Arbitrage-free multifactor term structure models: a theory based on stochastic control

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Abstract

We present an alternative approach to the pricing of bonds and bond derivatives in a multivariate linear-quadratic model for the term structure that is based on the solution of a linear-quadratic stochastic control problem. It leads also to an approach that is an alternative to that of computing forward prices by forward measures. We finally provide explicit formulas for the computation of bond options in a bivariate factor model.

Key words: Linear-quadratic term structures, bond option pricing, stochastic control.


1 Introduction

The use of the Girsanov transformation to obtain a martingale measure has become a fundamental tool of asset and bond pricing. The key feature of this technique is a change of drift which preserves trajectories. However, as it is well known, this is not the only way to change the drift of a stochastic process. In fact, the drift can also be changed by feedback, albeit with a change of the trajectories, but keeping the same measure. It turns

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out that in the case of a linear-quadratic factor model for the term structure (a term structure where the bond prices are exponentially quadratic in the factors with the latter satisfying linear-Gaussian dynamics, (see (2.1) and (2.2) below), a feedback approach resulting from a stochastic control methodology provides the same pricing model, which can be obtained in the traditional manner, without changing the measure at all.

Stochastic control techniques have been adopted quite early in finance (see e.g. Merton [1971]), but not in the context of derivative pricing. Here we show that stochastic control techniques can be fruitfully applied also to derivative pricing in the context of term structure models. The feedback approach used in the paper can, according to work in progress, accommodate at least partly also stock option pricing but, contrary to the term structure, in the stock market one ends up with an infinite-horizon control problem.

In the stochastic control approach the control turns out to be in feedback form, namely as a function of the state/factor process and this leads to a so-called closed-loop model. The trajectories of the factors in a closed-loop model are changed with respect to those of the corresponding open loop model, but for the bond pricing this is quite irrelevant. In fact, since the observed values are eventually the rates and bond prices, it is quite indifferent, as far as pricing is concerned, whether these values are generated by an original open loop model with different trajectories (which we never observe) and the same measure, or same trajectories and a different measure. What is relevant is that they produce the same prices. Moreover the fact that (see the Appendix) Riccati and related equations can be easily computed makes the approach quite appealing and leads also to a computable alternative approach to bond option pricing (see section 4.2)

The paper is structured as follows. In section 2, which is based on Gombani et al. [2001], we recall conditions which the arbitrage-free assumption entails on a quadratic multivariate model for the term structure. Section 3 is devoted to the linear-quadratic stochastic control approach. In Subsection 3.1 we first recall some basic facts concerning a generalized version of the classical linear-quadratic Gaussian control problem. These are then applied in subsection 3.2 to obtain an alternative to bond pricing and in subsection 3.3 to compute forward prices of a bond. Section 4 is aimed at forward measures and bond derivative pricing. To this effect, in subsection 4.1 we investigate the relationship between forward measures and our stochastic control approach and apply it to the pricing of general derivatives of the factors. The specific result for bond pricing is then presented in subsection 4.2. We provide explicit formulas for the computation of a call option on a bond in a bivariate factor model for the term structure. This requires to calculate simple line integrals and thus its computational complexity is comparable to that of the Black and Scholes model. In the Appendix we provide explicit (known) formulas for the computation of the matrix differential Riccati equation needed in the paper.
2 Arbitrage free derivation for the term structure

On a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\), consider a model of the form

\[
dx(t) = F x(t)dt + G dw(t) , \quad t \geq 0 , \quad x_0 = 0
\]

(2.1)

\[
f(t, T) = a(t, T) + b(t, T)x(t) + x'(t)c(t, T)x(t)
\]

where \(x(t)\) has dimension \(n\), \(w\) is a \(k\)-dimensional Wiener process w.r. to \((Q, \mathcal{F}_t)\), the symbol \(\cdot\)' denotes transposition, \(F\) and \(G\) are matrices of appropriate dimensions and \(a(t, T), b(t, T), c(t, T)\) are scalar and matrix-valued functions respectively, differentiable with respect to \(t\) and with \(c(t, T)\) symmetric. Expectation under \(Q\) will in the sequel simply be denoted by \(E\). In (2.1), combined with (2.2) below, we consider an exponentially quadratic output model as an instance of a non-affine model. By imposing the conditions of absence of arbitrage as in Proposition 2.1 below, it can be shown (see Filipovic [2001]) that, for a linear-Gaussian factor model, general exponentially polynomial output models with degree larger than 2 reduce to the quadratic output model, i.e. the coefficients of the powers of \(x\) larger than 2 have to be equal to zero. Notice, furthermore, that the term \(a(t, T)\) in (2.1) implicitly includes the observed forward rate curve \(f^*(0, T)\) for an initial time \(t = 0\). Our model (2.1) is thus a linear factor - nonlinear output model. Dually one could also consider nonlinear factor - affine output models and there is also some equivalence between the two possible settings.

Model (2.1) for the forward rates implies for the short rate \(r(t) = r(t, x)\) and the zero-coupon bond prices \(p(t, T) = p(t, T, x)\) the representations

\[
r(t) = a(t) + b(t)x(t) + x'(t)c(t)x(t)
\]

(2.2)

\[
p(t, T) = \exp [-A(t, T) - B(t, T)x(t) - x'(t)C(t, T)x(t)]
\]

with \(a(t) = a(t, t), A(t, T) = \int_t^T a(t, u)du\) and, analogously, for \(b(t), c(t), B(t, T), C(t, T)\).

Notice that, by the above definitions, \(p(t, T, x)\) means that the bond price depends also on the factor process \(x\), evaluated at time \(t\); analogously for \(r(t, x)\). Although obvious in the present context, this meaning of notation will be important in the sequel (see Section 3.3). In what follows, whenever it does not create confusion, we shall use the shorthand notations \(r(t)\) and \(p(t, T)\).

So far \(F, G\) and \(a(t, T), b(t, T), c(t, T)\) in (2.1) appear as parameters in our model and the latter induce the parametric functions \(a(t), A(t, T), \cdots, c(t), C(t, T)\) in (2.2). However, while \(F\) and \(G\) are essentially free, in order to exclude the possibility of arbitrage, \(a(t, T), b(t, T), c(t, T)\) cannot be chosen arbitrarily. We shall therefore impose on them conditions for absence of arbitrage, that can equivalently be imposed on their integrated variants \(A(t, T), \cdots, C(t, T)\) in (2.2). We have

**Proposition 2.1** A sufficient condition for the term structure model (2.1), (2.2) to be arbitrage-free is that the coefficients \(A(t, T), B(t, T), C(t, T)\) in (2.2) satisfy the system of
differential equations in $t$

\[
\begin{align*}
\frac{\partial}{\partial t} C(t, T) &+ F'C(t, T) + C(t, T)F - 2C(t, T)GG'C(t, T) + c(t) = 0 \\
\frac{\partial}{\partial t} B(t, T) &+ B(t, T)F - 2B(t, T)GG'C(t, T) + b(t) = 0 \\
\frac{\partial}{\partial t} A(t, T) &+ \text{tr} \left( G'C(t, T)G \right) - \frac{1}{2} B(t, T)GG'B'(t, T) + a(t) = 0
\end{align*}
\]

with terminal conditions $A(T, T) = 0, B(T, T) = 0, C(T, T) = 0$. The functions $b(t), c(t)$ are here to be considered as parameters, while for $a(t)$ we have

\[(2.4) \quad a(t) = f^*(0, t) + \frac{1}{2} \int_0^t \beta_T(s, t)ds\]

having put

\[(2.5) \quad \beta(t, T) := B(t, T)GG'B'(t, T) - 2\text{tr} \left( G'C(t, T)G \right)\]

and where $f^*(0, t)$ is the observed initial forward rate curve and the subscript $T$ denotes partial differentiation with respect to the second variable.

For the proof see Gombani et al. [2001].

Concerning the solutions $C(t, T)$ and $B(t, T)$ in system (2.3) see the Appendix A.1.

Notice that, under conditions (2.3), the given measure $Q$ is a martingale measure for the numeraire given by the money market account $B(t) := \exp \left\{ \int_0^t r(s, x)\,ds \right\}$.

3 The linear-quadratic stochastic control approach

3.1 Basic facts from linear-quadratic stochastic control

Here we recall some basic facts concerning the linear-quadratic regulator problem (see e.g. Fleming et al. [1975], Oksendal [1998], see also chapter 19 in Björk, [2003]) in the particular form that will be used in the sequel.

Given always a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, let the state $x(t) \in \mathbb{R}^n$ of a controlled system evolve according to

\[(3.1) \quad dx(t) = [F(t)x(t) + H(t) + Gu(t)]\,dt + G\,dw(t)\]

where the coefficients are given matrices/vectors, $u(t) \in \mathbb{R}^k$ is the control and $w(t)$ is a $(Q, \mathcal{F}_t)$-Wiener process with values in $\mathbb{R}^k$ for the same $k$ as for $u(t)$. For what follows we may without loss of generality assume that the controls $u(t)$ are of the feedback form, namely

\[(3.2) \quad u(t) = u(t, x(t))\]
and we shall call them *admissible* if equation (3.1) has a unique solution for \( u(t) = u(t, x(t)) \). Denote by \( \mathcal{U} \) the class of admissible controls.

Given a time horizon \( T \), we shall take as criterion, on the basis of which to select the control \( u \in \mathcal{U} \), the minimization of the expected total quadratic cost

\[
J(x, u) := E_x \left\{ \int_0^T \left( x'(s)c(s)x(s) + b(s)x(s) + a(s) + \frac{1}{2} u'(s)u(s) \right) ds + x'(T)Cx(T) + Bx(T) + A \right\}
\]

where \( x = x_0 \) is the initial condition of the process \( x(t) \) in (3.1) and \( c(t), b(t), a(t), C, B, A \) are given functions/ constants with \( c(t) > 0 \) and \( C > 0 \).

Define the expected cost-to-go when in state \( x \) at time \( t \) and using a control \( u \) as

\[
J(t, x, u) := E_x \left\{ \int_t^T \left( x'(s)c(s)x(s) + b(s)x(s) + a(s) + \frac{1}{2} u'(s)u(s) \right) ds + x'(T)Cx(T) + Bx(T) + A \right\}
\]

and the expected optimal cost-to-go as

\[
W(t, x) := \inf_{u \in \mathcal{U}} J(t, x, u)
\]

It follows that \( W(0, x) = \inf_{u \in \mathcal{U}} J(x, u) \). Sometimes we shall write \( J(t, T, x, u) \) to make explicit the dependence on the horizon.

To determine \( W(t, x) \) and an optimal \( u^* \) if it exists, i.e., such that \( W(0, x) = J(x, u^*) \), we shall follow the Dynamic Programming approach. To this effect recall (see e.g. Fleming et al. [1975]) that the Hamilton-Jacobi-Bellman (HJB) equation for \( W(t, x) \) defined in (3.5) with \( x(t) \) according to (3.1) is

\[
\begin{cases}
\frac{\partial W}{\partial t}(t, x) + \inf_{u \in \mathbb{R}^k} \left\{ x'c(t)x + b(t)x + a(t) + \frac{1}{2} u'u \right. \\
+ \nabla_x W(t, x)[F(t)x + H(t) + Gu] + \frac{1}{2} \text{tr} \left( \nabla_{xx} WGG' \right) \right\} = 0
\end{cases}
\]

By the so-called *Verification theorem* we then have that

i) if \( W(t, x) \) is a \( C^{1,2} \) solution of (3.6),

ii) if \( u^* = u^*(t, x) \in \mathbb{R}^k \) attains the infimum in (3.6)

then the control \( u^* \in \mathcal{U} \) that corresponds to the \( u^*(t, x) \) that are the minimizers in (3.6) is an optimal control in the sense that

\[
J(x, u^*) = W(0, x) = \inf_{u \in \mathcal{U}} J(x, u) \quad \forall x \in \mathbb{R}^n
\]

The procedure to apply the Verification theorem consists of two steps that in the case of (3.6) are
1. Fix \((t, x) \in [0, T] \times \mathbb{R}^n\) and solve the static optimization problem

\[
(3.8) \quad \inf_{u \in \mathbb{R}^n} \left\{ x'c(t)x + b(t)x + a(t) + \frac{1}{2} u'u + \nabla_x W(t,x)[F(t)x + H(t) + Gu] \right\}
\]

The solution, if it exists, depends on \((t, x)\) and also on the yet unknown function \(W\), i.e. \(u^* = u^*(t, x; W)\).

2. Substitute the so obtained expression for \(u = u^*\) into (3.6) and solve the resulting PDE.

For Step 1, we have the following

Lemma 3.1 The minimizing \(u\) in (3.8), and thus also in (3.6), is

\[
(3.9) \quad u^*(t, x; W) = -G' \left( \nabla_x W(t,x) \right)
\]

Proof: It is immediately seen that \(u^*\) in (3.9) is a stationary point of the function to be minimized in (3.8), but it is just a critical point and, if \(b(t) \notin \text{Im } c(t)\), then we cannot complete the squares in (3.8) to eliminate the linear term and use the standard argument of convexity of the quadratic function thus obtained to prove minimality. To see that \(u^*\) in (3.9) is indeed a minimum in the original problem (3.6), observe first that \(W(t,x)\) in (3.5) is bounded from below. In fact, we have

\[
(3.10) \quad J_0(t, T, x, u) := E_{t,x} \left\{ \int_t^T \left[ \frac{1}{2} u'(s)u(s) + b'(s)x(s) \right] ds \right\}
= J(t, T, x, u) - \int_t^T [x'(s)c(s)x(s) + a(s)]ds \leq J(t, x, u; T) - \int_t^T a(s)ds
\]

for any \(u(t)\). We can express \(J_0(t, T, x, u)\) as the sum of a non negative integral depending on \(u\) and a deterministic term independent of the control. In fact, denoting by \(\phi(s, t)\) the fundamental solution associated with (3.1), we get for \(s > t\)

\[
(3.11) \quad x(s) = \phi(s, t)x(t) + \int_t^s \phi(s, \tau)[(Gu(\tau) + H(\tau))d\tau + Gdw(\tau)]
\]

Since \(x(t)\) is measurable with respect to \(\mathcal{F}_t\), it can be brought outside the expected value and there is therefore no loss of generality in assuming \(x = x(t) = 0\). Thus substituting (3.11) into the expression (3.10), one obtains

\[
J_0(t, T, x, u) = E_{t,x} \left\{ \int_t^T [\frac{1}{2} u'(s)u(s) + b'(s)x(s)]ds \right\}
= E_{t,x} \left\{ \int_t^T \left[ \frac{1}{2} u'(s)u(s) + b'(s) \int_t^s \phi(s, \tau)[(Gu(\tau) + H(\tau))d\tau + Gdw(\tau)] \right] ds \right\}
= E_{t,x} \left\{ \int_t^T \frac{1}{2} u'(s)u(s)ds + \int_t^T \int_t^s b'(s)\phi(s, \tau)[Gu(\tau) + H(\tau)]d\tau ds \right\}
= E_{t,x} \left\{ \int_t^T \frac{1}{2} u'(s)u(s)ds + \int_t^T \int_t^s b'(s)\phi(s, \tau)ds Gu(\tau)d\tau + \int_t^T \int_t^s b'(s)\phi(s, \tau)ds H(\tau)d\tau \right\}
\]
Setting $\psi'(\tau) := \int_\tau^T b'(s)\phi(s, \tau)ds$, we can complete the squares and write:

$$J_0(t, T, x, u) = E_{t,x} \left\{ \int_t^T \frac{1}{2} u'(s)u(s)ds + \int_T^T \psi'(s)Gu(s)ds + \int_t^T \psi'(s)H(s)ds \right\}$$

$$= E_{t,x} \left\{ \int_t^T \left[ u'(s), 1 \right] \begin{bmatrix} \frac{1}{2}I_k & \frac{1}{2}G'\psi(s) \\ \frac{1}{2}\psi'(s)G'G\psi(s) \end{bmatrix} \begin{bmatrix} u(s) \\ 1 \end{bmatrix} ds \right\} - \int_t^T \frac{1}{2}\psi'(s)G'G\psi(s)ds + \int_t^T \psi'(s)H(s)ds$$

The matrix in the first integral is non negative definite and the second and third integrals do not depend on $u$. Thus $J_0(t, T, x, u)$ has a minimum. Since $J_0(t, T, x, u) + \int_t^T a(s)ds$ is bounding $J(t, T, x, u)$ from below, also $J(t, T, x, u)$ has a minimum given by $W(t, x)$ in (3.5). Given the quadratic nature of the function to be minimized, it has only one critical point which thus coincides with the minimum.

**Remark 3.2** Notice that this lengthy derivation could be somehow abridged if $b(t) \equiv 0$ (see Gombani et al. [2010]). In fact, the minimization problem becomes then the well known Linear Regulator Problem (see Anderson et al. [1971]).

**For Step 2.** one makes the usual Ansatz by putting

$$W(t, x) = x'C(t)x + B(t)x + A(t)$$

where $C(t), B(t)$ are deterministic matrix functions with $C(t)$ symmetric and $A(t)$ is a scalar function. One has

**Lemma 3.3** The functions $C(t), B(t), A(t)$ satisfy the following equations and terminal conditions at $t = T$

\[
\begin{aligned}
\frac{\partial}{\partial t}C(t) + F'(t)C(t) + C(t)F(t) - 2C(t)GG'C(t) + c(t) &= 0, \quad C(T) = C \\
\frac{\partial}{\partial t}B(t) + B(t)F(t) + 2H'(t)C(t) - 2B(t)GG'C(t) + b(t) &= 0, \quad B(T) = B \\
\frac{\partial}{\partial t}A(t) + B(t)H(t) + tr(GC(t)G') - \frac{1}{2}B(t)GG'B'(t) + a(t) &= 0, \quad A(T) = A
\end{aligned}
\]

where the values $A, B, C$ of the terminal conditions correspond to those of the terminal condition in (3.6).

**Proof:** The Ansatz (3.12) leads to

\[
\begin{aligned}
\frac{\partial W}{\partial t}(t, x) &= x'\frac{\partial}{\partial t}C(t)x + B(t)x + \frac{\partial}{\partial t}A(t) \\
\nabla_t W(t, x) &= 2x'C(t) + B(t) \\
\nabla_{xx} W(t, x) &= 2C(t)
\end{aligned}
\]
and (see (3.9)) \( u^* = -G'(2C(t)x + B(t)) \). Inserting these expressions into the HJB equation (3.6) one has

\[
x'[\frac{\partial}{\partial t}C(t) + c(t) + 2C(t)GG'C(t) + F'(t)C(t) + C(t)F(t) - 4C(t)GG'C(t)]x
\]

\[+ [\frac{\partial}{\partial t}B(t) + b(t) + B(t)GG'C(t) + B(t)GG'C(t) + B(t)F(t) - 2B(t)GG'C(t)]x\]

\[-2B(t)GG'C(t) + 2H'(t)C(t) - 2B(t)GG'C(t)\]

\[+ [\frac{\partial}{\partial t}A(t) + a(t) + \frac{1}{2}B(t)GG'B'(t) + B(t)H(t) - B(t)GG'B'(t) + tr(GC(t)G')] = 0\]

Simplifying and imposing that the equation has to hold for all \( x \in \mathbb{R}^n \), we obtain the result.

The terminal conditions follow immediately from the terminal condition in (3.6).

Notice that if \( C_1 \) and \( c(t) \) are symmetric non negative definite and \( N(t) \) is positive definite on the interval \( [t_0, t_1] \).

### 3.2 Bond pricing

With bond prices described by \( p(t, T, x) \), the Term Structure Equation (see Björk, [2003]), under the assumption that the dynamics of \( x(t) \) are given by (2.1), can be written as:

\[(3.14) \ \frac{\partial}{\partial t}p(t, T, x) + x'(t)F'(\nabla_x p(t, T, x))' + \frac{1}{2}trGG'\nabla_{xx}p(t, T, x) - p(t, T, x)r(t, x) = 0\]

with terminal condition \( p(T, T, x) = 1 \). It is well known Fleming [1982] (and it can be easily verified by direct computation) that (3.14) can be transformed, putting \( W(t, T, x) := -\ln p(t, T, x) \) (and dropping the variables) into

\[(3.15) \ \frac{\partial}{\partial t}W + x'F'(\nabla_x W)' - \frac{1}{2}\nabla_x WGG'(\nabla_x W)' + \frac{1}{2}trGG'\nabla_{xx}W + r = 0\]

Consider now the Hamilton-Jacobi-Bellman (HJB) equation:

\[(3.16) \ \frac{\partial}{\partial t}W + \inf_{u \in \mathbb{R}} \left\{ (x'F' + u'G') \cdot (\nabla_x W)' + \frac{1}{2}tr(GG'\nabla_{xx}W) + \frac{1}{2}u'^2 + r \right\} = 0\]

that has as solution \( u(t, x) := -G'\nabla_x W(t, x) \) and notice that, by substituting this solution into (3.16), this latter equation becomes (3.15). On the other hand, equation (3.16) is the HJB equation associated with a linear-quadratic regulator problem as discussed in the previous subsection. More precisely, this corresponds to a dynamics of \( x(t) \) of the form (3.1) where \( F(t) \equiv F \) with \( F \) as in (2.1) and \( H(t) = 0 \). Furthermore, the expected cost-to-go \( J(t, T, x, u) \), where we now indicate explicitly the dependence on \( T \), is here given by (3.4) where the expectation is with respect to the measure \( Q \), \( a(t), b(t), c(t) \) are those in
\( (2.2) \) and \( A = B = C = 0 \) (the terminal cost is here in fact \( J(T, T, x, u) = W(T, T, x) = -\log p(T, T, x) = 0 \).

It follows from (3.12) and Lemma 3.3 that the solution \( W(t, T, x) \) of (3.16) is given by (3.12). Notice that, due to the dependence also on \( T \) and the fact that \( a(t) \) plays no role in the minimization, \( W(t, T, x) \) can be rewritten here as

\[
W(t, T; x) = x'C(t, T)x + B(t, T)x + \tilde{A}(t, T) + \int_t^T a(s)ds
\]

where we put \( A(t, T) = \tilde{A}(t, T) + \int_t^T a(s)ds \). Now, for each \( T \), \( C(t, T), B(t, T) \) satisfy the first two equations in (3.13) respectively with \( C(T, T) = B(T, T) = 0 \) and \( \tilde{A}(t, T) \) satisfies the third of those equations with \( a(t) = 0 \) and \( \tilde{A}(T, T) = 0 \). In view of the definition of \( A(t, T) \), these are exactly the equations (2.3). We have now

**Theorem 3.4** Let \( r(t) \) be defined as in (2.2) and let the bond price \( p(t, T, x) \) be given by

\[
p(t, T, x) = E_{t, x}e^{-\int_t^T r(s)ds}
\]

Then

\[
p(t, T, x) = e^{-W(t, T, x)}
\]

where \( W \) is, for each \( T \), of the form (3.17) and \( a(t) \) is determined by the boundary condition

\[
W(0, T, 0) = \int_0^T f^*(0, s)ds
\]

where \( f^*(0, t) \) is the initially observed forward rate.

**Proof:** Due to the considerations preceding the statement of the theorem we only have to show that, by imposing (3.19), the function \( a(t) \) satisfies (2.4). Since

\[
W(0, T, 0) = \tilde{A}(0, T) + \int_0^T a(s)ds
\]

we have, in view of (3.19),

\[
\int_0^T a(s)ds = \int_0^T f^*(0, s)ds - \int_0^T \text{tr} (C(s, T)GG')ds - \frac{1}{2} \int_0^T B(s, T)GG'B'(s, T)ds + \int_0^T a(s)ds
\]

and thus, setting \( A(t, T) := \tilde{A}(t, T) + \int_t^T a(s)ds \), we can write:

\[
A(t, T) = \int_t^T \text{tr} (G'C(s, T)G)ds - \frac{1}{2} \int_t^T B(s, T)GG'B'(s, T)ds + \int_t^T a(s)ds
\]
Differentiating now with respect to $t$ we obtain the relations (2.3) with $a(t)$ satisfying (2.4).

Note that $A(t,T)$ has the explicit representation:

\begin{align}
(3.21) \quad A(t,T) &= \int_t^T \left[ \text{tr} \left( G'C(s,T)G \right) - \frac{1}{2} B(s,T)GG'B'(s,T) \right] ds + \int_t^T a(s) ds \\
&= \int_t^T \left[ \text{tr} \left( G'C(s,T)G \right) - \frac{1}{2} B(s,T)GG'B'(s,T) \right] ds \\
&\quad - \int_0^T \left[ \text{tr} \left( G'C(s,T)G \right) - \frac{1}{2} B(s,T)GG'B'(s,T) \right] ds + \int_0^T f^*(0,s) ds \\
&\quad + \int_0^T \left[ \text{tr} \left( G'C(s,t)G \right) - \frac{1}{2} B(s,t)GG'B'(s,t) \right] ds - \int_0^t f^*(0,s) ds \\
&= \int_t^T f^*(0,s) ds - \int_0^t \text{tr} \left[ G'C(s,T)G - G'C(s,t)G \right] ds \\
&\quad + \frac{1}{2} \int_0^t \left[ B(s,T)GG'B'(s,T) - B(s,t)GG'B'(s,t) \right] ds
\end{align}

3.3 Forward Prices

We compute now the forward price of a bond, that is the value $E_{t,x}^{Q,\tau} p(\tau, T, x)$ for a given $\tau$ with $t \leq \tau \leq T$, and where $Q_\tau$ is the forward measure with respect to the numeraire $p(t, \tau, x)$. Since prices expressed in units of $p(t, \tau, x)$ are $Q_\tau$-martingales, we have

\begin{align}
(3.22) \quad E_{t,x}^{Q,\tau} p(\tau, T, x) &= \frac{p(t, T, x)}{p(t, \tau, x)}
\end{align}

Recall that in $p(t, \tau, x)$, the dependence on the factor process $x$ is through its value at time $t$. We claim that we can derive this forward price as an expectation with respect to $Q$ assuming however that, instead of (2.1), the factors satisfy

\begin{align}
(3.23) \quad dx^\tau(t) &= \left[ (F - 2GG'C(t, \tau))x^\tau(t) - GG'B'(t, \tau) \right] dt + Gdw(t)
\end{align}

where $F$ and $G$ are as in (2.1) and $C(t, \tau), B(t, \tau)$ correspond to those in (3.17) for $T = \tau$. For this purpose introduce the following quantity

\begin{align}
(3.24) \quad p^\tau(t, T, x) := E_{t,x} p(\tau, T, x^\tau) = E_{t,x} e^{-\int_0^\tau (C(\tau, T)x^\tau(\tau) - B(\tau, T)x^\tau(\tau) - A(\tau, T))} dx^\tau(t)
\end{align}

where the second equality follows from (3.18) and (3.17). The Kolmogorov backward equation to compute this expected value is (dropping the variables):

\begin{align}
(3.25) \quad \frac{\partial}{\partial t} p^\tau + \left[ x'(F' - 2CG'C) - BGG' \right] \nabla_x p^\tau + \frac{1}{2} \text{tr} \left( GG' \nabla_{xx} p^\tau \right) = 0
\end{align}
with terminal condition $p^\tau(\tau, T, x) = e^{-x'C(\tau,T)x-B(\tau,T)x-A(\tau,T)}$. Setting

(3.26) \[ W^\tau(t, T, x) := -\ln p^\tau(t, T, x) \]

(3.25) becomes:

(3.27) \[
\frac{\partial}{\partial t} W^\tau + [x'(F' - 2CGG') - BGG'] \nabla_x W^\tau - \frac{1}{2}(\nabla_x W^\tau)' \cdot GG' \nabla_x W^\tau \\
+ \frac{1}{2} tr (GG' \nabla_{xx} W^\tau) = 0
\]

with terminal condition $W^\tau(\tau, T, x) = x'C(\tau,T)x + B(\tau,T)x + A(\tau,T)$. This equation is again the result of the substitution into the Hamilton-Jacobi-Bellman equation:

(3.28) \[
\frac{\partial}{\partial t} W^\tau + \min_{u \in \mathbb{R}^k} \left\{ [x'(F' - 2CGG') - BGG' + u'G'] \nabla_x W^\tau \\
+ \frac{1}{2} tr (GG' \nabla_{xx} W^\tau) + \frac{1}{2} u'u \right\} = 0
\]

of $u$ with the minimizer $u(t, x) := -G'\nabla_x W^\tau(t, x)$. Again, (3.28) is the HJB equation associated with a linear-quadratic regulator problem as discussed in section 3.1. This time it corresponds to a dynamics for the factors that are those of a process $x^\tau(t)$ that satisfies (3.1) with $F(t) = F - 2GG'C(t, \tau)$ (notice the slight abuse of notation concerning $F(t)$ and $F$) and $H(t) = -GG'B(t, \tau)$. The expected optimal cost-to-go is

(3.29) \[
W^\tau(t, T, x) = \inf_{u \in \mathcal{U}} E_t \mathcal{L} \left\{ \int_t^\tau \frac{1}{2} u'(s)u(s)ds + (x^\tau(\tau))'C(\tau, T)x^\tau(\tau) \\
+ B(\tau,T)x^\tau(\tau) \right\} + A(\tau,T)
\]

Again, from (3.12) and Lemma 3.3 it follows that the solution $W^\tau(t, T, x)$ is here given by

(3.30) \[ W^\tau(t, T, x) = x'C^\tau(t, T)x + B^\tau(t, T)x + A^\tau(t, T) \]

for $t \leq \tau \leq T$ where $C^\tau(t, T)$ satisfies

(3.31) \[
\frac{\partial}{\partial t} C^\tau(t, T) + (F' - 2C(t, \tau)GG')C^\tau(t, T) + C^\tau(t, T)(F - 2GG'C(t, \tau)) \\
- 2C^\tau(t, T)GG'C^\tau(t, T) = 0
\]

$C^\tau(\tau, T) = C(\tau, T)$.

The function $B^\tau(t, T)$ satisfies

(3.32) \[
\frac{\partial}{\partial t} B^\tau(t, T) + B^\tau(t, T)[F - 2GG'C(t, \tau) - 2GG'C^\tau(t, T)] \\
- 2B(t, \tau)GG'C^\tau(t, T) = 0
\]

$B^\tau(\tau, T) = B(\tau, T)$
and for $A^\tau(t, T)$ we have

\begin{equation}
\frac{\partial}{\partial t} A^\tau(t, T) + \text{tr} \left( G' C^\tau(t, T) G \right) - B^\tau(t, T) GG'B'(t, \tau) - \frac{1}{2} B^\tau(t, T) GG'B''(t, T) = 0
\end{equation}

\begin{equation}
A^\tau(\tau, T) = A(\tau, T)
\end{equation}

Note that, contrary to the case of bond prices, here we have the terminal condition $W^\tau(\tau, T, x) = x'C(\tau, T)x + B(\tau, T)x + A(\tau, T)$. The last equation (3.33) can be written in integral form as

\begin{equation}
A^\tau(t, T) = A(\tau, T) + \int_t^\tau \text{tr} \left( G' C^\tau(s, T) G \right) ds - \int_t^\tau B^\tau(s, T) GG'B'(s, \tau) ds - \frac{1}{2} \int_t^\tau B^\tau(s, \tau) GG'B''(s, \tau) ds
\end{equation}

We are thus led to formulate the following

**Proposition 3.5** The forward price at time $t$ of a bond is given by

\begin{equation}
E_t^{Q_t} p(\tau, T, x) = p^\tau(t, T, x)
\end{equation}

where

\begin{equation}
p^\tau(t, T, x) = E_t^{Q_t} \exp\{- (x^\tau(t))' C(\tau, T) x^\tau(t) - B(t, \tau) x^\tau(t) - A(\tau, T) \} = \exp\{- (x^\tau(t))' (t) C^\tau(t, T) x^\tau(t) - B^\tau(t, T) x^\tau(t) - A^\tau(t, T) \}
\end{equation}

The process $p^\tau(t, T, x)$ has dynamics

\begin{equation}
dp^\tau(\tau, T, x) = -2(x^\tau(t))' C^\tau(t, \tau) + B^\tau(t, \tau) \, Gdw(t)
\end{equation}

**Proof**: First, we claim that

\begin{equation}
C(t, \tau) + C^\tau(t, T) = C(t, T)
\end{equation}

for $t \leq \tau$. In fact, setting $\hat{C}(t) := C(t, \tau) + C^\tau(t, T)$ for $t \leq \tau$,

\begin{equation}
-\frac{d}{dt} \hat{C}(t) = -\frac{\partial}{\partial t} C(t, \tau) - \frac{\partial}{\partial t} C^\tau(t, T)
\end{equation}

\begin{equation}
= F' C(t, \tau) + C(t, \tau) F - 2C(t, \tau) G G'C(t, \tau) + c(t)
\end{equation}

\begin{equation}
+ (F' - 2C(t, \tau) G G') C^\tau(t, T) + C^\tau(t, T) (F - 2G G'C(t, \tau))
\end{equation}

\begin{equation}
- 2C^\tau(t, T) G G'C^\tau(t, T)
\end{equation}

\begin{equation}
= F' \hat{C}(t) + \hat{C}(t) F - 2\hat{C}(t) G G' \hat{C}(t) + c(t)
\end{equation}
which is the Riccati equation in (2.3) whose unique solution (with terminal condition
\( C(T, T) = 0 \)) is \( C(t, T) \). Since \( \dot{C}(\tau) = C(\tau, T) \), in view of uniqueness it must be \( \dot{C}(t) = C(t, T) \) for \( t \leq \tau \), as claimed. In the same fashion, we claim that

\[
(3.39) \quad B(t, \tau) + B^*(t, T) = B(t, T)
\]

for \( t \leq \tau \). In fact, setting \( \hat{B}(t) := B(t, \tau) + B^*(t, T) \) for \( t \leq \tau \),

\[
- \frac{d}{dt} \hat{B}(t) = - \frac{\partial}{\partial t} B(t, \tau) - \frac{\partial}{\partial t} B^*(t, T) \\
= B(t, \tau)[F - 2GG'C(t, \tau)] + b(t) \\
+ B^*(t, T)[F - 2GG'C(t, \tau) - 2GG'C^*(t, T)] - 2B(t, \tau)GG'C^*(t, T) \\
= \hat{B}(t)[F - 2GG'C(t, \tau) - 2GG'C^*(t, T)] + b(t) \\
= \hat{B}(t)[F - 2GG'C(t, T)] + b(t)
\]

(in view of (3.38)), which is the second equation in (2.3) whose unique solution (with terminal condition \( B(T, T) = 0 \)) is \( B(t, T) \). Since \( \dot{B}(\tau) = B(\tau, T) \), in view of uniqueness it must be \( \hat{B}(t) = B(t, T) \) for \( t \leq \tau \), as claimed. Similarly, \( A(t, \tau) + A^*(t, T) = A(t, T) \). In fact, in view of (3.21) and (3.34) as well as (3.38),

\[
A(t, \tau) + A^*(t, T) = A(t, \tau) \\
+ A(\tau, T) + \int_t^\tau \text{tr} (G'C^*(s, T)G) \, ds - \int_t^\tau B^*(s, T)GG'B'(s, \tau) \, ds \\
- \frac{1}{2} \int_t^\tau B^*(s, \tau)GG'B'^*(s, \tau) \, ds \\
= \int_t^\tau f^*(0, s) \, ds - \int_0^\tau \text{tr} (G'[C(s, \tau) - C(s, t)]G) \, ds \\
+ \frac{1}{2} \int_0^\tau [B(s, \tau)GG'B'(s, \tau) - B(s, t)GG'B'(s, t)] \, ds \\
+ \int_0^\tau f^*(0, s) \, ds - \int_0^\tau \text{tr} (G'[C(s, T) - C(s, \tau)]G) \, ds \\
+ \frac{1}{2} \int_0^\tau [B(s, T)GG'B'(s, T) - B(s, \tau)GG'B'(s, \tau)] \, ds \\
+ \int_0^\tau \text{tr} (G'[C(s, T) - C(s, \tau)]G) \, ds - \int_t^\tau B^*(s, T)GG'B'(s, \tau) \, ds \\
- \frac{1}{2} \int_t^\tau B^*(s, \tau)GG'B'^*(s, \tau) \, ds \\
= \int_t^\tau f^*(0, s) \, ds - \int_0^\tau \text{tr} (G'[C(s, T) - C(s, t)]G) \, ds \\
+ \frac{1}{2} \int_0^\tau [B(s, T)GG'B'(s, T) - B(s, t)GG'B'(s, t)] \, ds \\
= A(t, T)
\]
Therefore, recalling also (3.38), (3.39) and the fact that we had assumed for the process \(x^\tau\) the same initial condition, which here is at time \(t\), as the process \(x(\cdot)\) in (2.1), from (3.17), (3.18) and (3.36) we immediately have

\[
p(t, \tau, x) p^\tau(t, T, x) = \exp\{-x'(t)C(t, \tau)x(t) - B(t, \tau)x(t) - A(t, \tau)\}
\]

that is, in view of (3.22),

\[
p^\tau(t, T, x) = E_{t,x}^{Q_{\tau}}[p(\tau, T, x)]
\]

which proves (3.35). Expression (3.37) follows from differentiation of (3.36) together with (3.31), (3.32) and (3.33).

4 Forward measures and bond option pricing

4.1 Forward measures

The results of section 3.3 suggest that a deeper connection exists between the forward prices \(p^\tau(t, T)\) and the usual forward measure \(Q_{\tau}\) which is normally used to compute (3.22). We shall now derive this connection and show that pricing with the forward measure can be made equivalent to a pricing approach under the standard martingale measure \(Q\) by using the forward prices \(p^\tau(t, T)\). To this effect

- Let \(x(\tau)\) be the value in \(\tau\) of the solution to (2.1) with initial condition \(x(t) = x\).
- Let \(x^\tau(\tau)\) be the value in \(\tau\) of the solution to (3.23) with initial condition \(x(t) = x\).

**Proposition 4.1** Given \(\tau\), the two random variables \(x(\tau)\) and \(x^\tau(\tau)\) have the same Gaussian distribution, the first under the forward measure \(Q_{\tau}\) (with numeraire \(p(t, \tau, x)\)), the second under the standard martingale measure \(Q\) (with numeraire \(B(t)\)).

**Proof**: For the numeraire \(p(t, \tau, x)\) we have, under \(Q\),

\[
dp(t, \tau, x) = p(t, \tau, x) \left[ r(t) dt - \left( 2x'(t)C(t, \tau) + B(\tau, T) \right) G dw(t) \right]
\]

For the Radon-Nikodym derivative \(L(t) = E \left\{ \frac{dQ_{\tau}}{dQ} \mid \mathcal{F}_t \right\} = \frac{p(t, \tau, x)}{B(t)}\) one then has (it is easily checked that the Novikov condition holds)

\[
dL(t) = -\left[ 2x'(t)C(t, \tau) + B(t, \tau) \right] GL(t) dw(t)
\]
It follows that the process $w^\tau(t)$ defined by

\begin{equation}
(4.3)\quad dw^\tau(t) = dw(t) + [2G'C(t,\tau)x(t) + G'B'(t,\tau)]dt
\end{equation}

is a Wiener process under $Q_\tau$.

For $x(t)$ satisfying (2.1) under $Q$ one then has, under $Q_\tau$,

\begin{equation}
(4.4)\quad dx(t) = [(F - 2GG'C(t,\tau))x(t) - GG'B'(t,\tau)]dt + Gdw^\tau(t)
\end{equation}

Since the dynamics in (4.4) is identical to those in (3.23) and $x(t) = x$ in both cases, the distribution of $x(\tau)$ under $Q_\tau$ and $x^\tau(\tau)$ under $Q$ are the same and given by a Gaussian.

\begin{remark}
An alternative approach to obtain the same result as in the previous Proposition could be to show that from the following equality, that derives from (3.35), namely

\begin{equation}
(4.5)\quad E_{t,x}^{Q_\tau}\left\{ \exp \left[-x'(\tau)C(\tau,T)x(\tau) - B(\tau,T)x(\tau) - A(\tau,T) \right] \right\}
\end{equation}

follows the equality of the two Gaussian distributions, that of $x(\tau)$ under $Q_\tau$ and that of $x^\tau(\tau)$ under $Q$, given that $x(t) = x^\tau(t) = x$.
\end{remark}

Coming now to pricing, we have the following result, which generalizes a scalar result in Pelsser [2000]:

**Proposition 4.3** Given a maturity $\tau$ and an integrable claim $H(x(\tau))$, its arbitrage free price at $t < \tau$ is

\begin{equation}
(4.6)\quad \Pi_t = E_{t,x} \left\{ e^{-\int_t^\tau r(s)ds} H(x(\tau)) \right\} = p(t,\tau,x)E_{t,x}^{Q_\tau} \left\{ H(x(\tau)) \right\} = e^{-W(t,\tau,x)}E_{t,x} \left\{ H(x^\tau(\tau)) \right\}
\end{equation}

\begin{proof}
The first equality follows from the definition of $Q$ as martingale measure for the numeraire $B(t)$, the second from the definition of the forward measure $Q_\tau$ and the third follows from (3.18) and the equality of the distributions of $x(\tau)$ under $Q_\tau$ and of $x^\tau(\tau)$ under $Q$.
\end{proof}

**4.2 Pricing of a bond derivative**

We derive now an explicit formula for the pricing of a bond option that is based on (4.6) and on the representation of the factor process $x^\tau$ in section 3.3. First we have
Remark 4.4 If \( \Phi_{\tau}(r, t) \) denotes the fundamental solution of (3.23), we immediately see that, for \( \tau > t \) with \( \tau < T \) the conditional mean of the Gaussian process \( x^\tau \) given \( x^\tau(t) = x \) is expressed by:

\[
E_{t,x}x^\tau(\tau) = \Phi_{\tau}(\tau, t)x - \int_t^\tau \Phi_{\tau}(\tau, s)GG'B'(s, \tau)ds
\]

and its conditional variance by

\[
E_{t,x}[x^\tau(\tau) - E_{t,x}x^\tau(\tau)]^2 = E \int_t^\tau \Phi_{\tau}(\tau, s)Gdw(s)dw'(s)GG'\Phi_{\tau}(\tau, s)'ds = \int_t^\tau \Phi_{\tau}(\tau, s)GG'\Phi_{\tau}(\tau, s)'ds
\]

Integral (4.7) can be computed numerically and integral (4.8) can be computed explicitly, using the explicit representations of \( \Phi_{\tau}(\tau, s) \) and \( B(s, t) \) (see (A.9) and (A.13) in the Appendix), using the representation of the function \( \Phi_{\tau}(\tau, s) \) (see (A.5) below). In fact, from (A.9),

\[
\Phi_{\tau}(\tau, s) = \Phi_{\tau}(s, \tau)\frac{1}{e^{H(s-\tau)}} = \left(\begin{bmatrix} I \\ 0 \end{bmatrix}e^{H(s-\tau)}\begin{bmatrix} I \\ 0 \end{bmatrix}\right)^{-1}
\]

and from (A.13) and the fact that \( B(\tau, \tau) = 0 \) (assuming, for simplicity, that \( b \) is constant)

\[
B(t, \tau) = \left[ \int_\tau^t b\phi_{\tau}(s, \tau)ds \right] \phi_{\tau}(t, \tau)^{-1}
\]

The mean (4.7) can now be calculated from (4.9) and (4.10) by numerical computation of a line integral. As for the variance, we can use a numerical integrator using again (4.9).

In view of the previous section, we can express the arbitrage free price of a claim by formula (4.6) where now \( x^\tau \) is the Gaussian process with mean \( \mu \) given by (4.7) and variance \( \Sigma \) given by (4.8). In particular, the value of a call option with strike price \( K \) and expiration \( \tau \) on a bond with maturity \( T \) will be,

\[
\Pi_t = e^{-W(t, \tau, \sigma)}E_{t,x} \max \left\{ 0, p(\tau, T, x) - K \right\} = e^{-W(t, \tau, \sigma)}E_{t,x} \max \left\{ 0, e^{-W(t, \tau, \sigma)} - K \right\}
\]

where, in view of (3.17),

\[
E_{t,x} \max \left\{ 0, e^{-W(t, \tau, \sigma)} - K \right\} = \frac{1}{(2\pi)^n \det \Sigma} \int_{-W(\tau, T, x) > 0} e^{-\frac{1}{2}(\xi - \mu)\Sigma^{-1}(\xi - \mu)} e^{-\frac{1}{2}C(\tau, T)(\xi - B(\tau, T)x - A(\tau, T))} d\xi_1...d\xi_n
\]

\[
e^{-\frac{1}{2}(\xi - \mu)\Sigma^{-1}(\xi - \mu)} d\xi_1...d\xi_n
\]

\[
= \frac{1}{(2\pi)^n \det \Sigma} \int_{-W(\tau, T, x) > 0} e^{-\frac{1}{2}(\xi - \mu)\Sigma^{-1}(\xi - \mu)} d\xi_1...d\xi_n
\]

16
Assume now that Σ has full rank (it can be shown that this is always the case if the pair \((F,G)\) is controllable, that is if the matrix \(G, FG, \ldots F^{n-1}G\) has full column rank). We perform then a suitable change of variables remembering that: a) two positive definite matrices can be simultaneously diagonalized by congruence and the transformation can be chosen so that one is the identity; b) if Σ is invertible, we can complete the squares. In fact, the quadratic form in the exponent \(F(\xi) := \xi' C\xi + B\xi + A\) can be written as

\[
F(\xi) := \xi' C\xi + B\xi + A = (\xi' - \nu')C(\xi - \nu) - \nu' C\nu + A
\]

where we have set

\[
\nu := -\frac{1}{2} C^{-1} B'
\]

To clarify the procedure we introduce the following quadratic functions \(F_E\) and \(F_D\), which will represent the exponent in the integrand and define the domain \(D\) of integration, respectively:

**Lemma 4.5** Let \(\xi \in \mathbb{R}^n\) and let \(A_E, B_E, C_E\) and \(A_D, B_D, C_D\) be coefficients of suitable size for the quadratic functions

\[
F_E(\xi) := \xi' C_E\xi + B_E'\xi + A_E \quad F_D(\xi) := \xi' C_D\xi + B_D\xi + A_D
\]

with \(C_E\) and \(C_D\) positive definite and let

\[
D := \{\xi : F_D(\xi) \leq 0\}
\]

Then

\[
\int_D e^{-\frac{1}{2} F_E(\xi)} d\xi_1 \ldots d\xi_n = \lambda \int_D e^{-\frac{1}{2} |\tilde{\xi}|^2} d\tilde{\xi}_1 \ldots d\tilde{\xi}_n
\]

where, with \(L' L = C_E\) and \(\nu\) as in (4.14),

\[
\tilde{\xi} = L(\xi - \nu) \quad \tilde{\lambda} = e^{\nu'C_E\nu - A_E|\det L|^{-1}} \quad \tilde{D} = \{\xi : F_D(L^{-1}\xi + \nu) \leq - \ln K\}
\]

**Proof** The substitution (4.17) immediately yields \(d\tilde{\xi}_1 \ldots d\tilde{\xi}_n = \det L d\xi_1 \ldots d\xi_n\) and:

\[
F_E(\xi) = \xi' C_E\xi + B_E'\xi + A_E = (\xi' - \nu')C_E(\xi - \nu) - \nu' C_E\nu + A_E = |\tilde{\xi}|^2 - \nu' C_E\nu + A_E
\]

from which (4.16) immediately follows. Notice that, denoting by \(L^{-T}\) the transposed inverse of a matrix \(L\),

\[
\tilde{D} = \{\tilde{\xi} : \tilde{\xi} L^{-T} C_D L^{-1} \tilde{\xi} + (2\nu' C_D + B_D)L^{-1} \tilde{\xi} + \nu' C_D\nu + B_D\nu + A_D \leq - \ln K\}
\]
We can obviously set
\[
C_D := L^{-T} C_D L^{-1} \quad B_D := (2\nu' C_D + B_D)L^{-1} \quad A_D := \nu' C_D \nu + B_D \nu + A_D + \ln K
\]
so that
\[
\tilde{D} = \{ \tilde{\xi} : F_D(\tilde{\xi}) \leq 0 \}
\]
Notice that, since \( C(t,T) > 0 \) for \( t < T \) (see Remark A.3), the functions \( F_D(\xi) \) and \( F_D(\tilde{\xi}) \) are strictly convex and thus the sets \( D \) and \( \tilde{D} \) are convex.

Thus, in principle, derivative prices can be computed in a standard manner. Nevertheless, these are multiple integrals and thus their actual computation is quite demanding.

In the cases of \( n = 1 \) and \( n = 2 \), though, the formulas can be reduced to calculating the value of single integrals and require therefore a computational effort comparable with that of the Black and Scholes formula.

### 4.2.1 n=1: the CIR case

It is well known (see Rogers [1995]) that the CIR model is equivalent to a scalar Squared Gaussian model like those presented here. Nevertheless, known computations for such models (see e.g. Pelsser [2000]) involve a double integral for the computation of the mean \( \mu \). However, as we said, this is not needed.

In fact, if \( n = 1 \), we can compute \((4.12)\) directly: set, for given \( t \) and \( T \).

\[
q(\xi) := \xi^2 C(t,T) + \xi B(t,T) + A(t,T)
\]

Then the domain of integration is \( D = \{ \xi : q(\xi) \leq -\ln K \} \). Notice that the fact that \( C > 0 \) (see Remark A.3) makes \( q(\xi) \) a convex function and thus \( D \) is the finite interval delimited by the roots (which we assume real) of

\[
q(\xi) + \ln K = 0
\]

Letting \( \xi_1,2 \) to be these roots with \( \xi_1 \leq \xi_2 \), the computation for \((4.12)\) reduces to

\[
E^Q_{t,x} \max\{0, e^{-W(\tau,T,x(\tau))} - K \}
= \frac{1}{\sqrt{(2\pi)\Sigma}} \int_{\xi_1}^{\xi_2} e^{-\left(\frac{(\xi-\mu)^2}{2\Sigma} + q(\xi)\right)} d\xi
- \frac{K}{\sqrt{(2\pi)\Sigma}} \int_{\xi_1}^{\xi_2} e^{-\frac{(\xi-\mu)^2}{2\Sigma}} d\xi
\]

with \( \mu \) and \( \Sigma \) computed above. These integrals are standard and the formula is similar to that of Black and Scholes.
4.2.2 The case \( n = 2 \)

Similarly to the one factor model discussed above, in the case of \( n = 2 \), the formulas, although more complicated, can be reduced to calculating the value of five single integrals and again the computational effort required is comparable with that of Black and Scholes.

The integrals in (4.12) are of the same kind. We show here how to compute the first one. Denote it by \( I_1 \). Set, to this end

\[
C_E := \frac{1}{2} \Sigma^{-1} + C(t, T) \quad B_E := \mu' \Sigma^{-1} + B(t, T) \quad A_E := \frac{1}{2} \mu' \Sigma^{-1} \mu + A(t, T)
\]

Then

\[
F_E(\xi) := \frac{1}{2}(\xi - \mu)' \Sigma^{-1}(\xi - \mu) + \xi' C(t, T) \xi + B(t, T) \xi - A(t, T)
\]

(4.22)

\[
= \xi' C_E \xi + B_E' \xi + A_E
\]

Then, using the transformations (4.17)-(4.19), we can write:

\[
I_1 = \frac{1}{2\pi \sqrt{\det \Sigma}} \int_{D} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2
\]

Recalling now that

\[
F_D(\xi) = W(t, T) + \ln K = \xi' C(t, T) \xi + B(t, T) \xi + A(t, T) + \ln K
\]

and using the representation (4.20), we obtain:

\[
C_D = L^{-T} C(t, T) L^{-1} \quad B_D = (2\nu' C(t, T) + B(t, T)) L^{-1} \quad A_D = \nu' C(t, T) \nu + B(t, T) \nu + A(t, T) + \ln K
\]

Let now

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{bmatrix} := C_D \quad [\kappa_1, \kappa_2] := B_D \quad \alpha := A_D
\]

Since \( C_D \) is positive definite, the set \( \tilde{D} \) (if it is non empty) is an ellipse. Then, using a standard substitution in (4.20), that is \( \tilde{\xi}_1 = \rho \cos \theta \) and \( \tilde{\xi}_2 = \rho \sin \theta \), we see that the set \( \tilde{D}_{\xi_1, \xi_2} := \{ (\xi_1, \xi_2); F_D(\xi_1, \xi_2) \leq 0 \} \) is mapped into the set

\[
\tilde{D}_{\rho, \theta} := \{ (\rho, \theta); p(\rho, \theta) \leq 0, \rho \geq 0 \}
\]

where we have set (using the above substitution):

\[
p(\rho, \theta) := \rho^2 (\sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + 2\sigma_{12} \sin \theta \cos \theta) + \rho (\kappa_1 \cos \theta + \kappa_2 \sin \theta) + \alpha
\]

As a function of \( \rho \), the above is a second degree polynomial with roots \( \rho_1(\theta), \rho_2(\theta) \). Notice that, if \( p(\rho, \theta) = 0 \), also \( p(-\rho, \theta + \pi) = 0 \). Therefore (as long as the discriminant \( \Delta \) of (4.23) is non negative), the set of points satisfying (4.23) with \( \rho > 0 \) is non empty. There are two possibilities:
a) The ellipse $\Gamma$ contains the origin, and thus we need to integrate on $[0, 2\pi)$. Thus, for each $\theta$, only one root of (4.23) is positive. Denoting this root by $\rho_2(\theta)$ and setting $k := \frac{1}{2\pi \sqrt{det C}}$, we can write $I_1$ as

$$I_1 = k \int_{\xi_1,\xi_2} e^{-\frac{1}{2} (\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 = \int_{\rho, \theta} e^{-\frac{1}{2} \rho^2} \rho d\rho d\theta = -k \int_0^{2\pi} e^{-\frac{1}{2} \rho^2} \bigg|_0^{\rho(\theta)} d\theta$$

$$= k \int_0^{2\pi} \left[ 1 - e^{-\frac{1}{2} \rho_2^2(\theta)} \right] d\theta$$

b) The ellipse $\Gamma$ does not contain the origin so, if they are real, both roots $\rho_1(\theta), \rho_2(\theta)$ have the same sign. Denote therefore by $\Gamma_\theta$ the subinterval of $[0, 2\pi)$ where the discriminant $\Delta$ of (4.23) is non negative and $\rho \geq 0$. If we make the convention that $\rho_1(\theta) \leq \rho_2(\theta)$, we can write $I_1$ as:

$$I_1 = k \int_{\xi_1,\xi_2} e^{-\frac{1}{2} (\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 = \int_{\rho, \theta} e^{-\frac{1}{2} \rho^2} \rho d\rho d\theta = -k \int_{\Gamma_\theta} e^{-\frac{1}{2} \rho^2} \bigg|_{\rho_1(\theta)}^{\rho_2(\theta)} d\theta$$

$$= k \int_{\Gamma_\theta} \left[ e^{-\frac{1}{2} \rho_2^2(\theta)} - e^{-\frac{1}{2} \rho_1^2(\theta)} \right] d\theta$$

In both cases, this is a simple integral whose value is easily computed numerically. The second integral in (4.12) can be expressed analogously.

To find the extremes of integration (or whether we are in the first or second case), it is sufficient to look at $\alpha$ and at the discriminant

$$\Delta(\theta) = (\kappa_1 \cos \theta + \kappa_2 \sin \theta)^2 - 4\alpha (\sigma_{11} \cos^2 \theta + \sigma_{22} \sin^2 \theta + 2\sigma_{12} \sin \theta \cos \theta)$$

In fact, if $\alpha$ is negative, then the origin ($\rho = 0$) is in $\tilde{D}_{\rho, \theta}$ and, in view of the positivity of the matrix $C_{\tilde{D}}$, the quadratic term coefficient is positive and thus $\Delta(\theta) > 0$ for all $\theta$. We are thus in the first case. With a bit of care also the case $\alpha = 0$ falls within this situation.

If $\alpha$ is strictly positive, then the origin is not contained in $\tilde{D}_{\rho, \theta}$ and the discriminant will be negative for some $\theta$. To find thus the extremes of the interval we need to find the values $\theta$ for which $\Delta(\theta) = 0$; divide $\Delta(\theta)$ by, say, $\cos^2 \theta$ to obtain the equation:

$$\Delta(\theta) \frac{1}{\cos^2 \theta} = (\kappa_1 + \kappa_2 \tan \theta)^2 - 4\alpha (\sigma_{11} + \sigma_{22} \tan^2 \theta + 2\sigma_{12} \tan \theta) = 0$$

which is a quadratic equation in $\tan \theta$. This will provide four values of $\theta$ in $0, 2\pi$ for which $\Delta$ vanishes. Thus $\Delta$ is positive on two of these intervals. We need to pick the one where both $\rho_1$ and $\rho_2$ are positive.

It might happen that the quadratic term of $\Delta(\theta) \frac{1}{\cos^2 \theta}$ is 0. We then have to divide $\Delta$ by $\sin^2 \theta$. If its leading coefficient is also 0, it means that the ellipse is tangent both to vertical and horizontal axis: the integration interval is again well defined.

In conclusion, for $n = 2$, to compute a call option price at $t$ expiring at $\tau$ on a bond expiring at $T$ we need:
• One line integral to compute $\mu_{t,x}(\tau)$
• One line integral to compute $\Sigma_{t,x}(\tau)$
• One line integral to compute $A(t, T)$.
• Two line integrals to compute the expected value.

In conclusion we need five line integrals, which are quite fast to compute. The computation of the integrals in (4.12) is now straightforward.

A Differential Riccati equation and closed loop dynamics: computation of the solutions

The computation of bond prices and interest rates becomes definitely simple provided we have a solution to the equations (2.3). We provide now an explicit solution $C(t, T)$ to (2.3) provided we have a solution to two time invariant equations of which the solution can be easily computed numerically. This will suffice if $B \equiv 0$. Otherwise, a fundamental solution of the second equation in (2.3) is needed. It turns out that also this can be computed explicitly. We follow Anderson et al. [1971] for this approach.

Since, for each $T$, those in (2.3) are ordinary differential equations, to simplify the notation, in what follows we will drop the explicit dependence on this variable as well as the dependence on $t$ where possible.

A.1 A complete solution for matrix $C$ and vector $B$

We have the following general result that leads to a solution of the first equation in (2.3).

**Theorem A.1** Let $C(t)$ be a solution to

(A.1) \[ \begin{cases} \frac{dC(t)}{dt} + C(t)F(t) + F'(t)C(t) - C(t)GN^{-1}(t)G'C(t) + c(t) = 0 \\ C(t_1) = C_1 \end{cases} \]

where $C_1$ and $c(t)$ are symmetric non negative definite and $N(t)$ is positive definite on the interval $[t_0, t_1]$. Then a solution $C(t)$ to (A.1) always exists between $t_0$ and $t_1$ and it can be expressed as

(A.2) \[ C(t) = Y(t)X(t)^{-1} \]

where $X$ and $Y$ satisfy the following linear differential equation:

(A.3) \[ \frac{d}{dt} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F & -GN^{-1}G' \\ -c & -F' \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad \begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix} = \begin{bmatrix} I \\ C_1 \end{bmatrix} \]
Moreover, if $\Phi(t,s)$ denotes the fundamental solution associated with 

\[(A.4) \quad \frac{dx(t)}{dt} = [F(t) - GN^{-1}(t)G'C(t)]x(t)\]

then $X$ and $Y$ admit the following interpretation

\[(A.5) \quad X(t) = \Phi(t, t_1)\]

and (in view of (A.2))

\[(A.6) \quad Y(t) = C(t)\Phi(t, t_1)\]

**Proof** Let $\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$ be a solution to (A.3). Suppose first $X(t)$ is invertible. Then, we can write (dropping the dependence on $t$), in view of (A.3),

\[
\begin{align*}
\frac{d}{dt}C &= \frac{d}{dt}[YX^{-1}] = \left(\frac{d}{dt}Y\right)X^{-1} - YX^{-1}\frac{d}{dt}(X)X^{-1} \\
&= -cX^{-1} - F'YX^{-1} - YX^{-1}FXX^{-1} + YX^{-1}GN^{-1}G'YX^{-1} \\
&= -c - F'C - CF + CGN^{-1}G'C
\end{align*}
\]

as wanted according to (A.1). Conversely, if $C(t)$ exists in $(t_0, t_1)$, the differential equation (A.4) has the fundamental solution $\Phi(t, s)$, which is defined for $t_0 \leq t, s \leq t_1$ and thus it is invertible on that interval. Moreover,

\[(A.7) \quad \frac{d}{dt}\Phi(t, t_1) = [F(t) - GN^{-1}(t)G'C(t)]\Phi(t, t_1) = F(t)\Phi(t, t_1) - GN^{-1}(t)G'C(t)\Phi(t, t_1)\]

and

\[(A.8) \quad \frac{d}{dt}C(t)\Phi(t, t_1) = \frac{dC(t)}{dt}\Phi(t, t_1) + C(t)\frac{d}{dt}\Phi(t, t_1) \\
= -[C(t)F(t) + F'(t)C(t) - C(t)GN^{-1}(t)G'C(t) + c(t)]\Phi(t, t_1) \\
+ [C(t)F(t)\Phi(t, t_1) - C(t)GN^{-1}(t)G'C(t)\Phi(t, t_1)] \\
= -c(t)\Phi(t, t_1) - F'(t)C(t)\Phi(t, t_1) \\
= -c(t)\Phi(t, t_1) - F'(t)\Phi(t, t_1)
\]

Since $\Phi(t_1, t_1) = I$, setting $C(t_1) = C_1$, we get the second row in (A.3) by putting $Y(t) = C(t)\Phi(t, t_1)$. The conclusion then follows by the uniqueness of the solution to (A.8).

Notice that, if $F, G, N, c$ are constant, setting $H = \begin{bmatrix} F & -G\gamma \end{bmatrix}$ to be the Hamiltonian matrix in (A.3), we obtain the explicit representation

\[(A.9) \quad \Phi(t, t_1) = X(t) = [I, 0]e^{H(t-t_1)} \begin{bmatrix} X(t_1) \\ Y(t_1) \end{bmatrix}
\]
Therefore, the integral
\[ \int_{t}^{t_1} \Phi(t_1, s)GG' \Phi(t_1, s)' ds = [I, 0] \int_{t}^{t_1} e^{H(s-t)} \left[ X(t_1) \quad Y(t_1) \right] GG'[X(t_1)'Y(t_1)']e^{H'(s-t)} ds \left[ \begin{array}{c} I \\ 0 \end{array} \right] \]
is of the form \( \int_{t}^{t} e^{H(s-\tau)} Q e^{H'(s-\tau)} ds \) and can be computed, using the following simple result, as the solution to a Lyapunov equation:

**Lemma A.2** Let \( P(t) = \int_{0}^{t} e^{A_s} Q e^{A_s'} ds \). Then \( P(t) \) is the solution to the Lyapunov equation:

(A.10) \[ AP(t) + P(t)A' + Q - e^{At} Q e^{A't} = 0 \]

Similarly, let \( Q \) be invertible and \( \Pi := \int_{0}^{t} (e^{A_s} Q e^{A_s'})^{-1} ds \). Then \( \Pi \) satisfies:

(A.11) \[ A' \Pi(t) + \Pi(t) A + Q^{-1} - (e^{At} Q e^{A't})^{-1} = 0 \]

**Proof** We can write \( e^{At} Q e^{A't} - Q \) as the integral of its derivative, i.e.:

\[ e^{At} Q e^{A't} - Q = \int_{0}^{t} (A e^{As} Q e^{A's} + e^{As} Q e^{A's} A') ds = AP(t) + P(t)A' \]

from which the conclusion follows.

In the same manner,

\[ \frac{d}{ds} \left( e^{As} Q e^{A's} \right)^{-1} = A' \left( e^{As} Q e^{A's} \right)^{-1} + \left( e^{As} Q e^{A's} \right)^{-1} A \]

and thus

\[ \left( e^{As} Q e^{A's} \right)^{-1} - Q^{-1} = A' \left( e^{As} Q e^{A's} \right)^{-1} + \left( e^{As} Q e^{A's} \right)^{-1} A = A' \Pi(t) + \Pi(t) A \]

as wanted. \( \blacksquare \)

**Remark A.3** Suppose \( C_1 \) and \( c(t) \) are symmetric non negative definite and \( N(t) \) is positive definite on the interval \( [t_0, t_1] \). Then \( C(t) \geq 0 \) for \( t < t_1 \) (see e.g. Fleming et al. [1975]). If moreover \( F, G \) is controllable, then \( C(t) > 0 \) for \( t < t_1 \).

Concerning the solution of the second equation in (2.3) we have now (recall that in our case we have \( N = \frac{1}{2} \))

**Corollary A.4** Let \( B(t) \) be the solution to

(A.12) \[ \begin{cases} \frac{dB(t)}{dt} + B(t)[F(t) - G N^{-1}(t) G' C(t)] + b(t) = 0 \\ B(t_1) = B_1 \end{cases} \]

and let \( \Phi(t, s) \) be the fundamental solution for the system (A.12). Then \( B(t) \) can be written as

(A.13) \[ B(t) = B_1 \Phi(t_1, t) + \int_{t_1}^{t} b(s) \Phi(s, t) ds = B_1 X(t)^{-1} + \int_{t_1}^{t} b(s) X(s) X^{-1}(t) ds \]
Proof Since, in view of (A.4),
\[
\frac{d}{dt}\Phi(t, s) = [F(t) - GN^{-1}(t)G'C(t)]\Phi(t, s)
\]
and \(\Phi(t, s)^{-1} = \Phi(s, t)\), we get that
\[
\frac{d}{ds}\Phi(t, s) = \frac{d}{ds}[\Phi(s, t)^{-1}] = \Phi(t, s)[F(s) - G(s)N^{-1}(s)G'(s)C(s)]
\]
and thus the result.

Again, using the representation (A.9), we easily see that, if \(b\) is constant vector (or even a polynomial vector) the integral (A.13) can be computed explicitly.

References


