Market models

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Market models (discrete time $\Delta = 1$)

- A *locally riskless asset (money market account)*

$$B_{n+1} = B_n(1+r_n) \quad \leftrightarrow \quad \frac{B_{n+1} - B_n}{B_n} = r_n \quad (r_n \text{ known at } n)$$

- *Risky assets* (for the moment just one)

$$S_{n+1} = S_n(1 + a_{n+1}) \quad \leftrightarrow \quad \frac{S_{n+1} - S_n}{S_n} = a_{n+1}$$

$$\updownarrow \quad (a_{n+1} \text{ unknown at } n)$$

$$S_{n+1} = S_n \xi_{n+1}$$

- *Example:* $\xi_n = \begin{cases} u & \text{with probability } p \\ d & \text{with probability } 1-p \end{cases}$

Market models (transition to continuous time)

- *Locally riskless asset*

$$B_{t+\Delta} - B_t = B_t r_t \Delta \longrightarrow dB_t = B_t r_t dt \quad \textit{(cont. compounding)}$$

- *Risky asset* $(a_{t+\Delta} = a_t \Delta + \sigma_t \xi_{t+\Delta}$ with $\xi_{t+\Delta} \sim \mathcal{N}(0, \Delta))$

$$S_{t+\Delta} = S_t \left(1 + a_t \Delta + \sigma_t \xi_{t+\Delta}\right)$$

Let w_t be a process s.t.

$$\Delta w_t := w_{t+\Delta} - w_t \sim \mathcal{N}(0, \Delta) \quad \textit{(Wiener process)}$$

$$S_{t+\Delta} = S_t \left(1 + a_t \Delta + \sigma_t \xi_{t+\Delta}\right)$$

$$\downarrow$$

$$dS_t = S_t \left[a_t dt + \sigma_t dw_t\right]$$

Price processes with continuous trajectories

$$dS_t = S_t \left[a_t dt + \sigma_t dw_t \right]$$

$$(\Delta w_t := w_{t+\Delta} - w_t \sim \mathcal{N}(0, \Delta) \;\; \longrightarrow \;\; dw_t \sim \sqrt{dt}$$

$$\rightarrow \left\{ \begin{array}{lcl} d \log S_t & = & \left(a_t - \frac{1}{2}\sigma_t^2 \right) dt + \sigma_t dw_t \\ S_{t+\Delta} & = & S_t \exp \left[\int_t^{t+\Delta} \left(a_s - \frac{1}{2}\sigma_s^2 \right) ds + \int_t^{t+\Delta} \sigma_s dw_s \right] \\ & = & S_t \xi_{t+\Delta} \end{array} \right.$$

$\rightarrow$ *All trajectories of S_t are continuous functions of t*

Price processes with discontinuous trajectories

- Let T_n $(0 = T_0 < T_1 < \cdots)$ be a sequence of random times where a certain event happens

- Let $N_t = n$ if $t \in [T_n, T_{n+1})$ $\Leftrightarrow$ $N_t = \sum_{n \geq 1} 1_{\{T_n \leq t\}}$

  $\rightarrow$ N_t is a *counting process* and $dN_t \in \{0, 1\}$.

- Let S_t change only at times T_n with return $\gamma_n > -1$ at T_n

$$\frac{S_{T_n} - S_{T_n^-}}{S_{T_n^-}} = \gamma_n \;\leftrightarrow\; S_{T_n} = S_{T_n^-}(1+\gamma_n) \;\leftrightarrow\; dS_t = S_{t^-}\gamma_t dN_t$$

$$\Rightarrow \; S_t \;=\; S_0 \prod_{n=1}^{N_t}(1+\gamma_{T_n}) = S_0 \exp\left[\sum_{n=1}^{N_t} \log(1+\gamma_{T_n})\right]$$

$$= \; S_0 \exp\left[\int_0^t \log(1+\gamma_t) dN_t\right]$$

Combining the two (jump diffusion models)

$$dS_t = S_{t^-}\left[a_t dt + \sigma_t dw_t + \gamma_t dN_t\right]$$

implies

$$\begin{aligned} S_{t+\Delta} &= S_t \exp\left[\int_t^{t+\Delta}\left(a_s - \frac{1}{2}\sigma_s^2\right) ds + \int_t^{t+\Delta} \sigma_s dw_s\right] \\ &\qquad \prod_{n=N_t}^{N_{t+\Delta}} (1+\gamma_n) \\ &= S_t \exp\Big[\int_t^{t+\Delta}\left(a_s - \frac{1}{2}\sigma_s^2\right) ds + \int_t^{t+\Delta} \sigma_s dw_s \\ &\qquad + \int_t^{t+\Delta} \log(1+\gamma_t)\, dN_t\Big] \end{aligned}$$

More risky assets

- A certain number K of risky assets

$$dS_t^i = S_t^i a_t^i dt + S_t^i \sum_{j=1}^{M} \sigma_t^{i,j} dw_t^j \, ; \quad i = 1, \cdots, K$$

$$(dS_t = (diag\, S_t)\, A_t dt + (diag\, S_t)\, \Sigma_t dw_t)$$

with $w_t = [w_t^1, \cdots, w_t^M]'$ an $M-$ Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $(\mathcal{F}_t = \mathcal{F}_t^w)$.

$\rightarrow$ *In the classical Black-Scholes model* a_t^i *and* $\sigma_t^{i,j}$ *are deterministic* $\Rightarrow$ $S_t :$ *a lognormal process.*

- The *coefficients* may also be *stochastic processes* that are either:

  i) adapted to $\mathcal{F}_t^w$, or

  ii) Markov modulated *(regime switching models)*

$$dS_t = (diag\, S_t)\, A_t(Z_t)dt + (diag\, S_t)\, \Sigma_t(Z_t)dw_t$$

$\longrightarrow$ Z_t an exogenous multivariate Markov factor process *(volume of trade, level of interest rates or, generically, the "state of the economy")*.

$\longrightarrow$ Z_t may be directly observable or not.

- Price trajectories may exhibit a *jumping behaviour*, then

$$dS_t = (diag\, S_t)\, A_t dt + (diag\, S_t)\, \Sigma_t dw_t + (diag S_{t-}) \Psi_{t-} dN_t$$

  with $N_t = (N_t^1, \cdots, N_t^H)'$ a Poisson jump process and $\Psi_t^{i,j} > -1$ *(Jump-diffusion models)*.

  $\longrightarrow$ More general driving processes are possible *(Levy, fractional BM)*.

- On *small time scales* prices do not follow continuous trajectories, but rather piecewise constant ones with jumps at random points in time.

  $\longrightarrow$ *May be modeled by continuous trajectories sampled at the jumps of a Poisson process.*