context see e.g. [9], [10], [14], [17], [18], [19] where the authors consider illiquid markets, partly also with regime switching models as in this paper, but under complete information).

In the companion paper [8] we study the case of a log utility, for the solution of which one cannot simply carry over the approach that we develop here for the power-utility case, even if there are close analogies between the two approaches. In other words, for our nonclassical setup the approach that one has to use may depend on the specific case.

In section 2 we give a more precise definition of the model and of the investment strategy and, after recalling some of the main filtering results from [4], we also specify the objective

function. In section 3 we obtain some preliminary results consisting mainly in estimation results and establishing continuity properties. Sections 4 and 5 then prepare for the main results of the paper stated in section 6. More precisely, the results of Section 4 will be used to obtain an approximation of the optimal value function, and those of Section 5 will be useful also to obtain a Dynamic Programming principle by a specific approach that depends on the chosen criterion function. In addition to the approximation result and the DP principle, the results of section 6 serve also the purpose of obtaining a methodology to determine an optimal strategy.

2 Market model, basic tools and objective

2.1 Introductory remarks

As mentioned in the general Introduction, we consider here the problem of maximization of expected power-utility from terminal wealth, when the dynamics of the prices of the risky assets in which one invests are of the usual diffusion type but with the coefficients in the dynamics depending on an unobservable finite-state Markovian factor process (regime-switching model). In addition it is assumed that the risky asset prices S_t^i , or equivalently their logarithmic values $X_t^i := \log S_t^i$, are observed only at random times τ_0, τ_1, \dots for which the associated counting process forms a Cox process with an intensity $n(\theta_t)$ that also depends on the unobservable factor process θ_t .

2.2 The market model

Let θ_t be the hidden finite state Markovian factor process. With Q denoting its transition intensity matrix (Q -matrix) its dynamics are given by

$$d\theta_t = Q^* \theta_t dt + dM_t, \theta_0 = \xi. \quad (2.1)$$

where M_t is a jump-martingale on a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. If N is the number of possible values of θ_t , we may without loss of generality take as its state space the set $E = \{e_1, \dots, e_N\}$, where e_i is a unit vector for each $i = 1, \dots, N$ (see [6]).

The evolution of θ_t may also be characterized by the process π_t given by the state probability vector that takes values in the set

$$\mathcal{S}_N := \left\{ \pi \in \mathbb{R}^N \mid \sum_{i=1}^N \pi^i = 1, 0 \leq \pi^i \leq 1, i = 1, 2, \dots, N \right\} \quad (2.2)$$

namely the set of all probability measures on E and we have $\pi_0^i = P(\xi = e_i)$. Denoting by $\mathcal{M}(E)$ the set of all finite nonnegative measures on E , it follows that $\mathcal{S}_N \subset \mathcal{M}(E)$. In our study it will be convenient to consider on $\mathcal{M}(E)$ the Hilbert metric $d_H(\pi, \bar{\pi})$ defined (see [1] [12] [13]) by

$$d_H(\pi, \bar{\pi}) := \log \left(\sup_{\bar{\pi}(A) > 0, A \subset E} \frac{\pi(A)}{\bar{\pi}(A)} \sup_{\pi(A) > 0, A \subset E} \frac{\bar{\pi}(A)}{\pi(A)} \right). \quad (2.3)$$

Notice that, while d_H is only a pseudo-metric on $\mathcal{M}(E)$, it is a metric on \mathcal{S}_N ([1]). Furthermore, see Lemma 3.6 in [12], for $\forall \pi, \bar{\pi} \in \mathcal{S}_N$ we also have

$$\|\pi - \bar{\pi}\|_{TV} \leq \frac{2}{\log 3} d_H(\pi, \bar{\pi}), \quad (2.4)$$

where $\|\cdot\|_{TV}$ is the total variation on \mathcal{S}_N .

In our market we consider m risky assets, for which the price processes $S^i = (S_t^i)_{t \geq 0}, i = 1, \dots, m$ are supposed to satisfy

$$dS_t^i = S_t^i \{r^i(\theta_t) dt + \sum_j \sigma_j^i(\theta_t) dB_t^j\}, \quad (2.5)$$

for given coefficients $r^i(\theta)$ and $\sigma_j^i(\theta)$ and with B_t^j ($j = 1, \dots, m$) independent (\mathcal{F}_t, P) -Wiener processes. Letting $X_t := \log S_t$, by Itô's formula we have

$$X_t = X_0 + \int_0^t r(\theta_s) ds - d(\sigma\sigma^*(\theta_s)) + \int_0^t \sigma(\theta_s) dB_s, \quad (2.6)$$

where by $d(\sigma\sigma^*(\theta))$ we denote the diagonal matrix $d(\sigma\sigma^*(\theta)) := (\frac{1}{2}(\sigma\sigma^*)^{11}(\theta), \dots, \frac{1}{2}(\sigma\sigma^*)^{mm}(\theta))$. As usual there is also a locally non-risky asset (bond) with price S_t^0 satisfying

$$dS_t^0 = r_0 S_t^0 dt \quad (2.7)$$

where r_0 stands for the short rate of interest.

As already mentioned, the asset prices and thus also their logarithms are observed only at random times $\tau_0, \tau_1, \tau_2, \dots$ and we shall put $X_k = (X_k^1, \dots, X_k^m)$ with $X_k^i := X_{\tau_k}^i$, ($i = 1, \dots, m; k \in \mathbb{N}$). The observations are thus given by the sequence $(\tau_k, X_k)_{k \in \mathbb{N}}$ that forms a multivariate marked point process with counting measure

$$\mu(dt, dx) = \sum_k \mathbf{1}_{\{\tau_k < \infty\}} \delta_{\{\tau_k, X_k\}}(t, x) dt dx. \quad (2.8)$$

The corresponding counting process $\Lambda_t := \int_0^t \int_{\mathbb{R}^m} \mu(dt, dx)$ is supposed to be a Cox process with intensity $n(\theta_t)$, i.e. $\Lambda_t - \int_0^t n(\theta_s) ds$ is an (\mathcal{F}_t, P) -martingale. We consider two sub-filtrations related to $(\tau_k, X_k)_{k \in \mathbb{N}}$ namely

$$\begin{aligned} \mathcal{G}_t &:= \mathcal{F}_0 \vee \sigma\{\mu((0, s] \times B) : s \leq t, B \in \mathcal{B}(\mathbb{R}^m)\}, \\ \mathcal{G}_k &:= \mathcal{F}_0 \vee \sigma\{\tau_0, X_0, \tau_1, X_1, \tau_2, X_2, \dots, \tau_k, X_k\}. \end{aligned} \quad (2.9)$$

In our development below we shall often make use of the following notations. Let

$$\begin{aligned} m_0(\theta) &:= r_0, \quad \sigma_0(\theta) := 0 \\ m_i(\theta) &:= r^i(\theta) - \frac{1}{2}(\sigma\sigma^*)^{ii}(\theta) \quad \text{for } i = 1, \dots, m \\ \sigma_i(\theta) &:= \sqrt{(\sigma\sigma^*)^{ii}(\theta)} \quad \text{for } i = 1, \dots, m \end{aligned} \quad (2.10)$$

and put $m(\theta) := \{m_1(\theta), \dots, m_m(\theta)\}$. For the conditional (on \mathcal{F}^θ) mean and variance of $X_s - X_t = \{X_s^i - X_t^i\}_{i=1, \dots, m}$ with $t < s \leq T$ we then have

$$m_{t,s}^\theta = \int_t^s m(\theta_u) du \quad , \quad \sigma_{t,s}^\theta = \int_t^s \sigma \sigma^*(\theta_u) du \quad (2.11)$$

and, for $z \in \mathbb{R}^m$ and $t < s \leq T$, we set

$$\rho_{t,s}^\theta(z) \sim N(z; m_{t,s}^\theta, \sigma_{t,s}^\theta) \quad (2.12)$$

namely the joint conditional (on \mathcal{F}^θ) density of $X_s - X_t$. Notice that, although here and in the companion paper [8] we use the same symbol $\rho^\theta(z)$, the corresponding quantities are different in the two papers due to a different mean value which is a consequence of the fact that here we base ourselves on the process $X_t = \log S_t$ while in [8] it is $\tilde{X}_t = \log \tilde{S}_t$.

2.3 Investment strategies and portfolios,

As mentioned in the Introduction, since observations take place at random time points τ_k , we shall consider investment strategies that are rebalanced only at those same time points τ_k .

Let N_t^i be the number of assets of type i held in the portfolio at time t , $N_t^i = \sum_k \mathbf{1}_{[\tau_k, \tau_{k+1})}(t) N_{\tau_k}^i$. The wealth process is defined by

$$V_t := \sum_{i=0}^m N_t^i S_t^i.$$

Consider then the investment ratios

$$h_t^i := \frac{N_t^i S_t^i}{V_t}, \quad i = 1, \dots, m$$

and set $h_{\tau_k}^i := h_{\tau_k}^i$. The set of admissible investment ratios is given by

$$\bar{H}_m := \{(h^1, \dots, h^m); h^1 + h^2 + \dots + h^m \leq 1, 0 \leq h^i, i = 1, 2, \dots, m\}, \quad (2.13)$$

i.e. no shortselling is allowed and notice that \bar{H}_m is bounded and closed. Put $h = (h^1, \dots, h^m)$. Analogously to [15] define next a function $\gamma: \mathbb{R}^m \times \bar{H}_m \rightarrow \bar{H}_m$ by

$$\gamma^i(z, h) := \frac{h^i \exp(z^i)}{1 + \sum_{i=1}^m h^i (\exp(z^i) - 1)} \quad i = 1, \dots, m. \quad (2.14)$$

Putting $\tilde{X}_t^i := \log \tilde{S}_t^i$ with $\tilde{S}_t^i := \frac{S_t^i}{S_t^0}$ (discounted asset prices) and noticing that N_t is constant on $[\tau_k, \tau_{k+1})$, for $i = 1, \dots, m$, and $t \in [\tau_k, \tau_{k+1})$, let

$$\begin{aligned} h_t^i &= \frac{N_t^i S_t^i}{\sum_{i=0}^m N_t^i S_t^i} = \frac{N_k^i S_t^i}{\sum_{i=0}^m N_k^i S_t^i} \\ &= \frac{N_k^i S_k^i S_t^i / S_k^i}{\sum_{i=0}^m N_k^i S_k^i S_t^i / S_k^i} = \frac{h_k^i S_t^i / S_k^i}{\sum_{i=0}^m h_k^i S_t^i / S_k^i} = \frac{h_k^i S_k^0 / S_t^0 S_t^i / S_k^i}{\sum_{i=0}^m h_k^i S_k^0 / S_t^0 S_t^i / S_k^i} \\ &= \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)}{h_k^0 + \sum_{i=1}^m h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)} = \frac{h_k^i \exp(\tilde{X}_t^i - \tilde{X}_k^i)}{1 + \sum_{i=1}^m h_k^i (\exp(\tilde{X}_t^i - \tilde{X}_k^i) - 1)} \\ &= \gamma^i(\tilde{X}_t - \tilde{X}_k, h_k). \end{aligned} \quad (2.15)$$

The set of admissible strategies \mathcal{A} is defined by

$$\mathcal{A} := \{\{h_k\}_{k=0}^\infty | h_k \in \bar{H}_m, \mathcal{G}_k \text{ m'ble for all } k \geq 0\}. \quad (2.16)$$

Furthermore, for $n > 0$, we let

$$\mathcal{A}^n := \{h \in \mathcal{A} | h_{n+i} = h_{\tau_{n+i}-} \text{ for all } i \geq 1\}. \quad (2.17)$$

Notice that, by the definition of \mathcal{A}^n , for all $k \geq 1$, $h \in \mathcal{A}^n$ we have

$$\begin{aligned} h_{n+k}^i &= h_{\tau_{n+k}-}^i \\ \Leftrightarrow \frac{N_{n+k}^i S_{n+k}^i}{\sum_{i=0}^m N_{n+k}^i S_{n+k}^i} &= \frac{N_{n+k-1}^i S_{n+k}^i}{\sum_{i=0}^m N_{n+k}^i S_{n+k}^i} \\ \Leftrightarrow N_{n+k} &= N_{n+k-1}. \end{aligned}$$

Therefore, for $k \geq 1$

$$N_{n+k} = N_n.$$

and

$$\mathcal{A}^0 \subset \mathcal{A}^1 \subset \dots \subset \mathcal{A}^n \subset \mathcal{A}^{n+1} \dots \subset \mathcal{A}. \quad (2.18)$$

Remark 2.1. Notice that, for a given finite sequence of investment ratios h_0, h_1, \dots, h_n such that h_k is an \mathcal{G}_k -measurable, \bar{H}_m -valued random variable for $k \leq n$, there exists $h^{(n)} \in \mathcal{A}^n$ such that $h_k^{(n)} = h_k$, $k = 0, \dots, n$. Indeed, if N_t is constant on $[\tau_n, T)$, then for h_t we have $h_t = \gamma(\tilde{X}_t - \tilde{X}_n, h_n)$, $\forall t \geq \tau_n$. Therefore, by setting $h_\ell^{(n)} = h_\ell$, $\ell = 0, \dots, n$, and $h_{n+k}^{(n)} = h_{\tau_{n+k}}$, $k = 1, 2, \dots$, since the vector process S_t and the vector function $\gamma(\cdot, h_n)$ are continuous, we see that $h_{n+k}^{(n)} = h_{\tau_{n+k}-}$, $k = 1, 2, \dots$.

Finally, considering only self-financing portfolios, for their value process we have the dynamics

$$\frac{dV_t}{V_t} = [r_0 + h_t^* \{r(\theta_t) - r_0 \mathbf{1}\}] dt + h_t^* \sigma(\theta_t) dB_t. \quad (2.19)$$

2.4 Filtering

As mentioned in the Introduction, the usual approach to stochastic control problems under incomplete information is to first transform them into a so-called separated problem, where the unobservable part of the state is replaced by its conditional (filter) distribution. This implies that we first have to study this conditional distribution and its (Markovian) dynamics, i.e. we have to study the associated filtering problem.

The filtering problem for our specific case, where the observations are given by a Cox process with intensity expressed as a function of the unobserved state, has been studied in [4] (see also [5]). Here we briefly summarize the main results from [4] in view of their use in our control problem.

Recalling the definition of $\rho^\theta(z)$ in (2.12) and putting

$$\phi^\theta(\tau_k, t) = n(\theta_t) \exp\left(-\int_{\tau_k}^t n(\theta_s) ds\right), \quad (2.20)$$

for a given function $f(\theta)$ we let

$$\psi_k(f; t, x) := E[f(\theta_t)\rho_{\tau_k, t}^\theta(x - X_k)\phi^\theta(\tau_k, t)|\sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \quad (2.21)$$

$$\bar{\psi}_k(f; t) := \int \psi_k(f; t, x)dx = E[f(\theta_t)\phi^\theta(\tau_k, t)|\sigma\{\theta_{\tau_k}\} \vee \mathcal{G}_k] \quad (2.22)$$

$$\pi_t(f) = E[f(\theta_t)|\mathcal{G}_t] \quad (2.23)$$

with ensuing obvious meanings of $\pi_{\tau_k}(\psi_k(f; t, y))$ and $\pi_{\tau_k}(\bar{\psi}_k(f; t))$ where we consider $\psi_k(f; t, y)$ and $\bar{\psi}_k(f; t)$ as functions of θ_{τ_k} . The process $\pi_t(f)$ is called the *filter process* for $f(\theta_t)$.

We have (see Lemma 4.1 in [4])

Lemma 2.1. *The compensator of the random measure $\mu(dt, dy)$ in (2.8) on the σ -algebra $\tilde{\mathcal{P}}(\mathcal{G}) = \mathcal{P}(\mathcal{G}) \otimes \mathcal{B}(R^m)$ with $\mathcal{P}(\mathcal{G})$ the predictable σ -algebra on $\Omega \times [0, \infty)$, is given by the following nonnegative random measure*

$$\nu(dt, dy) = \sum_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t) \frac{\pi_{\tau_k}(\psi_k(1, t, y))}{\int_t^\infty \pi_{\tau_k}(\bar{\psi}_k(1, s))ds} dt dy. \quad (2.24)$$

The main filtering result is the following (see Theorem 4.1 in [4]).

Theorem 2.1. *For any bounded function $f(\theta)$, the differential of the optimal filter $\pi_t(f)$ is given by*

$$\begin{aligned} d\pi_t(f) &= \pi_t(Lf)dt \\ &+ \int \sum_k \mathbf{1}_{(\tau_k, \tau_{k+1}]}(t) \left[\frac{\pi_{\tau_k}(\psi_k(f; t, y))}{\pi_{\tau_k}(\bar{\psi}_k(1; t, y))} - \pi_{t-}(f) \right] (\mu - \nu)(dt, dy), \end{aligned} \quad (2.25)$$

where L is the generator of the Markov process θ_t (namely $L = Q$).

Corollary 2.1. *We have*

$$\pi_{\tau_{k+1}}(f) = \frac{\pi_{\tau_k}(\psi_k(f; t, x))}{\pi_{\tau_k}(\bar{\psi}_k(1; t, x))} \Big|_{t=\tau_{k+1}, x=X_{k+1}}. \quad (2.26)$$

Recall that in our setting θ_t is an N -state Markov chain with state space $E = \{e_1, \dots, e_N\}$, where e_i is a unit vector for each $i = 1, \dots, N$. One may then write $f(\theta_t) = f(e_i)\mathbf{1}_{e_i}(\theta_t)$. For $i = 1, \dots, N$ let $\pi_t^i = \pi_t(\mathbf{1}_{e_i}(\theta_t))$ and

$$r_{ji}(t, z) := E[\exp(\int_0^t -n(\theta_s)ds)\rho_{0, t}^\theta(z)|\theta_0 = e_j, \theta_t = e_i], \quad (2.27)$$

$$p_{ji}(t) := P(\theta_t = e_i|\theta_0 = e_j) \quad (2.28)$$

and, noticing that $\pi_t \in \mathcal{S}_N$, define the function $M : [0, \infty) \times \mathbb{R}^m \times \mathcal{S}_N \rightarrow \mathcal{S}_N$ by

$$M^i(t, x, \pi) := \frac{\sum_j n(e_j)r_{ji}(t, x)p_{ji}(t)\pi^j}{\sum_j n(e_j)r_{ji}(t, x)p_{ji}(t)\pi^j}, \quad (2.29)$$

$$M(t, x, \pi) := (M^1(t, x, \pi), M^2(t, x, \pi), \dots, M^N(t, x, \pi)). \quad (2.30)$$

For $A \subset E$

$$M(t, x, \pi)(A) := \sum_{i=1}^N M^i(t, x, \pi) 1_{\{e_i \in A\}}. \quad (2.31)$$

The following corollary, for the proof of which we refer to the companion paper [8] (see also [4]), describes the dynamics of the filter process.

Corollary 2.2. *For the generic i -th state one has*

$$\pi_{k+1}^i = M^i(\tau_{k+1} - \tau_k, X_{k+1} - X_k, \pi_k) \quad (2.32)$$

and the process $\{\tau_k, \pi_k, X_k\}_{k=1}^\infty$ is a Markov process with respect to \mathcal{G}_k .

Finally from results in [1], [12] and [13] it follows (see again the companion paper [8]) that

$$d_H(M(t, x, \pi), M(t, x, \bar{\pi})) \leq d_H(\pi, \bar{\pi}) \quad (2.33)$$

2.5 Objective

Given a finite planning horizon $T > 0$, our problem of maximization of expected terminal power utility, consists in determining

$$\begin{aligned} & \sup_{h \in \mathcal{A}} \frac{1}{\mu} \log E[V_T^\mu | \tau_0 = 0, \pi_0 = \pi] \\ & = \log v_0 + \sup_{h \in \mathcal{A}} \frac{1}{\mu} \log E\left[\frac{V_T^\mu}{V_0^\mu} | \tau_0 = 0, \pi_0 = \pi\right]. \end{aligned} \quad (2.34)$$

for $\mu < 0$ as well as an optimal maximizing strategy $\hat{h} \in \mathcal{A}$.

3 Preliminary analysis

To study the optimization problem (2.34) stated in subsection 2.5, it suffices to analyze the criterion function

$$W(t, \pi, h.) := \frac{1}{\mu} \log E\left[\frac{V_T^\mu}{V_t^\mu} | \tau_0 = t, \pi_0 = \pi\right] \quad (3.1)$$

In the following we shall use the notation such that

$$E^{t, \pi}[\cdot] \equiv E[\cdot | \tau_0 = t, \pi_0 = \pi]$$

for simplicity.

Lemma 3.1. *For $t \in [0, T], \pi \in \mathcal{S}_N$*

$$\begin{aligned} & W(t, \pi, h.) \\ & = \frac{1}{\mu} \log E^{t, \pi} \left[\exp\left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) \right], \end{aligned} \quad (3.2)$$

where,

$$D(h, x) := \log\left(\sum_{i=0}^m h^i \exp(x^i)\right). \quad (3.3)$$

Proof. Since $\sum_{i=0}^m h^i = 1$,

$$D(h, 0) = \log\left(\sum_{i=0}^m h^i\right) = 0 \quad (3.4)$$

for $h \in \bar{H}_m$. For $k \geq 0$, $T \in [\tau_k, \tau_{k+1}]$

$$\begin{aligned} \frac{V_T^\mu}{V_k^\mu} &= \left(\sum_{i=0}^m \frac{N_T^i S_T^i}{V_k} \right)^\mu = \left(\sum_{i=0}^m \frac{N_k^i S_k^i S_T^i}{V_k S_k^i} \right)^\mu = \left(\sum_{i=0}^m h_k^i \frac{S_T^i}{S_k^i} \right)^\mu \\ &= \left(\sum_{i=0}^m h_k^i \exp(X_T^i - X_k^i)\right)^\mu \\ &= \exp(\mu D(h_k, X_T - X_k)). \end{aligned} \quad (3.5)$$

For $k \geq 1$, $T < \tau_k$

$$\frac{V_{T \wedge \tau_{k+1}}^\mu}{V_{T \wedge \tau_k}^\mu} = \frac{V_T^\mu}{V_T^\mu} = 1, \quad (3.6)$$

and

$$\exp(\mu D(h_k, X_{T \wedge \tau_{k+1}} - X_{T \wedge \tau_k})) = \exp(\mu D(h_k, X_T - X_T)) = 1. \quad (3.7)$$

Therefore, we obtain

$$\begin{aligned} E^{t, \pi}[(V_T/V_t)^\mu] &= E^{t, \pi}\left[\prod_{k=1}^{\infty} \frac{V_{T \wedge \tau_k}^\mu}{V_{T \wedge \tau_{k-1}}^\mu}\right] \\ &= E^{t, \pi}\left[\exp\left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{T \wedge \tau_{k-1}})\right)\right] \\ &= E^{t, \pi}\left[\exp\left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right)\right]. \end{aligned} \quad (3.8)$$

□

3.1 First estimates

We introduce and study now a function that will play a crucial role in the sequel. Set

$$\bar{W}^0(t, \pi, h) := \frac{1}{\mu} \log E^{t, \pi}[\exp(\mu D(h, X_T - X_t))] \quad (3.9)$$

We have the following lemma.

Lemma 3.2. For $t \in [0, T]$, $h \in \bar{H}_m$, we have the estimate:

$$\exp(\mu\bar{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t) \leq E^{t,\pi}[\exp(\mu D(h, X_T - X_t))] \leq \exp(\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t), \quad (3.10)$$

where $\bar{m} := \max_{0 \leq i \leq m} \max_{1 \leq j \leq N} m_i(e_j)$, $\underline{m} := \min_{0 \leq i \leq m} \min_{1 \leq j \leq N} m_i(e_j) \wedge 0$, implying that $\underline{m} \leq 0$, and $\bar{\sigma} := \max_{0 \leq i \leq m} \max_{1 \leq j \leq N} \sigma_i(e_j)$.

Proof. Since x^μ is convex, Jensen's inequality applies and we obtain

$$E^{t,\pi}[\left(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\right)^\mu] \leq E^{t,\pi}[\sum_{i=0}^m h^i \exp(\mu(X_T^i - X_t^i))]. \quad (3.11)$$

For each i and $t \in [0, T]$,

$$\begin{aligned} \underline{m}(T-t) &\leq \int_t^T m_i(\theta_s) ds \leq \bar{m}(T-t), \\ \int_t^T \sigma_i^2(\theta_s) ds &\leq \bar{\sigma}^2(T-t). \end{aligned} \quad (3.12)$$

Thus, we have

$$\begin{aligned} E^{t,\pi}[\exp(\mu(X_T^i - X_t^i))] &= E^{t,\pi}[\exp(\mu \int_t^T m_i(\theta_s) ds + \frac{\mu^2}{2} \int_t^T \sigma_i^2(\theta_s) ds)] \\ &\leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)), \end{aligned} \quad (3.13)$$

and

$$E^{t,\pi}[\exp(\mu(X_T^0 - X_t^0))] = \exp(\mu r_0(T-t)) \leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)). \quad (3.14)$$

Therefore, from (3.11) it follows that

$$\begin{aligned} E^{t,\pi}[\left(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\right)^\mu] &\leq \sum_{i=0}^m h^i \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \\ &= \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)). \end{aligned} \quad (3.15)$$

To obtain the lower estimate, applying Jensen's inequality yields

$$(E^{t,\pi}[\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)])^\mu \leq E^{t,\pi}[\left(\sum_{i=0}^m h^i \exp(X_T^i - X_t^i)\right)^\mu]. \quad (3.16)$$

Since x^μ is a decreasing function, we have

$$\begin{aligned} \left(\sum_{i=0}^m h^i E^{t,\pi}[\exp(X_T^i - X_t^i)]\right)^\mu &= \left(\sum_{i=0}^m h^i E^{t,\pi}[\exp(\int_t^T m_i(\theta_s) ds + \frac{1}{2} \int_t^T \sigma_i^2(\theta_s) ds)]\right)^\mu \\ &\geq \left(\sum_{i=0}^m h^i \exp((\bar{m} + \frac{(\bar{\sigma})^2}{2})(T-t))\right)^\mu \\ &= \exp((\mu\bar{m} + \frac{\mu(\bar{\sigma})^2}{2})(T-t)). \end{aligned} \quad (3.17)$$

□

The following is the direct consequence of Lemma 3.2.

Corollary 3.1.

$$(\underline{m} + \frac{\mu\bar{\sigma}^2}{2})(T-t) \leq \bar{W}^0(t, \pi, h) \leq (\bar{m} + \frac{\bar{\sigma}^2}{2})(T-t). \quad (3.18)$$

Lemma 3.3. $\bar{W}^0(t, \pi, h)$ is a continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$ and the following estimates hold:

$$|\exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(t, \bar{\pi}, h))| \leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \frac{2}{\log 3} d_H(\pi, \bar{\pi}), \quad (3.19)$$

$$|\exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(\bar{t}, \pi, h))| \leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) l(t - \bar{t}) \text{ for } \bar{t} < t, \quad (3.20)$$

where, d_H is defined by (3) and

$$l(t) := E[|1 - \exp(\mu|X_t - X_0|)|]. \quad (3.21)$$

Proof. Let us first prove the continuity of $\bar{W}^0(t, \pi, h)$ with respect to π . Owing to (2.4) and recalling $p_{ji}(t)$ that was defined in (2.28), we have

$$\begin{aligned} & |\exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(t, \bar{\pi}, h))| \\ &= |\int_{\mathbb{R}^m} (h^0 \exp(r_0(T-t)) + \sum_{i=1}^m h^i \exp(y^i))^\mu \\ &\quad \times \sum_{ij} E[\rho_{0, T-t}^\theta(y) | \theta_0 = e_j, \theta_{T-t} = e_i] p_{ji}(T-t) (\pi^j - \bar{\pi}^j) dy ds| \\ &= |\sum_j \exp(\mu\bar{W}^0(t, e_j, h)) (\pi^j - \bar{\pi}^j)| \\ &\leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \sum_j |\pi^j - \bar{\pi}^j| \\ &\leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \|\pi^j - \bar{\pi}^j\|_{TV} \\ &\leq \exp((\mu\underline{m} + \frac{(\mu\bar{\sigma})^2}{2})(T-t)) \frac{2}{\log 3} d_H(\pi, \bar{\pi}). \end{aligned} \quad (3.22)$$

Next, we show the continuity of $\bar{W}^0(t, \pi, h)$ with respect to t . First notice that, due to the time homogeneity of the process (X_t, θ_t) ,

$$\begin{aligned} \exp(\mu\bar{W}^0(t, \pi, h)) &= E^{t, \pi}[\exp(\mu D(h, X_T - X_t))] \\ &= E^{0, \pi}[\exp(\mu D(h, X_{T-t} - X_0))]. \end{aligned} \quad (3.23)$$

Notice furthermore that

$$|D(h, x) - D(h, y)| \leq |x - y| \quad (3.24)$$

holds because

$$|\nabla_x D(h, x)| \leq 1. \quad (3.25)$$

Therefore,

$$\begin{aligned}
& |\exp(\mu\bar{W}^0(t, \pi, h)) - \exp(\mu\bar{W}^0(\bar{t}, \pi, h))| \\
&= |E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) - \exp(\mu D(h, X_{T-\bar{t}} - X_0))]| \\
&= |E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) \left(1 - \exp(\mu(D(h, X_{T-\bar{t}} - X_0) - D(h, X_{T-t} - X_0)))\right)]| \\
&\leq E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) |1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t})|] \\
&= E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t})||X_{T-t}]] \\
&= E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))] l(t - \bar{t}) \\
&\leq \exp((\underline{\mu}m + \frac{(\underline{\mu}\bar{\sigma})^2}{2})(T-t)) l(t - \bar{t}).
\end{aligned} \tag{3.26}$$

Finally, we prove the continuity with respect to h . By the definition of $D(h, x)$ and Jensen's inequality,

$$\exp(\mu D(h, x)) = \left(\sum_{i=0}^m h^i \exp(x^i)\right)^\mu \leq \sum_{i=0}^m h^i \exp(\mu x^i) \leq \sum_{i=0}^m \exp(\mu x^i). \tag{3.27}$$

Therefore, for $m \geq 1$

$$\exp(\mu D(h, X_T - X_t)) \leq \sum_{i=0}^m \exp(\mu(X_T^i - X_t^i)). \tag{3.28}$$

Since $X_T - X_t = \{X_T^i - X_t^i\}_{i=1,\dots,m}$ is, conditionally on \mathcal{F}^θ , Gaussian with mean $\{\int_t^T m_i(\theta_s) ds\}_{i=1,\dots,m}$ and covariance $\{\int_t^T (\sigma\sigma^*)^{ij}(\theta_s) ds\}_{i,j=1,\dots,m}$ we have

$$E^{t,\pi}[\exp(\mu(X_T^i - X_t^i))] < \infty. \tag{3.29}$$

Then, applying the dominated convergence theorem, for $h_j \in \bar{H}_m$, s.t. $\lim_{j \rightarrow \infty} h_j = h \in \bar{H}_m$

$$\begin{aligned}
\lim_{j \rightarrow \infty} \bar{W}^0(t, \pi, h_j) &= \frac{1}{\mu} \log E^{t,\pi}[\lim_{j \rightarrow \infty} \exp(\mu D(h_j, X_T - X_t))] \\
&= \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_T - X_t))] \\
&= \bar{W}^0(t, \pi, h).
\end{aligned} \tag{3.30}$$

□

3.2 Estimates for a basic criterion function

We shall now derive some useful inequalities. For this purpose it is convenient to introduce the function space

$$\mathcal{G} := \{g \in C([0, T] \times \mathcal{S}_N); (\underline{m} + \frac{\underline{\mu}\bar{\sigma}^2}{2})(T-t) \leq g(t, \pi) \leq (\bar{m} + \frac{\bar{\sigma}^2}{2})(T-t)\} \tag{3.31}$$

and set

$$\hat{\xi}(t, \pi, h; g) := \frac{1}{\mu} \log E^{t,\pi}[\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) + \mu 1_{\{\tau_1 < T\}} g(\tau_1, \pi_1))]. \tag{3.32}$$

Then we have the following lemma.

Lemma 3.4. For each $g \in \mathcal{G}$, we have the estimates

(i)

$$\begin{aligned} c \exp\left(\left(\mu\bar{m} + \frac{\mu(\bar{\sigma})^2}{2}\right)(T-t)\right) &\leq E^{t,\pi}[\exp\left(\mu D(h, X_{\tau_1} - X_t) + \mu g(\tau_1, \pi_1)\right) 1_{\{\tau_1 \leq T\}}] \\ &\leq c \exp\left(\left(\mu\underline{m} + \frac{\mu\bar{\sigma}^2}{2}\right)(T-t)\right), \end{aligned} \quad (3.33)$$

where $c = E[1_{\{\tau_1 \leq T\}}]$,

(ii)

$$\begin{aligned} (1-c) \exp\left(\left(\mu\bar{m} + \frac{\mu(\bar{\sigma})^2}{2}\right)(T-t)\right) &\leq E^{t,\pi}[\exp\left(\mu D(h, X_T - X_t)\right) 1_{\{\tau_1 > T\}}] \\ &\leq (1-c) \exp\left(\left(\mu\underline{m} + \frac{\mu\bar{\sigma}^2}{2}\right)(T-t)\right) \end{aligned} \quad (3.34)$$

and

(iii)

$$\left(\underline{m} + \frac{\mu\bar{\sigma}^2}{2}\right)(T-t) \leq \hat{\xi}(t, \pi, h; g) \leq \left(\bar{m} + \frac{\bar{\sigma}^2}{2}\right)(T-t). \quad (3.35)$$

Proof. Let us first set

$$I_1 = E^{t,\pi}[\exp\left(\mu D(h, X_{\tau_1} - X_t) + \mu g(\tau_1, \pi_1)\right) 1_{\{\tau_1 \leq T\}}]$$

and

$$I_2 = E^{t,\pi}[\exp\left(\mu D(h, X_T - X_t)\right) 1_{\{\tau_1 > T\}}].$$

Recall also that $n(\theta_t)$ is the intensity of the Cox process describing the observations and that the dynamics of the filter process π_t was given in Corollary 2.2 in terms of the function $M(t, x, \pi)$.

(i) Since $g \in \mathcal{G}$, from the definition of $\rho_{t,T}^\theta(z)$ in (2.12) and (3.12) we obtain

$$\begin{aligned}
I_1 &= E^{t,\pi} \left[\int_t^T \int_{\mathbb{R}^m} (h^0 \exp(r_0(s-t)) + \sum_{i=1}^m h^i \exp(z^i))^\mu \exp(\mu g(s, M(s-t, z, \pi))) \right. \\
&\quad \times \rho_{t,s}^\theta(z) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds dz \\
&\geq E^{t,\pi} \left[\int_t^T \int_{\mathbb{R}^m} \left(h^0 \exp(r_0(s-t)) + \sum_{i=1}^m h^i \exp(z^i) \right)^\mu \rho_{t,s}^\theta(z) dz \right. \\
&\quad \times \exp((\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&\geq E^{t,\pi} \left[\int_t^T (h^0 \exp(r_0(s-t)) + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(z^i) \rho_{t,s}^\theta(z) dz)^\mu \right. \\
&\quad \times \exp((\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&\geq E^{t,\pi} \left[\int_t^T \exp((\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2})(s-t)) \right. \\
&\quad \times \exp((\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&= \exp((\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2})(T-t)) E^{t,\pi} [1_{\{\tau_1 \leq T\}}],
\end{aligned} \tag{3.36}$$

by using Jensen's inequality. On the other hand, we obtain

$$\begin{aligned}
I_1 &\leq E^{t,\pi} \left[\int_t^T \int_{\mathbb{R}^m} (h^0 \exp(r_0(s-t)) + \sum_{i=1}^m h^i \exp(z^i))^\mu \rho_{t,s}^\theta(z) dz \right. \\
&\quad \times \exp((\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&\leq E^{t,\pi} \left[\int_t^T \left(h^0 \exp(\mu r_0(s-t)) + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(\mu z^i) \rho_{t,s}^\theta(z) dz \right) \right. \\
&\quad \times \exp((\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&\leq E^{t,\pi} \left[\int_t^T \exp((\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(s-t)) \right. \\
&\quad \times \exp((\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(T-s)) n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds \\
&= \exp((\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(T-t)) E^{t,\pi} [1_{\{\tau_1 \leq T\}}],
\end{aligned} \tag{3.37}$$

again by using Jensen's inequality and (3.12).

(ii) By using Jensen's inequality, we have

$$\begin{aligned}
I_2 &= E^{t,\pi}[\int_{\mathbb{R}^m} (h^0 \exp(r_0(T-t)) \\
&\quad + \sum_{i=1}^m h^i \exp(z^i)^\mu \rho_{t,T}^\theta(z) dz \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&\geq E^{t,\pi}[(h^0 \exp(r_0(T-t)) \\
&\quad + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(z^i) \rho_{t,T}^\theta(z) dz)^\mu \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&\geq E^{t,\pi}[\{\sum_{i=0}^m h^i \exp((\bar{m} + \frac{(\bar{\sigma})^2}{2})(T-t))\}^\mu \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&\geq E^{t,\pi}[\exp((\bar{m} + \frac{(\bar{\sigma})^2}{2})(T-t))^\mu \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&= \exp((\mu \bar{m} + \frac{\mu(\bar{\sigma})^2}{2})(T-t)) E^{t,\pi}[1_{\{\tau_1 > T\}}],
\end{aligned} \tag{3.38}$$

from (3.12), since the function x^μ is decreasing. On the other hand, by using Jensen's inequality, we have

$$\begin{aligned}
I_2 &= E^{t,\pi}[\int_{\mathbb{R}^m} (h^0 \exp(r_0(T-t)) \\
&\quad + \sum_{i=1}^m h^i \exp(z^i)^\mu \rho_{t,T}^\theta(z) dz \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&\leq E^{t,\pi}[h^0 \exp(\mu r_0(T-t)) \\
&\quad + \sum_{i=1}^m h^i \int_{\mathbb{R}^m} \exp(\mu z^i) \rho_{t,T}^\theta(z) dz \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&= E^{t,\pi}[\sum_{i=0}^m h^i \exp(\mu \int_t^T m_i(\theta_s) ds + \frac{\mu^2}{2} \int_t^T \sigma_i^2(\theta_s) ds) \\
&\quad \times \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&\leq E^{t,\pi}[\exp((\mu \underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T-t)) \int_T^\infty n(\theta_s) \exp(-\int_t^s n(\theta_u) du) ds] \\
&= \exp((\mu \underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T-t)) E^{t,\pi}[1_{\{\tau_1 > T\}}],
\end{aligned} \tag{3.39}$$

because of (3.12).

(iii) Since

$$\hat{\xi}(t, \pi, h; g) = \frac{1}{\mu} \log(I_1 + I_2). \tag{3.40}$$

The estimates in (3.35) follow from (i) and (ii). \square

Lemma 3.5.

(i) For all $g \in \mathcal{G}$, the function $\exp(\mu \hat{\xi}(t, \pi, h; g))$ is continuous with respect to h .

(ii) For each $g \in \mathcal{G}$ satisfying

$$|\exp(\mu g(t, \pi)) - \exp(\mu g(t, \bar{\pi}))| \leq \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\mu \underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T-t)), \tag{3.41}$$

we have

$$\begin{aligned} & |\exp(\mu\hat{\xi}(t, \pi, h; g)) - \exp(\mu\hat{\xi}(t, \bar{\pi}, h; g))| \\ & \leq \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t))(1+c), \end{aligned} \quad (3.42)$$

where c is defined in Lemma 3.4.

(iii) For each $g \in \mathcal{G}$ satisfying, for $\bar{t} < t$,

$$|\exp(\mu g(t, \pi)) - \exp(\mu g(\bar{t}, \pi))| \leq k(t - \bar{t}) \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)), \quad (3.43)$$

where $k(t)$ is some nonnegative function on \mathbb{R} such that $k(0) = 0$ and continuous at 0, we have

$$\begin{aligned} & |\exp(\mu\hat{\xi}(t, \pi, h; g)) - \exp(\mu\hat{\xi}(\bar{t}, \pi, h; g))| \\ & \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)) (2^{\frac{\bar{n}}{n}} (e^{\underline{n}(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})), \end{aligned} \quad (3.44)$$

where $\underline{n} := \min n(\theta) = \min_i n(e_i)$ and $l(t)$ is as defined in (3.21).

Proof. Let us first prove (i). From (3.27), we have for $m \geq 1$

$$\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) + \mu g(\tau_1, \pi_1)) \leq \sum_{i=0}^m \exp(\mu(X_{T \wedge \tau_1}^i - X_t^i) + \mu g(\tau_1, \pi_1)). \quad (3.45)$$

Similarly to (3.29), we have for each i

$$E^{t, \pi}[\exp(\mu(X_{T \wedge \tau_1}^i - X_t^i) + \mu g(\tau_1, \pi_1))] < \infty. \quad (3.46)$$

Applying the dominated convergence theorem, for $h_n \subset \bar{H}_m$, s.t. $\lim_{n \rightarrow \infty} h_n = h \in \bar{H}_m$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \exp(\mu\hat{\xi}(t, \pi, h_n; g)) \\ & = E^{t, \pi}[\lim_{n \rightarrow \infty} \exp(\mu D(h_n, X_{T \wedge \tau_1} - X_t) + \mu g(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})] \\ & = \exp(\mu\hat{\xi}(t, \pi, h; g)). \end{aligned} \quad (3.47)$$

Next, to prove (ii), recalling Corollary 2.2, we rewrite

$$\begin{aligned} & \exp(\mu\hat{\xi}(t, \pi, h; g)) \\ & = E^{t, \pi}[\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) + \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}})] \\ & = \sum_j E^{t, e_j}[\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) \\ & \quad + \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}})] \pi^j. \end{aligned} \quad (3.48)$$

Then, recalling the definition of $\hat{\xi}(\cdot)$ in (3.32), from Lemma 3.4(iii) and (2.4) it follows that

$$\begin{aligned}
& |\exp(\mu\hat{\xi}(t, \pi, h; g)) - \exp(\mu\hat{\xi}(t, \bar{\pi}, h; g))| \\
= & \left| \sum_j E^{t, e_j} [\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) \right. \\
& + \mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi)) 1_{\{\tau_1 \leq T\}})] (\pi^j - \bar{\pi}^j) \\
& + \sum_j E^{t, e_j} [\exp(\mu D(h, X_{\tau_1} - X_t)) \{ \exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi))) \\
& - \exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi}))) \} 1_{\{\tau_1 \leq T\}}] \bar{\pi}^j | \\
\leq & \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - t)) \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \\
& + E^{t, \bar{\pi}} [\exp(\mu D(h, X_1 - X_t)) | \exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi))) \\
& - \exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi}))) | 1_{\{\tau_1 \leq T\}}].
\end{aligned} \tag{3.49}$$

Furthermore, because of (3.41), using also (3.10)

$$\begin{aligned}
& |\exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \pi))) \\
& - \exp(\mu g(\tau_1, M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi})))| \\
\leq & \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - \tau_1)) \\
& \times \frac{2}{\log 3} d_H(M(\tau_1 - t, X_{\tau_1} - X_t, \pi), M(\tau_1 - t, X_{\tau_1} - X_t, \bar{\pi})) \\
\leq & \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - \tau_1)).
\end{aligned} \tag{3.50}$$

Therefore, we obtain

$$\begin{aligned}
& |\exp(\mu\hat{\xi}(t, \pi, h; g)) - \exp(\mu\hat{\xi}(t, \bar{\pi}, h; g))| \\
\leq & \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - t))(1 + c),
\end{aligned} \tag{3.51}$$

Finally, to prove (iii) we rewrite, using the time homogeneity of (X_t, θ_t) ,

$$\begin{aligned}
& \exp(\mu\hat{\xi}(t, \pi, h; g)) \\
= & E^{t, \pi} [\exp(\mu D(h, X_T - X_t)) 1_{\{\tau_1 > T\}} + \exp(\mu D(h, X_{\tau_1} - X_t) + \mu g(\tau_1, \pi_1)) 1_{\{\tau_1 \leq T\}}] \\
= & E^{t, \pi} [\exp(\mu D(h, X_{T-t} - X_0)) 1_{\{\tau_1 > T-t\}} \\
& + \exp(\mu D(h, X_{\tau_1} - X_0) + \mu g(\tau_1 + t, \pi_1)) 1_{\{\tau_1 \leq T-t\}}].
\end{aligned} \tag{3.52}$$

Therefore, recalling that $\bar{t} < t$,

$$\begin{aligned}
& |\exp(\mu\hat{\xi}(t, \pi, h; g)) - \exp(\mu\hat{\xi}(\bar{t}, \pi, h; g))| \\
\leq & |E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))\{1_{\{\tau_1 > T-t\}} - 1_{\{\tau_1 > T-\bar{t}\}}\}]| \\
& + |E^{0,\pi}[\exp(\mu D(h, X_{\tau_1} - X_0) + \mu g(\tau_1 + t, \pi_1))\{1_{\{\tau_1 \leq T-t\}} - 1_{\{\tau_1 \leq T-\bar{t}\}}\}]| \\
& + |E^{0,\pi}[\{\exp(\mu D(h, X_{T-t} - X_0) - \exp(\mu D(h, X_{T-\bar{t}} - X_0))\}1_{\{\tau_1 > T-\bar{t}\}}]| \\
& + |E^{0,\pi}[\exp(\mu D(h, X_{\tau_1} - X_0))\{\exp(\mu g(\tau_1 + t, \pi_1)) - \exp(\mu g(\tau_1 + \bar{t}, \pi_1))\}1_{\{\tau_1 \leq T-\bar{t}\}}]| \\
\equiv & J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.53}$$

Now we have

$$\begin{aligned}
J_1 & \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t))P^{0,\pi}(T-t < \tau_1 < T-\bar{t}) \\
& = \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t))E^{t,\pi}[\int_{T-t}^{T-\bar{t}} n(\theta_s) \exp(\int_0^s -n(\theta_u)du)ds] \\
& \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)) \int_{T-t}^{T-\bar{t}} \bar{n} \exp(-\underline{n}s)ds \\
& \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)) \left(\frac{\bar{n}}{\underline{n}}\right) (e^{\underline{n}(t-\bar{t})} - 1).
\end{aligned} \tag{3.54}$$

We also have, using Lemma 3.4(iii),

$$J_2 \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)) \left(\frac{\bar{n}}{\underline{n}}\right) (e^{\underline{n}(t-\bar{t})} - 1). \tag{3.55}$$

Further, since $|D(h, x) - D(h, y)| \leq |x - y|$ holds from (3.24), we obtain

$$\begin{aligned}
J_3 & \leq E^{0,\pi}[|\exp(\mu D(h, X_{T-t} - X_0)) - \exp(\mu D(h, X_{T-\bar{t}} - X_0))|] \\
& = E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0)) \\
& \quad \times |1 - \exp(\mu(D(h, X_{T-\bar{t}} - X_0) - D(h, X_{T-t} - X_0)))|] \\
& \leq E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)] \\
& = E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)||X_{T-t}]].
\end{aligned} \tag{3.56}$$

Since (X_t, θ_t) is a time homogeneous process, we have

$$\begin{aligned}
& E[|1 - \exp(\mu|X_{T-\bar{t}} - X_{T-t}|)||X_{T-t}] \\
& = E[|1 - \exp(\mu|\int_{T-t}^{T-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_{T-t}^{T-\bar{t}} \sigma(\theta_s)dB_s||X_{T-t}] \\
& = E_{X_{T-t}}[|1 - \exp(\mu|\int_{T-t}^{T-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_{T-t}^{T-\bar{t}} \sigma(\theta_s)dB_s|)] \\
& = E_{X_{T-t}}[|1 - \exp(\mu|\int_0^{t-\bar{t}} r(\theta_s) - d(\sigma\sigma^*(\theta_s))ds + \int_0^{t-\bar{t}} \sigma(\theta_s)dB_s|)] \\
& = E_{X_{T-t}}[|1 - \exp(\mu|X_{t-\bar{t}} - X_0|)] \\
& = l(t - \bar{t}),
\end{aligned} \tag{3.57}$$

where l is the function defined in (3.21). Hence, we obtain

$$\begin{aligned}
J_3 & \leq E^{0,\pi}[\exp(\mu D(h, X_{T-t} - X_0))l(t - \bar{t})] \\
& \leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t))l(t - \bar{t}).
\end{aligned} \tag{3.58}$$

Since g satisfies assumption (3.43), we have

$$\begin{aligned} J_4 &\leq E^{0,\pi}[\exp(\mu D(h, X_{\tau_1} - X_0)) \\ &\quad \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - t - \tau_1))k(t - \bar{t})1_{\{\tau_1 \leq T - \bar{t}\}}] \\ &\leq c \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - t))k(t - \bar{t}), \end{aligned} \quad (3.59)$$

by using Lemma 3.4(i).

Putting all estimates together, we finally obtain

$$\begin{aligned} &|\exp(\mu \hat{\xi}(t, \pi, h; g)) - \exp(\mu \hat{\xi}(\bar{t}, \pi, h; g))| \\ &\leq \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - t))(2 \left(\frac{\bar{n}}{n}\right) (e^{n(t - \bar{t})} - 1) + l(t - \bar{t}) + ck(t - \bar{t})). \end{aligned} \quad (3.60)$$

□

4 Basic approximation results

Our main results will contain an approximation of the optimal value function. This section is intended to prepare for that result. Let us define an operator J_μ on $C([0, T] \times \mathcal{S}_N)$ by

$$J_\mu g(t, \pi) := \sup_{h \in \bar{H}_m} \hat{\xi}(t, \pi, h; g) \quad (4.1)$$

and the norm $\|g\|$ for a function $g \in C([0, T] \times \mathcal{S}_N)$

$$\|g\| := \sup_{(t, \pi) \in [0, T] \times \mathcal{S}_N} |g(t, \pi)|. \quad (4.2)$$

Lemma 4.1. *Let us assume that g is a function belonging to \mathcal{G} (see (3.31)) and satisfying (3.41), (3.43) in Lemma 3.5. Then $J_\mu g(t, \pi)$ is continuous with respect to t, π and $J_\mu g \in \mathcal{G}$.*

Proof. For $a, b \geq \gamma > 0, |a - b| < \varepsilon, \varepsilon > 0$, one has

$$\begin{aligned} \log a - \log b &= \log(a/b) = \log(1 + (a - b)/b) \leq \varepsilon/\gamma \\ \log b - \log a &= \log(b/a) = \log(1 + (b - a)/a) \leq \varepsilon/\gamma \\ \Rightarrow |\log a - \log b| &\leq \varepsilon/\gamma, \end{aligned} \quad (4.3)$$

where we have used the inequality $\log(1 + x) \leq x$ for $x \geq 0$. We set $a = \exp(\mu \hat{\xi}(t, \pi, h; g))$, $b = \exp(\mu \hat{\xi}(t, \bar{\pi}, h; g))$ and use Lemma 3.4(iii), setting $\gamma = \exp(\{\mu \bar{m} + \frac{\mu \bar{\sigma}^2}{2}\}(T - t))$. From Lemma 3.5(ii), it then follows that

$$\begin{aligned} &|\hat{\xi}(t, \pi, h; g) - \hat{\xi}(t, \bar{\pi}, h; g)| \\ &\leq \frac{1}{|\mu|} \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu - 1)(\bar{\sigma})^2}{2})(T - t))(1 + c) \\ &\leq \frac{1}{|\mu|} \frac{2}{\log 3} d_H(\pi, \bar{\pi}) \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu - 1)(\bar{\sigma})^2}{2})T)(1 + c). \end{aligned} \quad (4.4)$$

On the other hand, by analogous reasoning, from Lemma 3.5(iii), it follows that

$$\begin{aligned}
|\hat{\xi}(t, \pi, h; g) - \hat{\xi}(\bar{t}, \pi, h; g)| &\leq \frac{1}{|\mu|} \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})(T-t)) \\
&\quad \times (2 \left(\frac{\bar{n}}{n}\right) (e^{n(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})) \\
&\leq \frac{1}{|\mu|} \exp((\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)(\bar{\sigma})^2}{2})T) \\
&\quad \times (2 \left(\frac{\bar{n}}{n}\right) (e^{n(t-\bar{t})} - 1) + l(t-\bar{t}) + ck(t-\bar{t})).
\end{aligned} \tag{4.5}$$

Namely, $\hat{\xi}(t, \pi, h; g)$ is continuous with respect to (t, π) , uniformly with respect to h , and hence $J_\mu g(t, \pi)$ is continuous. Further, because of Lemma 3.4(iii), we see that $J_\mu g \in \mathcal{G}$. \square

Definition 4.1. For all $g \in \mathcal{G}$, define

$$J_\mu^0 g(t, \pi) := g(t, \pi), \tag{4.6}$$

and for $n \geq 1$

$$J_\mu^n g(t, \pi) := J_\mu(J_\mu^{n-1} g(t, \pi)). \tag{4.7}$$

Corollary 4.1. Under the assumptions of Lemma 4.1, $J_\mu^n g \in \mathcal{G}$ for $n \geq 0$. Furthermore, there exists a Borel function $\hat{h}^{(n)}(t, \pi)$ such that

$$\sup_{h \in \bar{H}_m} \hat{\xi}(t, \pi, h; J_\mu^n g) = \hat{\xi}(t, \pi, \hat{h}^{(n)}(t, \pi); J_\mu^n g), \quad n \geq 0. \tag{4.8}$$

Proof. Similar arguments to the proof of Lemma 4.1 apply to see that $J_\mu^n g \in \mathcal{G}$. Moreover, since \bar{H}_m is compact and $\hat{\xi}(t, \pi, h; g)$ is a bounded continuous function on $[0, T] \times \mathcal{S}_N \times \bar{H}_m$, there exists a Borel function $\hat{h}^{(0)}(t, \pi)$ such that $\sup_{h \in \bar{H}_m} \hat{\xi}(t, \pi, h; g) = \hat{\xi}(t, \pi, \hat{h}^{(0)}(t, \pi); g)$. By the same reasoning we have (4.8) for general $n \geq 1$. \square

Lemma 4.2. For each $g \in \mathcal{G}$ and $n \geq 1$, we have the following estimate

$$\begin{aligned}
&\|J_\mu^{n+1} g(t, \pi) - J_\mu^n g(t, \pi)\| \\
&\leq \frac{c^n}{|\mu|} \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T) |1 - \exp(\mu\|J_\mu g(t, \pi) - g(t, \pi)\||)
\end{aligned} \tag{4.9}$$

where $0 < c < 1$ is the constant defined in Lemma 3.4.

Proof. Let us first prove that, for $n \geq 1$,

$$|\exp\{\mu J_\mu^{n+1} g(t, \pi)\} - \exp\{\mu J_\mu^n g(t, \pi)\}| \leq c^n e^{(\mu \underline{m} + \frac{\mu \bar{\sigma}^2}{2})(T-t)} |1 - e^{\mu\|J_\mu g - g\||}. \tag{4.10}$$

To prove it for $n = 1$, using Lemma 3.4(i), we see that

$$\begin{aligned}
&|\exp(\mu \hat{\xi}(t, \pi, h; J_\mu g)) - \exp(\mu \hat{\xi}(t, \pi, h; g))| \\
&\leq E^{t, \pi} [e^{\mu D(h, X_{\tau_1} - X_t)} |e^{\mu J_\mu g(\tau_1, \pi_1)} - e^{\mu g(\tau_1, \pi_1)}| \mathbf{1}_{\{\tau_1 \leq T\}}] \\
&= E^{t, \pi} [e^{\mu D(h, X_{\tau_1} - X_t) + \mu J_\mu g(\tau_1, \pi_1)} |1 - e^{\mu(g(\tau_1, \pi_1) - J_\mu g(\tau_1, \pi_1))}| \mathbf{1}_{\{\tau_1 \leq T\}}] \\
&\leq |1 - e^{\mu\|J_\mu g - g\||} E^{t, \pi} [e^{\mu D(h, X_{\tau_1} - X_t) + \mu J_\mu g(\tau_1, \pi_1)} \mathbf{1}_{\{\tau_1 \leq T\}}] \\
&\leq c e^{\mu \underline{m}(T-t) + \frac{\mu \bar{\sigma}^2}{2}(T-t)} |1 - e^{\mu\|J_\mu g - g\||}.
\end{aligned} \tag{4.11}$$

Then, we have

$$\begin{aligned}
|e^{\mu J_\mu^2 g(t, \pi)} - e^{\mu J_\mu g(t, \pi)}| &= |e^{\mu \sup_h \hat{\xi}(t, \pi, h; J_\mu g)} - e^{\mu \sup_h \hat{\xi}(t, \pi, h; g)}| \\
&= |\inf_h e^{\mu \hat{\xi}(t, \pi, h; J_\mu g)} - \inf_h e^{\mu \hat{\xi}(t, \pi, h; g)}| \\
&\leq \sup_h |e^{\mu \hat{\xi}(t, \pi, h; J_\mu g)} - e^{\mu \hat{\xi}(t, \pi, h; g)}| \\
&\leq c e^{\underline{\mu m}(T-t) + \frac{(\underline{\mu} \bar{\sigma}^2)}{2}(T-t)} |1 - e^{\mu \|J_\mu g - g\|}|.
\end{aligned}$$

Assuming that (4.10) holds for $n-1$, we will prove it for n . Using again Lemma 3.4(i)

$$\begin{aligned}
|e^{\mu J_\mu^{n+1} g(t, \pi)} - e^{\mu J_\mu^n g(t, \pi)}| &= |e^{\mu \sup_h \hat{\xi}(t, \pi, h; J_\mu^n g)} - e^{\mu \sup_h \hat{\xi}(t, \pi, h; J_\mu^{n-1} g)}| \\
&= |\inf_h e^{\mu \hat{\xi}(t, \pi, h; J_\mu^n g)} - \inf_h e^{\mu \hat{\xi}(t, \pi, h; J_\mu^{n-1} g)}| \\
&\leq \sup_h |e^{\mu \hat{\xi}(t, \pi, h; J_\mu^n g)} - e^{\mu \hat{\xi}(t, \pi, h; J_\mu^{n-1} g)}| \\
&\leq \sup_h E^{t, \pi} [e^{\mu D(h, X_{\tau_1} - X_t)} |e^{\mu J_\mu^n g(\tau_1, \pi_1)} - e^{\mu J_\mu^{n-1} g(\tau_1, \pi_1)}| \mathbf{1}_{\{\tau_1 \leq T\}}] \\
&\leq \sup_h E^{t, \pi} [e^{\mu D(h, X_{\tau_1} - X_t)} c^{n-1} e^{(\underline{\mu m} + \frac{(\underline{\mu} \bar{\sigma}^2)}{2})(T-\tau_1)} \mathbf{1}_{\{\tau_1 \leq T\}} |1 - e^{\mu \|J_\mu g - g\|}|] \\
&\leq c^n e^{(\underline{\mu m} + \frac{(\underline{\mu} \bar{\sigma}^2)}{2})(T-t)} |1 - e^{\mu \|J_\mu g - g\|}|.
\end{aligned} \tag{4.12}$$

Thus, (4.10) has been proved. Now we can complete the proof. Indeed, by using (4.3), we have

$$\begin{aligned}
|J_\mu^{n+1} g(t, \pi) - J_\mu^n g(t, \pi)| &= \frac{1}{|\underline{\mu}|} |\underline{\mu} J_\mu^{n+1} g(t, \pi) - \underline{\mu} J_\mu^n g(t, \pi)| \\
&\leq \frac{1}{|\underline{\mu}|} e^{-\underline{\mu} \bar{m}(T-t) - \frac{\underline{\mu} \bar{\sigma}^2}{2}(T-t)} |e^{\underline{\mu} J_\mu^{n+1} g(t, \pi)} - e^{\underline{\mu} J_\mu^n g(t, \pi)}| \\
&\leq \frac{c^n}{|\underline{\mu}|} e^{\underline{\mu}(\underline{m} - \bar{m})(T-t) + \frac{(\underline{\mu}^2 - \underline{\mu}) \bar{\sigma}^2}{2}(T-t)} |1 - e^{\mu \|J_\mu g - g\|}| \\
&\leq \frac{c^n}{|\underline{\mu}|} e^{\underline{\mu}(\underline{m} - \bar{m})T + \frac{(\underline{\mu}^2 - \underline{\mu}) \bar{\sigma}^2}{2}T} |1 - e^{\mu \|J_\mu g - g\|}|
\end{aligned} \tag{4.13}$$

and hence obtain the present lemma by taking supremum with respect to (t, π) . \square

Corollary 4.2. *Let us assume that g is a function belonging to \mathcal{G} and satisfies (3.41), (3.43) in Lemma 3.5. Then $\{J_\mu^n g(t, \pi)\}$ is a Cauchy sequence in \mathcal{G} . Therefore, $\exists \lim_{n \rightarrow \infty} J_\mu^n g(t, \pi) \in \mathcal{G}$*

Proof. By Corollary 4.1, $\{J_\mu^n g(t, \pi)\} \subset \mathcal{G}$. Further, by using Lemma 4.2, we can see that $\{J_\mu^n g(t, \pi)\}$ is a Cauchy sequence in \mathcal{G} . \square

Corollary 4.3. *For each $g_1, g_2 \in \mathcal{G}$, we have the estimates:*

$$\begin{aligned}
&\|J_\mu g_1 - J_\mu g_2\| \\
&\leq \frac{c}{|\underline{\mu}|} \exp(\{\underline{\mu}(\underline{m} - \bar{m}) + \frac{\underline{\mu}(\underline{\mu}-1)\bar{\sigma}^2}{2}\}T) |1 - \exp(\mu \|g_1 - g_2\|)| \\
&\leq c \exp(\{\underline{\mu}(\underline{m} - \bar{m}) + \frac{\underline{\mu}(\underline{\mu}-1)\bar{\sigma}^2}{2}\}T) \|g_1 - g_2\|
\end{aligned} \tag{4.14}$$

Proof. The proof is similar to that of Lemma 4.2. \square

Definition 4.2. *Let us set*

$$\bar{W}^0(t, \pi) := \sup_{h \in \bar{H}_m} \bar{W}^0(t, \pi, h) = \sup_{h \in \bar{H}_m} \frac{1}{\mu} \log E[e^{\mu D(h, X_T - X_t)}], \quad (4.15)$$

and

$$\bar{W}^n(t, \pi) := J_\mu^n \bar{W}^0(t, \pi). \quad (4.16)$$

Lemma 4.3. $\bar{W}^0(t, \pi)$ belongs to \mathcal{G} . Furthermore, it satisfies (3.41) and (3.43) by setting $k(t) = l(t)$.

Proof. From Lemma 3.3, $\bar{W}^0(t, \pi)$ satisfies (3.41) and (3.43) with $k(t) = l(t)$ in Lemma 3.5 and we see that $\bar{W}^0(t, \pi) \in \mathcal{G}$ by using Corollary 3.1. \square

Corollary 4.4. $\{\bar{W}^n(t, \pi)\}$ is a Cauchy sequence in \mathcal{G} and therefore $\exists \lim_{n \rightarrow \infty} \bar{W}^n(t, \pi) \in \mathcal{G}$

Proof. From Corollary 4.2 and Lemma 4.3, we see that $\{\bar{W}^n(t, \pi)\}$ is a Cauchy sequence in \mathcal{G} . \square

5 Passage to the limit

Definition 5.1. *We set*

$$\bar{W}(t, \pi) := \lim_{n \rightarrow \infty} \bar{W}^n(t, \pi), \quad (5.1)$$

$$W(t, \pi) := \sup_{h \in \mathcal{A}} W(t, \pi, h.), \quad (5.2)$$

and

$$W^n(t, \pi) := \sup_{h \in \mathcal{A}^n} W(t, \pi, h.), \quad (5.3)$$

where $W(t, \pi, h.)$ is the criterion function defined by (3.1).

Lemma 5.1. *For all $n \geq 0$ and $h \in \mathcal{A}^n$, we have the following equation*

$$\begin{aligned} W(t, \pi, h.) &= \frac{1}{\mu} \log E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \right. \\ &\quad \left. \left. + \mu D(h_n, X_T - X_{\tau_n}) 1_{\{\tau_n < T\}} \right) \right] \\ &= \frac{1}{\mu} \log E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \right. \\ &\quad \left. \left. + \mu \bar{W}^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}} \right) \right]. \end{aligned} \quad (5.4)$$

Proof. By Lemma 3.1 and the definition of $\bar{W}^0(t, \pi, h)$ it suffices to prove the following equation for all $n > 0$, $k \geq n + 1$, $h \in \mathcal{A}^n$. For $\tau_k < T \leq \tau_{k+1}$

$$\begin{aligned} & \exp\left(\mu \sum_{i=n}^{k-1} D(h_i, X_{\tau_{i+1}} - X_{\tau_i}) + \mu D(h_k, X_T - X_{\tau_k})\right) \\ &= \exp(\mu D(h_n, X_T - X_{\tau_n})). \end{aligned} \quad (5.5)$$

It can be seen as follows.

$$\begin{aligned} & \exp\left(\mu \sum_{i=n}^{k-1} D(h_i, X_{\tau_{i+1}} - X_{\tau_i}) + \mu D(h_k, X_T - X_{\tau_k})\right) \\ &= \prod_{i=n}^{k-1} \left(\sum_{j=0}^m h_i^j \frac{S_{i+1}^j}{S_i^j}\right)^\mu \left(\sum_{j=0}^m h_k^j \frac{S_T^j}{S_k^j}\right)^\mu = \prod_{i=n}^{k-1} \left(\sum_{j=0}^m \frac{N_i^j S_{i+1}^j}{V_i}\right)^\mu \left(\sum_{j=0}^m \frac{N_k^j S_T^j}{V_k}\right)^\mu \\ &= \prod_{i=n}^{k-1} \left(\sum_{j=0}^m \frac{N_{i+1}^j S_{i+1}^j}{V_i}\right)^\mu \left(\sum_{j=0}^m \frac{N_k^j S_T^j}{V_k}\right)^\mu = \prod_{i=n}^{k-1} \left(\frac{V_{i+1}}{V_i}\right)^\mu \left(\sum_{j=0}^m \frac{N_T^j S_T^j}{V_k}\right)^\mu \\ &= \left(\frac{N_T S_T}{V_n}\right)^\mu = \left(\frac{V_T}{V_n}\right)^\mu = \exp(\mu D(h_n, X_T - X_{\tau_n})), \end{aligned} \quad (5.6)$$

where, we have used (3.5), the definition of \mathcal{A}^n and the self financing property of the investment strategy. \square

Corollary 5.1. *We have the following equation*

$$\begin{aligned} W^n(t, \pi) &= \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t, \pi} \left[\exp\left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \right. \\ &\quad \left. \left. + \mu \bar{W}^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}} \right) \right], \end{aligned} \quad (5.7)$$

for $n \geq 0$, $t \in [0, T]$, $\pi \in \mathcal{S}_N$.

Lemma 5.2. *For each $n \geq 0$, we have the following equation*

$$\bar{W}^n(t, \pi) = W^n(t, \pi). \quad (5.8)$$

Proof. By definition we have

$$\bar{W}^0(t, \pi) = W^0(t, \pi). \quad (5.9)$$

Moreover, $\bar{W}^0(t, \pi)$ belongs to \mathcal{G} and satisfies (3.41) and (3.43) because of Lemma 4.3. Therefore, in Corollary 4.1, we set $g(t, \pi) = \bar{W}^0(t, \pi)$ and obtain a Borel function $\hat{h}^{(n)}(t, \pi)$ satisfying (4.8) for $n \geq 0$. Then,

$$\bar{W}^n(t, \pi) = J_\mu^n \bar{W}^0(t, \pi) = \sup_{h \in H_m} \hat{\xi}(t, \pi, h; J_\mu^{n-1} \bar{W}^0) = \hat{\xi}(t, \pi, \hat{h}^{(n-1)}(t, \pi); J_\mu^{n-1} \bar{W}^0).$$

We also have a Borel function $\bar{h}(t, \pi)$ such that

$$\bar{W}^0(t, \pi) = \sup_h \bar{W}^0(t, \pi, h) = \bar{W}^0(t, \pi, \bar{h}(t, \pi)).$$

We define a strategy $\bar{h}^{(n)} \in \mathcal{A}^n$ as follows.

$$\begin{aligned}
\bar{h}_k^{(n)} &= \hat{h}^{(n-1-k)}(\tau_k, \pi_k), \quad k = 0, \dots, n-1, \\
\bar{h}_n^{(n)} &= \bar{h}(\tau_n, \pi_n) \\
\bar{h}_k^{(n)} &= \gamma(\tilde{X}_{\tau_k} - \tilde{X}_{\tau_n}, \bar{h}_n^{(n)}), \quad k \geq n+1.
\end{aligned} \tag{5.10}$$

First, to show that $\bar{W}_n(t, \pi) \leq W_n(t, \pi)$, we rewrite \bar{W}^n as follows.

$$\begin{aligned}
&\bar{W}^n(t, \pi) \\
&= \sup_{h \in \bar{H}_m} \hat{\xi}(t, \pi, h; J_\mu^{n-1} \bar{W}^0) \\
&= \hat{\xi}(t, \pi, \hat{h}^{(n-1)}(t, \pi); J_\mu^{n-1} \bar{W}^0) \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(\hat{h}^{(n-1)}(t, \pi), X_{T \wedge \tau_1} - X_t) + \mu \bar{W}^{n-1}(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})] \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(\hat{h}^{(n-1)}(t, \pi), X_T - X_t)) 1_{\{\tau_1 > T\}} \\
&\quad + \exp(\mu D(\hat{h}^{(n-1)}(t, \pi), X_{\tau_1} - X_t) + \mu \bar{W}^{n-1}(\tau_1, \pi_1)) 1_{\{\tau_1 \leq T\}}] \\
&= \frac{1}{\mu} \log E^{t, \pi} [e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_T - X_t)} 1_{\{\tau_1 > T\}} + e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_{\tau_1} - X_t)} \\
&\quad \cdot E^{\tau_1, \pi_1} [e^{\mu D(\hat{h}^{(n-2)}(\tau_1, \pi_1), X_{T \wedge \tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2, \pi_2) 1_{\{\tau_2 \leq T\}}}]] 1_{\{\tau_1 \leq T\}}].
\end{aligned} \tag{5.11}$$

Noting that

$$\begin{aligned}
&E^{t, \pi} [e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_{\tau_1} - X_t)} \\
&\quad \cdot E^{\tau_1, \pi_1} [e^{\mu D(\hat{h}^{(n-2)}(\tau_1, \pi_1), X_{T \wedge \tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2, \pi_2) 1_{\{\tau_2 \leq T\}}}]] 1_{\{\tau_1 \leq T\}}] \\
&= E^{t, \pi} [e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_{\tau_1} - X_t)} E^{\tau_1, \pi_1} [e^{\mu D(\hat{h}^{(n-2)}(\tau_1, \pi_1), X_T - X_{\tau_1})} 1_{\{\tau_2 > T\}}] 1_{\{\tau_1 \leq T\}}] \\
&\quad + E^{t, \pi} [e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_{\tau_1} - X_t)} \\
&\quad \cdot E^{\tau_1, \pi_1} [e^{\mu D(\hat{h}^{(n-2)}(\tau_1, \pi_1), X_{\tau_2} - X_{\tau_1}) + \mu \bar{W}^{n-2}(\tau_2, \pi_2) 1_{\{\tau_2 \leq T\}}}]] 1_{\{\tau_1 \leq T\}}],
\end{aligned} \tag{5.12}$$

we have

$$\begin{aligned}
&\bar{W}^n(t, \pi) \\
&= \frac{1}{\mu} \log E^{t, \pi} [e^{\mu D(\hat{h}^{(n-1)}(t, \pi), X_{T \wedge \tau_1} - X_t) + \mu D(\hat{h}^{(n-2)}(\tau_1, \pi_1), X_{T \wedge \tau_2} - X_{\tau_1})} 1_{\{\tau_1 \leq T\}} + \mu \bar{W}^{n-2}(\tau_2, \pi_2) 1_{\{\tau_2 \leq T\}}] \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(\bar{h}^{(n)}(\tau_{k-1}, \pi_{k-1}), X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\
&\quad + \mu \bar{W}^0(\tau_n, \pi_n, \bar{h}^{(n)}(\tau_n, \pi_n)) 1_{\{\tau_n < T\}})],
\end{aligned} \tag{5.13}$$

inductively. By Corollary 5.1 we then have

$$\begin{aligned}
& \bar{W}^n(t, \pi) \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(\bar{h}_n(\tau_{k-1}, \pi_{k-1}), X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \\
&\quad + \mu \bar{W}^0(\tau_n, \pi_n, \bar{h}_n(\tau_n, \pi_n)) 1_{\{\tau_n < T\}})] \\
&\leq \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t, \pi} [\exp\left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \\
&\quad \left. + \mu \bar{W}^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}}\right)] \\
&= W^n(t, \pi).
\end{aligned} \tag{5.14}$$

Next, we shall prove the converse inequality. By applying Lemma 5.1, we have for $h \in \mathcal{A}^n$

$$\begin{aligned}
& W(t, \pi, h) \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} + \mu \bar{W}^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}})] \\
&\leq \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} + \mu \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}})] \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}) \\
&\quad \times \exp(\mu D(h_{n-1}, X_{T \wedge \tau_n} - X_{\tau_{n-1}}) + \mu \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}) 1_{\{\tau_{n-1} \leq T\}} \\
&\quad + \exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}) 1_{\{\tau_{n-1} > T\}}] \\
&= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}) \\
&\quad \times E^{\tau_{n-1}, \pi_{n-1}} [\exp(\mu D(h_{n-1}, X_{T \wedge \tau_n} - X_{\tau_{n-1}}) + \mu \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}})] 1_{\{\tau_{n-1} \leq T\}} \\
&\quad + \exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}) 1_{\{\tau_{n-1} > T\}}].
\end{aligned} \tag{5.15}$$

By the definition of $\hat{\xi}$ and \bar{W}^1 we have

$$\begin{aligned}
& E^{\tau_{n-1}, \pi_{n-1}} [\exp(\mu D(h_{n-1}, X_{T \wedge \tau_n} - X_{\tau_{n-1}}) + \mu \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}})] \\
&= \exp(\mu \hat{\xi}(\tau_{n-1}, \pi_{n-1}, h_{n-1}; \bar{W}^0)) \\
&\geq \exp(\mu \sup_{h \in \bar{H}_m} \hat{\xi}(\tau_{n-1}, \pi_{n-1}, h; \bar{W}^0)) = \exp(\mu \bar{W}^1(\tau_{n-1}, \pi_{n-1})).
\end{aligned} \tag{5.16}$$

Therefore, for $h \in \mathcal{A}^n$, we have inductively

$$\begin{aligned}
& W(t, \pi, h.) \\
& \leq \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} + \mu \bar{W}^1(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}})] \\
& \leq \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(h, X_{T \wedge \tau_1} - X_0) + \mu \bar{W}^{n-1}(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})] \\
& \leq \bar{W}^n(t, \pi).
\end{aligned} \tag{5.17}$$

□

Lemma 5.3. For all $h \in \mathcal{A}$, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \\
& \quad + \mu W^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}})] \\
& = W(t, \pi, h.).
\end{aligned} \tag{5.18}$$

Proof.

$$\begin{aligned}
& E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\
& \quad + \mu W^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}})] \\
& = E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}}) + \mu W^0(\tau_n, \pi_n, h_n)) 1_{\{\tau_n \leq T\}}] \\
& \quad + E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}}) 1_{\{\tau_n > T\}}] \\
& =: I_1(n) + I_2(n).
\end{aligned} \tag{5.19}$$

We shall first give an estimate for $I_1(n)$. From Corollary 3.1 it follows that for $h \in \bar{H}_m$

$$\exp(\mu W^0(t, \pi, h)) \leq \exp((\underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T - t)). \tag{5.20}$$

Therefore, we have

$$\begin{aligned}
I_1(n) & \leq E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}}) + \mu(\underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T - \tau_n)) 1_{\{\tau_n \leq T\}}] \\
& \leq E^{t, \pi} [\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}})) E^{\tau_{n-1}, \pi_{n-1}} [\exp(\mu D(h_{n-1}, X_{\tau_n} - X_{\tau_{n-1}}) \\
& \quad + \mu(\underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T - \tau_n)) 1_{\{\tau_n < T\}}] 1_{\{\tau_{n-1} \leq T\}}] \\
& \leq c E^{t, \pi} [\exp(\mu \sum_{k=1}^{n-1} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}})) \exp((\underline{m} + \frac{(\mu \bar{\sigma})^2}{2})(T - \tau_{n-1})) 1_{\{\tau_{n-1} \leq T\}}],
\end{aligned} \tag{5.21}$$

by using Lemma 3.4 (i) because clearly $\exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)) \in \mathcal{G}$. Thus, we obtain inductively

$$\begin{aligned}
I_1(n) &\leq cE^{t,\pi} \left[\exp\left(\mu \sum_{k=1}^{n-2} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}})\right) \right. \\
&\quad \times E^{\tau_{n-2}, \pi_{n-2}} \left[\exp\left(\mu D(h_{n-2}, X_{\tau_{n-1}} - X_{\tau_{n-2}}) \right. \right. \\
&\quad \left. \left. + (\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - \tau_{n-1})\right) 1_{\{\tau_{n-1} < T\}} 1_{\{\tau_{n-2} \leq T\}} \right] \\
&\leq c^2 E^{t,\pi} \left[\exp\left(\mu \sum_{k=1}^{n-2} D(h_{k-1}, X_{\tau_k} - X_{\tau_{k-1}})\right) \right. \\
&\quad \left. \times \exp\left((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T - \tau_{n-2})\right) 1_{\{\tau_{n-2} \leq T\}} \right] \\
&\leq c^n \exp((\underline{\mu m} + \frac{(\underline{\mu \bar{\sigma}})^2}{2})(T-t)),
\end{aligned} \tag{5.22}$$

and therefore we see that

$$\lim_{n \rightarrow \infty} I_1(n) = 0.$$

On the other hand, since $1_{\{\tau_n > T\}} = \sum_{j=0}^{n-1} 1_{\{\tau_j < T \leq \tau_{j+1}\}}$, we have

$$\begin{aligned}
I_2(n) &= E^{t,\pi} \left[\exp\left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) 1_{\{\tau_n \geq T\}} \right] \\
&= E^{t,\pi} \left[\sum_{j=0}^{n-1} \exp\left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \right] \\
&= E^{t,\pi} \left[\sum_{j=0}^{n-1} \exp\left(\mu \sum_{k=1}^{j+1} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \right. \\
&\quad \left. \left. + \mu \sum_{k=j+2}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \right].
\end{aligned} \tag{5.23}$$

Noting that $\{\tau_k < T\} \cap \{T \leq \tau_{j+1}\} = \emptyset$ for all $k \geq j+1$, we have

$$\begin{aligned}
&\exp\left(\mu \sum_{k=j+2}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \\
&= \exp\left(\mu \sum_{k=j+2}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}}\right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \\
&= 1_{\{\tau_j < T \leq \tau_{j+1}\}}
\end{aligned} \tag{5.24}$$

and that

$$\begin{aligned}
\lim_{n \rightarrow \infty} I_2(n) &= \lim_{n \rightarrow \infty} E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right) 1_{\{\tau_n \geq T\}} \right] \\
&= \lim_{n \rightarrow \infty} E^{t, \pi} \left[\sum_{j=0}^{n-1} \exp \left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \right] \\
&= E^{t, \pi} \left[\sum_{j=0}^{\infty} \exp \left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right) 1_{\{\tau_j < T \leq \tau_{j+1}\}} \right] \\
&= E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^{\infty} D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right) \right].
\end{aligned} \tag{5.25}$$

Therefore, we obtain

$$\begin{aligned}
&\lim_{n \rightarrow \infty} E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} + \mu W^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}} \right) \right] \\
&= \lim_{n \rightarrow \infty} I_1(n) + I_2(n) = \exp(\mu W(t, \pi, h.)),
\end{aligned} \tag{5.26}$$

having used Lemma 3.1. This completes the proof. \square

Lemma 5.4.

$$\bar{W}(t, \pi) = W(t, \pi). \tag{5.27}$$

Proof. By the definition of \mathcal{A}^n , the inclusions $\mathcal{A}^n \subset \mathcal{A}^{n+1} \subset \mathcal{A}$ hold for $n \geq 0$ and we have

$$\sup_{h \in \mathcal{A}^n} W(t, \pi, h.) \leq \sup_{h \in \mathcal{A}^{n+1}} W(t, \pi, h.) \leq \sup_{h \in \mathcal{A}} W(t, \pi, h.). \tag{5.28}$$

From the definition of $W^n(t, \pi)$ and $W(t, \pi)$ it follows that

$$W^n(t, \pi) \leq W^{n+1}(t, \pi) \leq W(t, \pi). \tag{5.29}$$

Therefore, from Lemma 5.2 we have

$$\bar{W}^n(t, \pi) \leq \bar{W}^{n+1}(t, \pi) \leq W(t, \pi). \tag{5.30}$$

Thus, from Corollary 4.4 and (5.1), we obtain

$$\bar{W}(t, \pi) \leq W(t, \pi). \tag{5.31}$$

On the other hand, for $h \in \mathcal{A}$

$$\begin{aligned}
&\bar{W}^n(t, \pi) = W^n(t, \pi) \\
&\geq \frac{1}{\mu} \log E^{t, \pi} \left[\exp \left(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \right. \right. \\
&\quad \left. \left. + \mu W^0(\tau_n, \pi_n, h_n) 1_{\{\tau_n \leq T\}} \right) \right].
\end{aligned} \tag{5.32}$$

Letting $n \rightarrow \infty$ and applying Lemma 5.3,

$$\bar{W}(t, \pi) \geq W(t, \pi, h.) \quad (5.33)$$

and hence, we obtain

$$\bar{W}(t, \pi) = W(t, \pi). \quad (5.34)$$

□

Lemma 5.5. *We have $W = J_\mu W$. Namely, $W(t, \pi)$ satisfies the following equation*

$$\begin{aligned} & W(t, \pi) \\ &= \sup_{h \in \bar{H}_n} \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) + \mu W(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})]. \end{aligned} \quad (5.35)$$

Proof. We have, by using (4.14),

$$\begin{aligned} \|W - J_\mu W\| &\leq \|W - J_\mu \bar{W}^n\| + \|J_\mu \bar{W}^n - J_\mu W\| \\ &\leq \|W - \bar{W}^{n+1}\| + C_1 \|\bar{W}^n - W\|, \end{aligned}$$

where $C_1 = c \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T)$. Hence, by sending n to ∞ , we see that $\|W - J_\mu W\| = 0$. □

6 Main Theorem

From the previous Lemmas we obtain now the main result of this paper, namely a Dynamic Programming-type approach to solve the power-utility maximization problem.

Theorem 6.1.

(i) *Approximation theorem*

\bar{W}^n computed according to (4.16) are approximations to the solution of the original problem in the sense that, for any $\epsilon > 0, n > n_\epsilon$,

$$\|W - \bar{W}^n\| < \epsilon, \quad (6.1)$$

where,

$$n_\epsilon := \frac{\log(1 - c)|\mu| + \log \epsilon - \log |1 - \exp(\mu \|J_\mu^1 \bar{W}^0 - J_\mu^0 \bar{W}^0\|)| - \{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T}{\log c}.$$

(ii) *Dynamic programming principle:*

for $n \geq 0$

$$\begin{aligned} & W(t, \pi) \\ &= \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}} \\ &+ \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}})]. \end{aligned} \quad (6.2)$$

(iii) *Optimal value and optimal strategy for the Power Utility Maximization Problem*
For the utility maximization under initial condition $V_0 = v_0, \tau_0 = 0, \pi_0 = \pi$ we have

$$\begin{aligned} \sup_{h \in \mathcal{A}} \frac{1}{\mu} \log E^{0, \pi} [V_T^\mu] &= \log v_0 \\ &+ \frac{1}{\mu} \log E^{0, \pi} [\exp(\mu \sum_{k=1}^{\infty} D(\hat{h}_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} < T\}})], \end{aligned} \quad (6.3)$$

where the optimal strategy in the k -th period, namely \hat{h}_k is given by

$$\hat{h}_k = \hat{h}(\tau_k, \pi_k) \quad (6.4)$$

with $\hat{h}(\tau, \pi)$ defined by

$$\begin{aligned} &\sup_{h \in \mathcal{H}_m} \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(h, X_{T \wedge \tau_1} - X_t) + \mu W(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})] \\ &= \frac{1}{\mu} \log E^{t, \pi} [\exp(\mu D(\hat{h}(t, \pi), X_{T \wedge \tau_1} - X_t) + \mu W(\tau_1, \pi_1) 1_{\{\tau_1 \leq T\}})]. \end{aligned} \quad (6.5)$$

Proof. First, we prove (i). For any n

$$\begin{aligned} \|\bar{W} - \bar{W}^n\| &= \left\| \lim_{k \rightarrow \infty} \bar{W}^{n+k} - \bar{W}^n \right\| = \lim_{k \rightarrow \infty} \|\bar{W}^{n+k} - \bar{W}^n\| \\ &\leq \lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} \|\bar{W}^{n+i+1} - \bar{W}^{n+i}\| = \lim_{k \rightarrow \infty} \sum_{i=0}^{k-1} \|J_\mu^{n+i+1} \bar{W}^0 - J_\mu^{n+i} \bar{W}^0\| \\ &\leq \frac{1}{|\mu|} |1 - \exp(\mu \|J_\mu^1 \bar{W}^0 - J_\mu^0 \bar{W}^0\|)| \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T) \sum_{i=0}^{\infty} c^{i+n} \\ &= \frac{1}{|\mu|} |1 - \exp(\mu \|J_\mu^1 \bar{W}^0 - J_\mu^0 \bar{W}^0\|)| \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T) c^n \sum_{i=0}^{\infty} c^i \\ &= \frac{c^n}{(1-c)|\mu|} |1 - \exp(\mu \|J_\mu^1 \bar{W}^0 - J_\mu^0 \bar{W}^0\|)| \exp(\{\mu(\underline{m} - \bar{m}) + \frac{\mu(\mu-1)\bar{\sigma}^2}{2}\}T), \end{aligned} \quad (6.6)$$

where we have used Lemma 4.2.

Next, we prove (ii). Proceeding analogously to the proof of Lemma 5.2, for $n, m \in \mathbf{N}$, we take a sequence of functions $\tilde{h}^{n,k}(t, \pi)$, $k = 0, 1, \dots, n$ such that

$$\sup_{h \in \mathcal{H}^m} \hat{\xi}(t, \pi, h; J_\mu^{n+m-k-1} \bar{W}^0) = \hat{\xi}(t, \pi, \tilde{h}^{n,k}(t, \pi); J_\mu^{n+m-k-1} \bar{W}^0)$$

and set

$$\tilde{h}_k^{n,k} := \tilde{h}^{n,k}(\tau_k, \pi_k), \quad k = 0, 1, \dots, n$$

and

$$\tilde{h}_k^{n,k} = \gamma(\tilde{X}_{\tau_k} - \tilde{X}_{\tau_n}, \tilde{h}_n^{n,n}), \quad k \geq n+1.$$

Then, $\tilde{h}^{(n)} = \{\tilde{h}_k^{n,k}\}_k \in \mathcal{A}^n$. Therefore, similarly to the proof of Lemma 5.2, it follows that

$$\begin{aligned} \bar{W}^{n+m} &= \frac{1}{\mu} \log E^{t, \pi} [\exp\{\mu \sum_{k=1}^n D(\tilde{h}_k^{n,k-1}(\tau_{k-1}, \pi_{k-1}), X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu J_\mu^m \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.7)$$

Since $J_\mu^m \bar{W}^0 \leq W$, we have

$$\begin{aligned} \bar{W}^{n+m} &\leq \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(\tilde{h}^{n,k-1}(\tau_{k-1}, \pi_{k-1}), X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}] \\ &\leq \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.8)$$

Therefore, we obtain

$$\begin{aligned} W(t, \pi) &\leq \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.9)$$

by letting $m \rightarrow \infty$. On the other hand, for each $h \in \mathcal{A}^n$, it follows that

$$\begin{aligned} \bar{W}^{n+m}(t, \pi) &\geq \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu J_\mu^m \bar{W}^0(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.10)$$

Then, by letting $m \rightarrow \infty$, we have

$$\begin{aligned} W(t, \pi) &\geq \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.11)$$

Hence,

$$\begin{aligned} W(t, \pi) &\geq \sup_{h \in \mathcal{A}^n} \frac{1}{\mu} \log E^{t,\pi} [\exp\{\mu \sum_{k=1}^n D(h_{k-1}, X_{T \wedge \tau_k} - X_{\tau_{k-1}}) 1_{\{\tau_{k-1} \leq T\}} \\ &\quad + \mu W(\tau_n, \pi_n) 1_{\{\tau_n \leq T\}}\}]. \end{aligned} \quad (6.12)$$

and thus we obtain (ii).

Part (iii) is an immediate consequence of the previous results in particular of the proof of point (ii) of this same theorem. \square

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