# CONSTRUCTIVE DOMAIN THEORY AS A BRANCH OF INTUITIONISTIC POINTFREE TOPOLOGY

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Abstract. In this paper, the notions of information base and of translation between information bases are introduced; they have a very simple intuitive interpretation and can be taken as an alternative approach to domain theory. Technically, they form a category which is equivalent to the category of Scott domains and approximable mappings.

All the definitions and most of the results are inspired by the intuitionistic approach to pointfree topology as developed mainly by Martin-Löf and the first author.

As in intuitionistic pointfree topology, constructivity is guaranteed by adopting the framework of Martin-Löf's intuitionistic type theory, equipped with a few abbreviations which allow to use a standard set theoretic notation.

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**Foreword.**

Domains were introduced, in algebraic terms, by D. Scott and J. Ershov in early 70's to obtain models of pure $\lambda$-calculus; Scott himself in early 80's proposed in [Scott 81a], [Scott 82] and [Scott 81b] more intuitive presentations of domains. The first conception of the similar but refined presentation developed here is due to P. Martin-Löf and goes back to 1983 [Martin-Löf 83]. Shortly later, in Spring 1984, he suggested to the first author a joint work on what later became an intuitionistic approach to pointfree topology; such work itself was influenced by some ideas contained in Scott's presentations. By the end of 1986, stable definitions of the basic notions were reached, in such a way that domain theory could be seen as a special case (cf. [Sambin 87] and [Sambin 88] or the Appendix 1). Then the subject remained latent until the third author, under suggestion and supervision of the second author, wrote his thesis [Virgili 90].

Here we finally give a complete and detailed exposition, which is due to fresh joint work of the three authors; novelties include a constructive proof of the main theorem, a simpler definition of morphisms and a deeper understanding of the use of the predicate *Pos* of positivity. The presentation is also new, and it is centered on a very simple notion which is called information base, because of its clear intuitive meaning in terms of information.

In analogy with consistent filters on neighbourhood systems or points on pointfree topologies, we introduce the notion of concept over an information base. We show in section 2 that the collection of concepts over an information base is a Scott domain, and that every Scott domain can be presented in this way. Similarly, in analogy with approximable relations between neighbourhood systems, we introduce the notion of translation from an information base into another. We show in section 3 that translations can be used to obtain all morphisms between Scott domains. More formally, we prove that the category of Scott domains is equivalent to the category of information bases and translations. Thus, in purely mathematical terms, the two approaches are interchangeable; however, we believe that our presentation is based on simpler and more natural intuitions. Such intuitions are close to those which led D. Scott to introduce his presentations through the notions of information system and neighbourhood system; but our notion of information base is definitely simpler than the former, and more abstract than the latter (even if they form equivalent categories, see Appendix 2). In conclusion, our proposal is to adopt information bases, concepts and translations as basic notions on which domain theory can be built up.

In the whole paper, constructivity is guaranteed by adopting Martin-Löf's intuitionistic type theory (see e.g. [Martin-Löf 84]), henceforth abbreviated ITT, as ground theory for sets. We have been careful to check that all definitions (except in section 1) can be expressed and all proofs can be carried out within the framework of ITT (up to a degree of accuracy not too far from formalization); on the other hand, to increase readability we have provided

ITT with some pieces of standard set theoretic notation, introduced by means of abbreviatory definitions given below in the preliminaries. However, to deal with the standard "axiomatic" definition of Scott domains, one is compelled to expand ITT with a couple of notions, which again can be found in the preliminaries. The overall result is a paper which can be read equally well also ignoring ITT and which, like stereoscopic pictures, automatically changes according to the foundational glasses the reader wears; of course, the classically minded reader in some occasions will be puzzled by what seems to him a uselessly complicated proof or definition.

Beside the foundational interest of putting ITT at work on a piece of existing mathematics, the advantage of adopting ITT is that it allows a third reading of all the results in terms of recursive presentations: all results remain true if one systematically reads computable (or effective, or recursively presented) domain instead of domain, computable function instead of family of elements, etc. (but this has still to be worked out precisely, probably through a realizability interpretation of ITT, as suggested by Martin-Löf).

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## Preliminaries

To remain within the framework of ITT and to avoid at the same time hardly readable sequences of lengthy expressions, some pieces of standard notation are here introduced as abbreviations and briefly explained. Moreover, in order to deal with subsets (which is necessary in domain theory), a specific notion for subsets is adopted, together with notation to deal with them. Here we illustrate only the main definitions we are going to use, while for a deeper treatment the reader is referred to [Sambin-Valentini 95].

The distinction between sets and collections[1] is basic in ITT; a collection is a set only if one can effectively produce its elements. For instance $\mathbf{N}$, i.e. natural numbers, is a set since its element are (equivalent to) 0 or the successor of any element already known to be in $\mathbf{N}$, while, in general, the collection of all the subsets of a set $S$ cannot be a set since one cannot produce all of its elements. On the other hand, the collection of the finite subsets of $S$ is obviously a set, which in the following will be denoted by $\mathcal{P}_{Fin}(S)$ (for a formal definition of $\mathcal{P}_{Fin}(S)$, see [Sambin-Valentini 95]; here we only remark that, given two finite subsets of $S$, their union can be defined within ITT).

In general not only is the collection of all the subsets of a set not a set, but even *one* subset of a set can be only a collection but no set; as an example, consider the subset of $\mathbf{N}$ of the code numbers of the recursive functions which do not halt on 0: because of the unsolvability of the "halting problem", there is no way to effectively produce all the elements of this subset. On the other

[1] In this paper we systematically use the word "collection" for what is called "category" in ITT, in order to keep "category" with its common mathematical meaning.

hand, being able to deal with subsets is necessary to develop almost any piece of mathematics.

For this reason, we introduce the following notion of subset, which is suggested by the axiom of separation of ZF set theory. Let $S$ be a set; then we put

$$U \subseteq S \equiv U(x)\ prop\ [x : S]$$

that is, we say that $U$ is a *subset of* $S$ whenever $U$ is a propositional function on elements of $S$. In what follows also the alternative notation $\{x \in S : U(x)\ true\}$ will be used for the subset $U$ of $S$, in order to make the exposition clearer.

Even if the notation $U$ is not formally correct within ITT, it is really convenient and we can easily introduce some definitions which allow to recover the fragment of set theory we need. The result is a sort of local set theory, since all the relations and operations we introduce are always relativized to a set; to make this fact explicit, we will indicate the set as an index (even if in the following we will sometime omit it when it is clear from the context).

The first definition is membership; this definition is an immediate consequence of the fact that $U$ is a propositional function:

$a \,\epsilon_S\, U \equiv a \in S$ and $U(a)\ true$, i.e. there is a proof of the proposition $U(a)$.

The next step is the definition of inclusion between subsets of $S$, which is an obvious consequence of the previous definition of membership:

$$U \subseteq_S V \equiv (\forall x \in S)(U(x) \rightarrow V(x))$$

which in turn gives

$$U =_S V \equiv (\forall x \in S)(U(x) \leftrightarrow V(x)).$$

As usual, given $A(x)\ prop\ [x : S]$, quantification over a subset $U \subseteq S$, i.e. $(\forall x \,\epsilon_S\, U)A(x)$, is nothing but an abbreviation for $(\forall x \in S)(U(x) \rightarrow A(x))$; similarly $(\exists x \,\epsilon_S\, U)A(x) \equiv (\exists x \in S)(U(x) \,\&\, A(x))$.

Then subset operations can be introduced: in this paper we only need binary intersection and arbitrary union. Given two subsets $U$, $V$ of $S$ and a family $(V_i)_{i \in I}$ of subsets of $S$, i.e. a propositional function $V(i, x)\ prop\ [i : I, x : S]$, we put:

$$U \cap_S V \equiv U(x) \,\&\, V(x)\ prop\ [x : S]$$
$$\cup_{i \in I} V_i \equiv (\exists i \in I)V(i, x)\ prop\ [x : S]$$

It is interesting to recall that ITT allows to “convert” any propositional function $U$ on elements of $S$ into a “proper” set by means of the type of the disjoint union of the sets $U(x)$ for $x \in S$, which is denoted in ITT by $(\Sigma x \in S)U(x)$ or $\Sigma(S,U)$. In fact $\Sigma(S,U)$ is a set formed with pairs whose first element is an element $a$ of $S$ and the second one is a proof of the proposition $U(a)$; now it is easy to see that a subset $U$ of a set $S$ can be identified with the set $\Sigma(S,U)$, provided that one "forgets" all the proof-elements, i.e. he identifies all the pairs that have the same first element[2].

In this way we deal with subsets, but we also need to consider some sub-collections in order to express the usual approach to domain theory. To this aim we have to introduce the notion of set-indexed family: if $I$ is a set and $C$ is a collection, then we write $(x_i)_{i\in I}$ to mean a sub-collection of elements in $C$ indexed by the set $I$, i.e. a "function" from $I$ into $C$.

Of course any set is a set-indexed family, since it is indexed by itself by means of the identity function, while in general a set-indexed family is not a set since nothing similar to the replacement axiom can be assumed over ITT without lowering its level of constructivity.

Quantification is allowed within ITT only over elements of a set; so, given a set-indexed family $D = (x_i)_{i\in I}$ of elements in $C$ and a "proposition" $U(x)$ with argument $x$ in $C$, even if we write $(\forall x \in D)U(x)$ and $(\exists x \in D)U(x)$ in order to simplify the notation, what we mean is actually $(\forall i \in I)U(x_i)$ and $(\exists i \in I)U(x_i)$.

We can also give specific meaning to families indexed by a subset: given a set $S$ and $U \subseteq S$, a family $(x_{(a,b)})_{(a,b)\in\Sigma(S,U)}$ such that $x_{(a,b)} = x_{(a,b')}$ for any $a$, $b$, $b'$, is called a family indexed over the subset $U$, and then we will write $(x_a)_{a \epsilon U}$.

## 1. Scott Domains and their morphisms

In this section we will recall the basic facts about the "axiomatic" approach to domain theory (see e.g. [Barendregt 84]) which we are going to use in the following and at the same time we will express them according to the above preliminaries. The reader acquainted with the subject should probably skip this section or, rather, read it just to see what must be changed to develop the topic within the framework described in the previous section.

To help the intuition of the reader to whom the subject is new we recall that the aim of domain theory is to model, in an algebraic structure, the order relation "to be more defined than" between states of knowledge about an (abstract) object: for this reason we are mainly concerned with partially ordered collections.

Let $\mathcal{D} = \langle D, \leq \rangle$ be a partially ordered collection. A family $(x_i)_{i\in I}$ of elements in $D$ is called *(upper) bounded* whenever there exists an element $x \in D$ such that $(\forall i \in I)(x_i \leq x)$ (briefly $(x_i)_{i\in I} \leq x$) and *directed* if $I$ is inhabited and $(\forall i, j \in I)(\exists k \in I)(x_i \leq x_k \,\&\, x_j \leq x_k)$.

[2]More formally, one should first define the *image* of a function $f : I \to S$ by putting $Im(f) \equiv (\exists i \in I)(x = f(i))$ *prop* $[x : S]$. Then one can easily prove that any subset $U$ of $S$ is equal, in the sense of $=_S$ above, to the image under the first projection of the "proper" set $\Sigma(S,U)$.

**Definition 1.1.** A partially ordered collection $\mathcal{D}$ is called *coherent*, or sometimes *bounded complete*, if every bounded family of elements in $D$ has a supremum and *complete* (briefly *cpo*) if $D$ has a minimum element $\perp$ and every directed family has a supremum.

An element $a$ of a cpo $\mathcal{D}$ is called *compact*, or *finite*, if, for any directed family $(x_i)_{i\in I}$ of elements of $D$, $a \leq \vee_{i\in I} x_i$ implies that $(\exists k \in I)(a \leq x_k)$. Note that $\perp$ is trivially compact and that, whenever it exists, the supremum of any finite family of compact elements is compact. We will write $K(D)$ for the collection of compact elements of $D$ and we will reserve $a$, $b$, $c$,... to denote its elements, while we keep $x$, $y$, $z$,... for generic elements of $D$.

**Definition 1.2.** A cpo $\mathcal{D}$ is called *algebraic* if, for every $x \in D$, the collection $\{a \in K(D) : a \leq x\}$ of compact lower bounds of $x$ is a family of elements $(a_i)_{i\in I}$, for a suitable index set $I$, such that $(a_i)_{i\in I}$ is directed and $x = \vee_{i\in I} a_i$

The usual intuition behind the definition of algebraic cpo is that every element can be recovered by means of the compact elements below it, which then may be thought of as its approximations: in our approach we consider only the supremum over (set-indexed) families of elements, and this is why we require $\{a \in K(D) : a \leq x\}$ to be set-indexed. Only when $\mathcal{D}$ is algebraic, we then write $\downarrow_K(x)$ for the family $\{a \in K(D) : a \leq x\}$, and thus the equation $x = \vee{\downarrow_K}(x)$ makes sense.

**Definition 1.3.** A *Scott domain*, or simply *domain*, is a coherent algebraic cpo. In particular, we call *set-based* any Scott domain such that the collection of the compact elements is a family.

The following lemma simplifies the task of proving an algebraic cpo to be a Scott domain.

**Lemma 1.4.** *Any algebraic cpo such that any bounded pair of compact elements has a supremum, is a Scott domain.*

*Proof.* Let $(x_i)_{i\in I}$ be a family of elements of $D$ bounded by $z$. Since $\mathcal{D}$ is algebraic, for each $i \in I$, the collection of compact lower bounds of $x_i$ is a family $(a_j)_{j\in J(i)}$. Now we can consider the family of compact elements $\{a_j : j \in J(i), i \in I\}$ which is indexed by the disjoint union $\Sigma(I, J)$ of all the index-sets of such families; obviously any of its finite sub-families is bounded by $z$ and hence, by the assumption, it has a supremum. Then we construct the family of all such suprema which is indexed by $\mathcal{P}_{Fin}(\Sigma(I, J))$. Finally, it is easy to see that this family is directed and its supremum is also the supremum of $(x_i)_{i\in I}$.

The following lemma shows that in a Scott domain $\mathcal{D}$ not only the elements but also their ordering can be recovered from the structure of $K(D)$.

**Lemma 1.5.** *Let $\mathcal{D}$ be a set-based Scott domain; then for any $x, y \in D$, $x \leq y$ iff $\downarrow_K(x) \subseteq \downarrow_K(y)$, i.e. $(\forall a \in K(D))(a \leq x \rightarrow a \leq y)$.*

*Proof.* The implication from left to right is obvious; the other one holds since $\downarrow_K(x) \subseteq \downarrow_K(y)$ gives $\vee\downarrow_K(x) \leq \vee\downarrow_K(y)$, and hence $x \leq y$ since $\mathcal{D}$ is algebraic.

A *homomorphism* between Scott domains is defined as a map which respects the structure, that is the order relation and suprema of directed families. So in order that $f$ be a homomorphism it is sufficient that $f$ is monotonic and that, for any directed family $(x_i)_{i\in I}$, $f(\vee_{i\in I}x_i) \leq \vee_{i\in I}f(x_i)$; in fact by monotonicity $(f(x_i))_{i\in I}$ is directed, hence $\vee_{i\in I}f(x_i)$ exists and $\vee_{i\in I}f(x_i) \leq f(\vee_{i\in I}x_i)$.

The behaviour of a homomorphism too is completely determined by its values on the compact elements:

**Lemma 1.6.** *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be Scott domains and $f : \mathcal{D}_1 \to \mathcal{D}_2$; then $f$ is a homomorphism if, for every $x \in D_1$, $f(\downarrow_K(x))$ is a directed family and $f(x) = \vee f(\downarrow_K(x))$.*

*Proof.* If $x \leq y$ then $\downarrow_K(x) \subseteq \downarrow_K(y)$ hence $f(\downarrow_K(x)) \subseteq f(\downarrow_K(y))$ and therefore $f(x) = \vee f(\downarrow_K(x)) \leq \vee f(\downarrow_K(y)) = f(y)$, i.e. $f$ is monotonic. So in order to show that $f$ is a homomorphism, it is sufficient to prove that $f(\vee_{i\in I}x_i) \leq \vee_{i\in I}f(x_i)$ for each directed family $(x_i)_{i\in I}$ of elements of $D_1$. Let us put $y \equiv \vee_{i\in I}x_i$. For any $c \in \downarrow_K(y)$, $c \leq \vee_{i\in I}x_i$ and hence $c \leq x_k$ for some $k \in I$, since $c$ is compact. Then, by monotonicity of $f$, $f(c) \leq f(x_k)$ and hence $\vee f(\downarrow_K(y)) \leq \vee_{i\in I}f(x_i)$ from which the claim follows since $f(y) = \vee f(\downarrow_K(y))$ by assumption.

The property expressed by this lemma suggests the name *approximable function* [Scott 81a] for a homomorphism between domains, since the value on any element is completely determined by the value on its "approximations".

Finally note that Scott domains and approximable functions form a category, here called **ScDom**; in fact, it is simple to show that the composition of two approximable functions is approximable and trivially the identity function is approximable.

## 2. Information bases and concepts

In the previous section, we stressed the fact that a domain is completely described by the structure of its compact elements. The aim of this section is to introduce the notion of information base and to show that it plays exactly the role of the structure of compact elements in a domain and hence it is sufficient to reconstruct the whole domain. The definition of information base has moreover an independent intuitive motivation, which has been inspired by the pointfree approach to topology (see Appendix 1).

An information base is a set $S$ of pieces of information, provided with a little bit of structure. Intuitively, pieces of information may be thought of as neighbourhoods of a point, or as constituents of a concept. Thus the set $S$ is given together with a relation $\triangleleft$, $a \triangleleft b$ meaning that $a$ is more informative, or a better approximation of a concept, than $b$. Then it is natural to assume that $\triangleleft$ is reflexive and transitive. Moreover, we assume that $S$ contains an element $\Delta$ which gives no information, and thus $a \triangleleft \Delta$ holds for any $a$.

Also it is assumed that two pieces of information $a$, $b$ can *always* be put together in order to obtain a piece of information $a \cdot b$, which combines the information given by $a$ and $b$; the only requirement is that a weak form of compatibility with the relation $\triangleleft$ holds. But note that so, even if $a$ and $b$ are individually consistent, it may well happen that they are not compatible with each other, in the sense that their combination $a \cdot b$ gives an overload of information. To deal with this, it is convenient to introduce a predicate *Pos*, defined on elements of $S$, which expresses their consistency or *positivity*; two elements $a$ and $b$ will be considered compatible if their combination is positive, i.e. $Pos(a \cdot b)$ holds. Then $\Delta$ is trivially consistent, and if $a$ is more informative than $b$, the consistency of $a$ implies that of $b$. Moreover the fact that $a$ is not consistent means exactly that $a$ expresses "too much" information, hence that $a$ is more informative than any other piece of information.

The following definition is the formal outcome of the above intuitive explanations:

**Definition 2.1.** An *information base* $\mathcal{S}$ is a structure

$$\langle S, \cdot, \Delta, Pos, \triangleleft \rangle,$$

where $S$ is a set, $\cdot$ a binary associative operation called *combination*, $\Delta$ a distinguished element called *unit*, *Pos* a property on $S$ called *positivity* or *consistency*, and $\triangleleft$ a binary relation between elements of $S$ called *cover*, which satisfy the following conditions[3] for all $a, b, c \in S$:

(properness) $Pos(\Delta)$

(monotonicity) $\dfrac{Pos(a) \qquad a \triangleleft b}{Pos(b)}$  (positivity) $\dfrac{Pos(a) \rightarrow a \triangleleft b}{a \triangleleft b}$

(unity) $a \triangleleft \Delta$

(reflexivity) $a \triangleleft a$  (transitivity) $\dfrac{a \triangleleft b \qquad b \triangleleft c}{a \triangleleft c}$

($\cdot$-left) $\dfrac{a \triangleleft b}{a \cdot c \triangleleft b}$ and $\dfrac{a \triangleleft b}{c \cdot a \triangleleft b}$  ($\cdot$-right) $\dfrac{a \triangleleft b \qquad a \triangleleft c}{a \triangleleft b \cdot c}$

All the conditions are a straightforward rephrasing of the preceding intuitive considerations, except Positivity. In order to explain it, let us recall that the intuitive meaning of the non-positivity of $a$ is that $a$ is "too much" informative, which is formally expressed by

(*ex falso quodlibet*) $$\frac{\neg Pos(a)}{a \triangleleft b}$$

[3]We have not explicitly stated the conditions $\dfrac{a = b \qquad a \triangleleft c}{b \triangleleft c}$ and $\dfrac{c \triangleleft b \qquad a = b}{c \triangleleft a}$ since they are valid in ITT for any relation.

Hence the first reason to introduce positivity is that *ex falso quodlibet* is one of its consequences, as it is easy to find out by intuitionistic logic. On the other hand we cannot simply require *ex falso quodlibet* since we need to infer $B$, at least for some specific proposition $B$, from $\neg Pos(a) \to B$ and $Pos(a) \to B$, i.e. by applying the *principle of proof by cases* on $Pos(a)$. Since in general it is not decidable whether a piece of information is consistent or not, in a constructive approach we may not know $Pos(a) \vee \neg Pos(a)$, namely the decidability of $Pos$, and hence we may not use the $\vee$-elimination rule in order to obtain proofs by cases. On the other hand we must be careful not to assume the principle of proof by cases on $Pos$ in unrestricted form (i.e. for arbitrary $B$), since it would imply, when $B$ is $Pos(a) \vee \neg Pos(a)$, that $Pos$ is decidable (and even, together with *ex falso quodlibet*, that $Pos(a)$ holds for any $a$). Positivity expresses exactly both *ex falso quodlibet* and the principle of proof by cases on $Pos$ when $B$ is restricted to be of the form $a \lhd b$; in fact, since for any propositions $A$ and $B$ $(*)$ $(A \to B) \to B$ and $(**)$ $\neg A \to B$ and $(\neg A \to B) \to ((A \to B) \to B)$ are equivalent over intuitionistic logic, we have:

**Proposition 2.2.** *The following are equivalent:*
*i. Positivity, i.e.* $(Pos(a) \to a \lhd b) \to a \lhd b$
*ii. Proofs by cases on Pos, i.e.*

$$(\neg Pos(a) \to a \lhd b) \to ((Pos(a) \to a \lhd b) \to a \lhd b)$$

*and ex falso quodlibet, i.e.* $\neg Pos(a) \to a \lhd b$

Note that compatibility of the relation $\lhd$ with the operation $\cdot$ can equivalently be expressed by some other conditions. For instance, it is easily checked that in a structure satisfying reflexivity and transitivity, $\cdot$-left and $\cdot$-right together are equivalent to:

(stability) $\dfrac{a \lhd b \qquad c \lhd d}{a \cdot c \lhd b \cdot d}$

(idempotency) $a \lhd a \cdot a$ (weakening) $a \cdot b \lhd a$, $a \cdot b \lhd b$

Also note that the equivalence relation $\cong_{\mathcal{S}}$ induced on $S$ by putting

$$a \cong_{\mathcal{S}} b \equiv a \lhd b \,\&\, b \lhd a \qquad [a, b : S]$$

is a congruence, i.e. respects the whole structure. Then the quotient of the information base under such equality is a meet semilattice with unity equipped with a non-empty subset $Pos$ satisfying monotonicity and positivity with respect to the partial order induced by the meet operation.

In the development of domain theory, information bases play the role which, in the customary approach, is played by two notions introduced by Dana Scott, namely information systems [Scott 82] and neighbourhood systems [Scott 81a]. The connection between such notions and information bases takes technically the form of an equivalence of categories, which will be given in Appendix 2. Here we show how neighbourhood systems and information systems yield a special case of information bases.

From a constructive point of view, a neighbourhood system $\mathcal{N}$ is a structure $\langle \Delta, (X_i)_{i\in I}\rangle$ where $\Delta$ is a set and $(X_i)_{i\in I}$ is a family of subsets of $\Delta$ which contains $\Delta$ and the intersection of two elements $X_i$, $X_j$ if they have a lower bound $X_k$ for some $k \in I$. Intuitively, an information base is then obtained from $\mathcal{N}$ by closing $(X_i)_{i\in I}$ under *all* the finite intersections, and contextually by stating that only the elements of the original family $(X_i)_{i\in I}$ are positive. The intersection $X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n}$ corresponds to the finite subset $\{i_1, i_2, \ldots, i_n\}$ of the set of indexes $I$. Thus we consider the structure

$$\langle \mathcal{P}_{Fin}(I), \cup, \emptyset, Pos, \lhd \rangle$$

where

$$Pos(\{i_1, i_2, \ldots, i_n\}) \equiv (\exists k \in I)(X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_n} = X_k)$$

and

$$\{i_1, \ldots, i_n\} \lhd \{j_1, \ldots, j_m\} \\ \equiv Pos(\{i_1, \ldots, i_n\}) \to X_{i_1} \cap \cdots \cap X_{i_n} \subseteq X_{j_1} \cap \cdots \cap X_{j_m}$$

It is readily checked that this gives in fact an information base. In other words, the notion of information base may be seen as the formal counterpart of the notion of neighbourhood system (in much the same way as pointfree topologies are more generally the formal counterpart of topological spaces).

Given an information system $\langle D, \Delta, Con, \vdash \rangle$, where $D$ is the set of tokens, $\Delta$ is a distinguished token, $Con$ is a set of finite subsets of $D$, $\vdash$ is the entailment relation between two elements of $Con$ (cf. [Scott 82]), we construct an information base as follows. First of all, any finite set of tokens, that is any element of $\mathcal{P}_{Fin}(D)$, is a piece of information. Obviously we declare that $Pos(u)$ holds exactly when $u \in Con$. Then the operation of combination will be simply the union, and hence $\emptyset$ will be the unit. In order to obtain a covering relation $\vdash^+$, the entailment relation $\vdash$, which is defined only on consistent finite sets of tokens, is extended to all finite subsets by putting[4]

$$u \vdash^+ v \equiv Pos(u) \to Pos(v) \,\&\, u \vdash v$$

which simultaneously means that $\vdash^+$ and $\vdash$ coincide on positive elements and guarantees that $\vdash^+$ satisfies positivity. The conditions defining information systems ensure that $\langle \mathcal{P}_{Fin}(D), \cup, \emptyset, Con, \vdash^+ \rangle$ is an information base; we leave the details to the reader.

[4]Here, like often in the sequel, we are using the fact (see [Martin-Löf 84], p. 26 and p. 43) that any implication $A \to B$ and any conjunction $A \,\&\, B$ is a proposition when $B$ is a proposition under the assumption that the proposition $A$ is true.

Anyhow, apart from the previous examples which just illustrate the way to obtain an information base starting with a "concrete" structure, the definition of information base is much more general. In fact any (small) Cartesian category for which a notion of positivity can be provided is an example of information base where the objects of the category are the elements, and the cover relation between $A$ and $B$ means that there is an arrow between them; as an example, let us consider the category of finite sets where $Pos(A)$ might mean that $A \neq \emptyset$.

As well as information systems and neighbourhood systems, also information bases are used to construct domains. The basic idea is that any subset $\alpha$ of consistent and mutually compatible pieces of information determines a *concept*, that is is an element of the domain. The intended meaning of $a \epsilon \alpha$ is that $\alpha$ includes the information given by $a$. Since we wish to identify two concepts when they convey the same amount of information, we assume that $\alpha$ is closed under deducible information; namely, we require that if $\alpha$ includes both $a$ and $b$, then it includes their combination $a \cdot b$, and that if $\alpha$ includes $a$, then it includes also any $b$ which is less informative than $a$. We adopt here also the alternative name *point* for such notion, because of its close connection with the general notion of point in a pointfree topology (see Appendix 1). Then we have:

**Definition 2.3.** Let $\mathcal{S}$ be an information base. Then a *concept*, or a *point*, $\alpha$ of $\mathcal{S}$ is a subset of $S$ which for all $a, b \in S$ satisfies the following conditions:

i. 1. $\Delta \,\epsilon\, \alpha$ 2. $\dfrac{a \,\epsilon\, \alpha \qquad b \,\epsilon\, \alpha}{a \cdot b \,\epsilon\, \alpha}$ 3. $\dfrac{a \,\epsilon\, \alpha \qquad a \lhd b}{b \,\epsilon\, \alpha}$

ii. $\dfrac{a \,\epsilon\, \alpha}{Pos(a)}$

Note that in mathematical terms i. expresses closure under $\Delta$, $\cdot$, and $\lhd$; so a subset satisfying i. is just a filter of $\mathcal{S}$. A concept is then a filter of positive pieces of information (and hence it is the formal counterpart of the notion of filter in a neighbourhood system).

The following lemma, whose proof is an exercise in intuitionistic logic, sums up the basic properties of concepts of an information base.

**Lemma 2.4.** *For any information base $\mathcal{S}$, any $a, b \,\epsilon\, Pos$ and any concept $\alpha$ of $\mathcal{S}$, the following hold:*

i. $\uparrow a \equiv \{c \in S : a \lhd c\}$ *is a concept;*
ii. $a \lhd b$ *iff* $\uparrow b \subseteq \uparrow a$*;*
iii. $a \,\epsilon\, \alpha$ *iff* $\uparrow a \subseteq \alpha$*;*
iv. $\alpha = \cup_{a \epsilon \alpha} \uparrow a$.

For any $a \,\epsilon\, Pos$, $\uparrow a$ is called the *concept generated by* $a$; hence the collection of generated concepts is a family of elements indexed by the subset $Pos$.

Now we are going to show that, for any information base $\mathcal{S}$, the collection of concepts of $\mathcal{S}$ equipped with the inclusion ordering is a set-based Scott domain. The underlying idea is that the concept $\alpha$ is less specified than $\beta$ if any information met by $\alpha$ is also met by $\beta$; this is why inclusion between concepts is often called the *specialisation ordering*. Note that $\alpha \subseteq \beta$ means that $\beta$ is more informative than $\alpha$, which is to be contrasted with $a \lhd b$, which means that $a$ is more informative than $b$ and is equivalent to $\uparrow b \subseteq \uparrow a$. From now on we will write $Pt(\mathcal{S})$ to mean the collection of concepts of $\mathcal{S}$ equipped with the specialisation ordering.

**Lemma 2.5.** *For any information base $\mathcal{S}$, $Pt(\mathcal{S})$ is a cpo.*

*Proof.* The specialisation ordering is obviously a partial order and $\uparrow\Delta$ is its bottom since $\Delta \,\epsilon\, \alpha$ for any concept $\alpha$. In order to show that a directed family of concepts $(\alpha_i)_{i\in I}$ has a supremum, it is sufficient to note that the union $\cup_{i\in I}\alpha_i$ is a concept: the only non-trivial condition is closure under combination, which requires that $(\alpha_i)_{i\in I}$ be directed.

**Lemma 2.6.** *If $\mathcal{S}$ is an information base and $\alpha \,\epsilon\, Pt(\mathcal{S})$, then*

i. *the family $(\uparrow a)_{a\epsilon\alpha}$ is directed;*
ii. *$\alpha$ is compact iff $\alpha = \uparrow a$ for some $a \,\epsilon\, Pos$;*
iii. *any bounded couple of generated concepts has a supremum.*

*Proof.* i. The family $(\uparrow a)_{a\epsilon\alpha}$ is directed since $\alpha$ is closed under $\cdot$ and $\uparrow a, \uparrow b \subseteq \uparrow(a \cdot b)$ for any $a, b \,\epsilon\, \alpha$. ii. If $\alpha$ is a generated concept, i.e. of the form $\uparrow a$ for some $a \,\epsilon\, Pos$, and $(\alpha_i)_{i\in I}$ is a directed family of concepts, then $\uparrow a \subseteq \cup_{i\in I}\alpha_i$ iff $a \,\epsilon\, \cup_{i\in I}\alpha_i$ iff $(\exists k \in I)(a \,\epsilon\, \alpha_k)$ iff $(\exists k \in I)(\uparrow a \subseteq \alpha_k)$. Conversely, let $\alpha$ be compact; since $(\uparrow a)_{a\epsilon\alpha}$ is directed by i. and $\alpha \subseteq \cup_{a\in\alpha}\uparrow a$ by 2.4.iv, then $(\exists c \,\epsilon\, a)(\alpha = \uparrow c)$. iii. Let $\uparrow a$ and $\uparrow b$ be generated concepts; then it is easy to see that $a \cdot b$ is positive whenever they are bounded and $\uparrow(a \cdot b)$ is obviously their supremum.

**Proposition 2.7.** *For any information base $\mathcal{S}$, $Pt(\mathcal{S})$ is a set-based Scott domain.*

*Proof.* For any concept $\alpha$, by lemma 2.6.ii, a lower bound $\beta \subseteq \alpha$ is compact iff $\beta = \uparrow a$ for some $a \,\epsilon\, \alpha$; hence compact lower bounds of $\alpha$ form the family $(\uparrow a)_{a\epsilon\alpha}$, which is directed by 2.6.i and has $\alpha$ as supremum by 2.4.iv; hence $Pt(\mathcal{S})$ is algebraic. Hence lemmas 1.4 and 2.6.iii tell that $Pt(\mathcal{S})$ is a Scott domain. Moreover, $Pt(\mathcal{S})$ is set-based since the family of compact concepts is indexed by $Pos$.

To prove the converse, we show now how any set-based Scott domain $\langle D, \leq\rangle$ is (isomorphic to) the collection of concepts over an information base which is built up directly from $K(D)$. The hint to find the correct definition comes once again from the topological intuition. To this aim let us recall the definition of Scott topology on a cpo.

**Definition 2.8.** In any cpo $\mathcal{D}$, a sub-collection $O$ is called *(Scott) open* if it is *hereditary*, or *upward closed*, that is if $x \in O$ and $x \leq y$ then $y \in O$, and *smooth*, that is, for each directed subset $U$, if $\vee U \in O$ then $(\exists u \in U)u \in O$.

It is easy to check (see for instance [Barendregt 84]) that Scott opens of $D$ form a topology on $\mathcal{D}$ , which is called the *Scott topology*.

We show now that, if $\mathcal{D}$ is not only a cpo but a Scott domain, then it is completely determined by its Scott topology. To this aim let us observe that, given a base $\mathcal{B}$ for the Scott topology of $\mathcal{D}$, its order relation can be completely recovered since $x \leq y$ if and only if $(\forall O \in \mathcal{B})(x \in O \rightarrow y \in O)$. In fact from left to right the result is an obvious consequence of hereditarity. To prove the other implication, one must note that, for any $a \in K(D)$, $\uparrow a$ is Scott open and hence $\uparrow a$ can be expressed as a suitable union of elements of the base $\mathcal{B}$. Hence if $x \in \uparrow a$ then there is $O \in \mathcal{B}$ such that $O \subseteq \uparrow a$ and $x \in O$ and so, by hypothesis, $y \in O$ which implies $y \in \uparrow a$; this proves that $(\forall a \in K(D))(a \leq x \rightarrow a \leq y)$ which, by lemma 1.5, is equivalent to $x \leq y$.

The above remark suggests that we need a base, in the usual topological sense, in order to find the information base we are looking for.

A base for the Scott topology on $\mathcal{D}$ is usually obtained by considering all the sub-collections $\uparrow a$ for $a \in K(D)$ and possibly adding $\emptyset$ (cf. [Barendregt 84]). Here this must be refined a little to avoid any definition or proof based on the distinction between the cases $\uparrow a \cap \uparrow b = \uparrow(a \vee b)$ and $\uparrow a \cap \uparrow b = \emptyset$, i.e. between $\{a, b\}$ bounded or not. Then the idea is to consider directly, for any $U \in \mathcal{P}_{Fin}(K(D))$, the sub-collection of upper bounds $O_U \equiv \{x \in D : U \leq x\}$, where $U \leq x$ is an abbreviation for $a \leq x$ for any $a \in U$. It is easy to check that $\{O_U : U \in \mathcal{P}_{Fin}(K(D))\}$ is a base for the Scott topology on $\mathcal{D}$. In fact, $O_U$ is Scott-open because it is obviously closed upwards, and it is smooth since, for any directed subset $W$ of $D$, if $U \leq \vee W$ then, being $U$ bounded, $\vee U$ exists and hence $\vee U \leq \vee W$, but since $\vee U$ is compact, because $U$ is finite, there exists $w \in W$ such that $\vee U \leq w$, that is $U \leq w$. Moreover, for any Scott open $O$, the equation $O = \cup\{O_U : O_U \subseteq O\}$ holds since, supposing $x \in O$, that is $x = \vee{\downarrow_K}(x) \in O$, then by smoothness of $O$ there exists $a \in {\downarrow_K}(x)$ such that $a \in O$, so that $O_{\{a\}}$ is a subset which contains $x$ and $O_{\{a\}} \subseteq O$ because of hereditarity of $O$. Finally, in this approach a proof of $O_U \cap O_V = O_{U \cup V}$ becomes straightforward and no argument by cases is needed.

So, apart from foundational matters, the information base is now disclosed; the foundational problem is that $\{O_U : U \in \mathcal{P}_{Fin}(K(D))\}$ is no set, but a family.

The standard way out is to build up an information base $\mathcal{S}_\mathcal{D}$ by pulling the structure of the base $\{O_U : U \in \mathcal{P}_{Fin}(K(D))\}$ back to the index set $\mathcal{P}_{Fin}(K(D))$ (pedantically, $\mathcal{P}_{Fin}(I)$ where $I$ is the index set for $K(D)$). In detail, we provide $\mathcal{P}_{Fin}(K(D))$ with an operation of combination $\cdot_{\mathcal{S}_\mathcal{D}}$ in such a way that $O_{U \cdot_{\mathcal{S}_\mathcal{D}} V} = O_U \cap O_V$, that is we put

$$U \cdot_{\mathcal{S}_\mathcal{D}} V \equiv U \cup V.$$

Then the unit element of $\mathcal{S}_\mathcal{D}$ is $\emptyset \in \mathcal{P}_{Fin}(K(D))$, which can also be seen observing that $O_\emptyset = D$ and hence $O_\emptyset \cap O_U = O_U$ for any $U$. We say that $U$

is positive when $O_U$ is inhabited, that is when $U$ is bounded; so we put

$$Pos_{\mathcal{S_D}}(U) \equiv (\exists a \in K(D))(U \leq a)$$

and $Pos_{\mathcal{S_D}}$ is a subset of $\mathcal{P}_{Fin}(K(D))$. Note that $U$ is positive if and only if $\vee U$ exists. Finally, we want $U$ to be covered by $W$ when $O_U \subseteq O_W$, which is clearly equivalent to: if $\vee U$ exists, then $W \leq \vee U$. Thus we put

$$U \lhd_{\mathcal{S_D}} W \equiv Pos_{\mathcal{S_D}}(U) \to W \leq \vee U$$

It is obvious now that

$$\mathcal{S_D} \equiv \langle \mathcal{P}_{Fin}(K(D)), \cdot_{\mathcal{S_D}}, \emptyset, Pos_{\mathcal{S_D}}, \lhd_{\mathcal{S_D}} \rangle$$

is an information base.

Now we can show that $\mathcal{S_D}$ is the information base we are looking for, since the domains $\mathcal{D}$ and $Pt(\mathcal{S_D})$ are isomorphic. The easiest way to find out an isomorphism, is to specialise to the base $\{O_U : U \in \mathcal{P}_{Fin}(K(D))\}$ the fact that a domain is completely determined by a base for its Scott topology. In fact in this way we obtain that $x \leq y$ if and only if $(\forall O_U)(x \in O_U \to y \in O_U)$, which can equivalently be expressed in our framework as $(\forall U \in \mathcal{P}_{Fin}(K(D)))(U \leq x \to U \leq y)$, i.e. $\{U : U \leq x\} \subseteq \{U : U \leq y\}$. It is easy to check that, for any $x \in D$, the subset[5] $\{U \in \mathcal{P}_{Fin}(K(D)) : U \leq x\}$ is a concept of $\mathcal{S_D}$ (which is the formal counterpart of the filter of opens $O_U$ containing $x$). Hence putting

$$f : x \mapsto \{U \in \mathcal{P}_{Fin}(K(D)) : U \leq x\}$$

defines a map from $\mathcal{D}$ into $Pt(\mathcal{S_D})$, which, by the above remark, is monotonic and one-one; to conclude we must only show that $f$ is onto and hence an isomorphism (since any bijective monotonic function respects all suprema). To this aim, observe that if $\alpha$ is a concept of $\mathcal{S_D}$ then $W \;\epsilon\; \alpha$ iff $(\forall a \in W)(\{a\} \;\epsilon\; \alpha)$, i.e. $\alpha$ is determined by the singletons it contains; hence the element of $D$ whose image under $f$ is $\alpha$ must be $\vee\{a \in K(D) : \{a\} \;\epsilon\; \alpha\}$, which exists since $\{a \in K(D) : \{a\} \;\epsilon\; \alpha\}$ is directed. So we have proved:

**Theorem 2.9.** *Any set-based Scott domain $\mathcal{D}$ is isomorphic to the concepts of a suitable information base $\mathcal{S_D}$.*

It is worthwhile to recall that, in the classical conception, any Scott domain is set-based, and thus the above theorem gives a representation theorem for Scott domains with no restriction. On the other hand, from a strictly constructive point of view, it could be argued that what we have called information base is nothing but a truly constructive introduction of Scott

[5]The fact that $\{U : U \leq x\}$ is a subset, i.e. a propositional function over $\mathcal{P}_{Fin}(K(D))$, is not so immediate. Given $x \in D$, suppose $I_x$ is the index set for $\downarrow_K(x)$; then $U \leq x$ means that $(\exists i \in I_x)(U \leq x_i)$ which is a propositional function with $U$ free.

domains; in this perspective, one could even say that the above theorem allows to forget the "axiomatic" definition of Scott domains, and hence that the theorem itself is to be forgotten, as soon as it has been proved.

This would have also the noteworthy advantage of forgetting all the troublesome basic notions introduced in the preliminaries which were needed to treat domains in the "axiomatic" approach.[6]

## 3. Translations

In the previous section we proved that the collection of Scott domains can be completely reconstructed using information bases and concepts. Now, we want to do the same for morphisms, that is reconstruct in our framework also the notion of approximable function between Scott domains. The hint for the correct definition comes again from topology. In fact it can be proved that domain homomorphisms are exactly those maps between domains which are continuous in the Scott topology of the domains (see e.g. [Barendregt 84]). In formal topology a general definition of continuous map between two formal topologies can be given [Sambin 87], but here we prefer to introduce a simpler definition which is equivalent to the general one in the particular case of information bases (see Appendix 1) and which moreover can be given a direct and intuitive motivation.

Let $\mathcal{S}$ and $\mathcal{T}$ be two information bases; at first, we consider the transformation of the information in $\mathcal{S}$ into information in $\mathcal{T}$ by means of a translation $t$ which sends every piece of information $a \in S$ into a single piece of information $t(a) \in T$. Then it is natural to require that $t$ should respect consistency, i.e. $Pos_{\mathcal{S}}(a)$ implies $Pos_{\mathcal{T}}(t(a))$, and the amount of information, i.e. if $a \lhd b$ then $t(a) \lhd t(b)$. In general, however, this notion of translation is too strong, since it may well happen that the correct translation of $a$ is only approximated by the pieces of information which are available in $T$; if we call the subset of all such pieces of information in $T$ translation of $a$, it is natural to require that the translation of $a$ be a concept, at least when $a$ is positive. However, to deal uniformly also with non positive information, instead of a function from $Pos_{\mathcal{S}}$ into $Pt(\mathcal{T})$, we are lead to consider a binary relation $aFb$ $[a : S, b : T]$, whose meaning is that $b$ is in the translation of $a$. We will get back the concept of $\mathcal{T}$ which is the translation of an element $a$ by putting $Fa \equiv \{b \in T : aFb\}$, i.e. writing $b \epsilon Fa$ for $aFb$. Basing on the idea that a non positive piece of information $a$ is "too much" informative, in its translation $Fa$ we put *all* the pieces of information available in $T$. This fact is expressed uniformly, in analogy with what was done for $\lhd$, by the condition that $Pos(a) \rightarrow aFb$ implies $aFb$.

All of this is formally expressed by the following definition:

**Definition 3.1.** Let $\mathcal{S}$, $\mathcal{T}$ be information bases. Then a relation $F$ between $\mathcal{S}$ and $\mathcal{T}$ is called a *translation*, or an *approximable relation*, if for all $a, c \in S$

[6] One should not be too radical, however, in forgetting things, otherwise one could argue that the present paper itself should be forgotten as soon as it has been read.

and $b, d \in T$:

i. 1. $aF\Delta_T$ 2. $\dfrac{aFb \quad aFd}{aFb \cdot d}$ 3. $\dfrac{aFb \quad b \lhd d}{aFd}$ 4. $\dfrac{Pos(a) \quad aFb}{Pos(b)}$

ii. $\dfrac{a \lhd c \qquad cFb}{aFb}$

iii. $\dfrac{Pos(a) \to aFb}{aFb}$

Such a definition agrees perfectly with the intuitive description given above. In fact the conditions in i. are exactly those which the translation $Fa$ must satisfy in order to be a concept, when $a$ is positive, while ii, which is equivalent to $\dfrac{a \lhd c}{Fc \subseteq Fa}$, tells that a translation respects the amount of information.

The two conditions ii. and iii. could equivalently be expressed by the single condition

$$\frac{a \lhd c \qquad Pos(c) \to cFb}{aFb};$$

however we preferred to keep them apart, since in this way the analogy of translations with covers is more perspicuous.

Note that in particular, if $\mathbb{1}$ is the trivial information base with just one positive element $1_{\mathbb{1}}$ (and the trivial combination and cover), then translations from $\mathbb{1}$ into $\mathcal{S}$ can be identified with concepts of $\mathcal{S}$: the concept $\alpha$ of $\mathcal{S}$ is associated with the translation $F_\alpha \equiv \{(1_{\mathbb{1}}, a) : a \,\epsilon\, \alpha\}$, so that $\alpha = F_\alpha 1_{\mathbb{1}}$, and conversely any translation $F$ from $\mathbb{1}$ into $\mathcal{S}$ is of this form, since $F = \{(1_{\mathbb{1}}, a) : a \,\epsilon\, F1_{\mathbb{1}}\}$ and $F1_{\mathbb{1}}$ is a concept.[7]

Also note that condition iii. is logically equivalent to

$$aFb \text{ iff } Pos(a) \to aFb$$

which is therefore true for every translation. So, to determine a translation $F : \mathcal{S} \to \mathcal{T}$, it is sufficient to define a relation $F_0$ in such a way that i. and ii. are satisfied, and then to force iii. to hold by putting $aFb \equiv Pos(a) \to aF_0b$; it is easy to see that i. and ii. continue to hold for $F$. Moreover this is the only translation which coincides with $F_0$ on all positive elements. In fact, if $F$, $G$ are two translations $\mathcal{S} \to \mathcal{T}$ such that $Fa = Ga$ for all positive $a$, then for arbitrary $a$ it is $Pos(a) \to aFb$ iff $Pos(a) \to aGb$, and hence $F = G$ by iii.

In a completely similar way one obtains that

$$Fc \subseteq Fa \text{ iff } Pos(a) \to Fc \subseteq Fa$$

since $Pos(a) \to Fc \subseteq Fa$ implies, assuming $Pos(a)$, that if $b \,\epsilon\, Fc$ then $b \,\epsilon\, Fa$ and hence $Pos(a) \to b \,\epsilon\, Fa$ which, by the previous observation, is

[7]Since $\mathbb{1}$, as we will see, is the terminal object of the category of information bases, this remark tells that the given definition of point of $\mathcal{S}$ agrees with the general definition of *global point* in a category, i.e. $Pt(\mathcal{S})$ can be identified with $Hom(\mathbb{1}, \mathcal{S})$.

equivalent to $b \epsilon Fa$. So condition ii. of the definition of translation can be rephrased by requiring $\dfrac{a \lhd c}{Fc \subseteq Fa}$ to hold only for any positive $a$, rather than for arbitrary $a$.

These remarks will often be used in the whole section, beginning now with the very definition of composition of translations. In fact, the first idea which comes into mind is to compose two translations $F : \mathcal{S} \to \mathcal{T}$ and $G : \mathcal{T} \to \mathcal{U}$ by applying them one after another, that is to consider the subset $G(Fa) \equiv \cup\{Gb : b \epsilon Fa\}$ as a composite translation of $a$; this would mean that the composite of $F$ and $G$ is the usual composition of relations $G \circ F$, i.e. the relation $\{(a, c) \in S \times U : (\exists b \in T)(aFb \,\&\, bGc)\}$. This works perfectly well for positive elements of $S$; it may happen, however, that *all* the pieces of information in $T$ are positive and thus they are not sufficient to produce a faithful composite translation of some non-positive information of $S$ into $U$. Formally, $G \circ F$ does satisfy i. and ii., but it may happen that it does not satisfy condition iii. of the definition of translation. Hence we define the composition of $F$ and $G$ by putting

$$a \, G \star F \, c \equiv Pos(a) \to a \, G \circ F \, c$$

that is

$$G \star F \equiv \{(a, c) \in S \times U : Pos(a) \to (\exists b \in T)(aFb \,\&\, bGc)\}.$$

The identical translation of $\mathcal{S}$ into $\mathcal{S}$ associates the concept which exactly contains the information given by $a$, that is the generated concept $\uparrow a$, with any positive piece of information $a$. Since for any $b \in S$, $b \epsilon \uparrow a$ iff $a \lhd b$, it turns out that the identity morphism $I_{\mathcal{S}}$ is simply the covering relation $\lhd$ itself. It is immediate to see that the conditions on translations, when written for the relation $\lhd$, are exactly the requirements for $\lhd$ to be a cover; it is also immediate, by i.3 and ii. of the very definition of translation, that any translation $F$ is not affected by the usual composition with $I_{\mathcal{S}}$ and so $I_{\mathcal{S}}$ is the identity morphism of $S$ also with respect to $\star$ because of iii. Thus we have:

**Theorem 3.2.** *Information bases and translations, with composition $\star$ and identities as above, form a category called* **InfBas***.*

*Proof.* After the above remarks, only associativity of $\star$ is left out; its proof, where i.4 and iii. are put in use, is a rather tedious exercise in intuitionistic logic.

Our next aim is to prove that the category of information bases **InfBas** is equivalent to the category of Scott domains **ScDom** by extending $Pt$ of the previous section to a dense, full and faithful functor $Pt$ : **InfBas** $\to$ **ScDom**.

A morphism $F : \mathcal{S} \to \mathcal{T}$, that is a translation of positive pieces of information of $\mathcal{S}$ into concepts of $\mathcal{T}$, is easily lifted to a translation of concepts of $\mathcal{S}$ into concepts of $\mathcal{T}$: a concept $\alpha$ is translated into the union of all the concepts which are the translation of some element $a$ of $\alpha$. Formally, $Pt(F)(\alpha) \equiv \cup\{Fa : a \epsilon \alpha\}$. Recalling that $\alpha$ is associated with the translation $F_\alpha : \mathbb{1} \to \mathcal{S}$, we have $\cup\{Fa : a \epsilon \alpha\} = \cup\{Fa : a \epsilon F_\alpha 1_{\mathbb{1}}\} \equiv (F \circ F_\alpha)(1_{\mathbb{1}})$ and hence $Pt(F)(\alpha)$ is a concept of $\mathcal{T}$, because it is the image of $1_{\mathbb{1}}$ under the composite translation $F \star F_\alpha$ (which is obviously equal to $F \circ F_\alpha$, since the only element of $\mathbb{1}$ is positive).[8]

It is immediate to see that $Pt(F)$ is monotonic and respects (arbitrary) unions, since it is defined in terms of union. So $Pt(F) : Pt(\mathcal{S}) \to Pt(\mathcal{T})$ is a morphism of domains. It is now easy to prove that:

**Lemma 3.3.** *Pt is a functor from* **InfBas** *to* **ScDom***.*

*Proof.* What has been left out is only that $Pt$ respects identities and composition. $Pt(I_{\mathcal{S}}) = 1_{Pt(\mathcal{S})}$ because $I_{\mathcal{S}}(a) = \uparrow a$ and hence $Pt(I_{\mathcal{S}})(\alpha) \equiv \cup\{\uparrow a : a \epsilon \alpha\} = \alpha$ and $Pt(G \star F) = Pt(G) \circ Pt(F)$ because, for any $\alpha$, $Pt(G\star F)(\alpha) = (G\star F)\star F_\alpha(1_{\mathbb{1}}) = G\star(F\star F_\alpha)(1_{\mathbb{1}}) = Pt(G)(Pt(F)(\alpha))$.

Associating a function on concepts with any translation provides the notion of translation itself with a new intuitive meaning. In fact we can show now that, for any translation $F : \mathcal{S} \to \mathcal{T}$,

(∗) $aFb$ iff for any concept $\alpha$ of $\mathcal{S}$, $a \epsilon \alpha \to b \epsilon Pt(F)(\alpha)$;

in other words, $aFb$ means that the associated function maps each concept including $a$ into a concept including $b$.

The direction from left to right in the proof of (∗) is clear by definition of $Pt(F)$, since $aFb$ means that $b \epsilon Fa$ and $Fa \subseteq Pt(F)(\alpha)$ whenever $a \epsilon \alpha$.

The crucial step of the proof of the converse is actually a formulation of its antecedent within the framework of ITT; indeed, the quantification over subsets, although of a specific kind like concepts, has no general meaning in ITT. However, in this case, the meaning of

(1) for every concept $\alpha$ of $\mathcal{S}$, $a \epsilon \alpha \to b \epsilon Pt(F)(\alpha)$

is expressed, inside ITT, by

(2) $(\forall c \in S)(Pos(c) \,\&\, a \epsilon \uparrow c \;\to\; b \epsilon Pt(F)(\uparrow c))$.

In fact, though (2) is only a special case, it is sufficient to recover (1) since every concept $\alpha$ is the union of generated concepts $\cup\{\uparrow c : c \epsilon \alpha\}$ and hence $a \epsilon \alpha$ means that $a \epsilon \uparrow c$ for some positive $c \epsilon \alpha$, from which $b \epsilon Pt(F)(\uparrow c)$ by (2) and so $b \epsilon Pt(F)(\alpha)$ because $Pt(F)$ is monotone and $\uparrow c \subseteq \alpha$.

The right to left direction of (∗) can now be proved, since $aFb$ follows from (2). In fact, instantiating (2) on $a$, we obtain $Pos(a) \to b \epsilon Pt(F)(\uparrow a)$, which is equivalent to $Pos(a) \to b \epsilon Fa$ since, when $a$ is positive, $Pt(F)(\uparrow a) \equiv \cup\{Fc : a \lhd c\} = Fa$; but we know that $Pos(a) \to b \epsilon Fa$ iff $aFb$.

[8] Together with the identification of $Pt(\mathcal{S})$ with $Hom(\mathbb{1}, \mathcal{S})$, this tells that $Pt(-)$ is just the functor $Hom(\mathbb{1}, -) :$ **InfBas** $\to$ **Set**.

The detour to show $(*)$ is so complete; even if it may appear a bit too long, two facts which are basic in proving that $Pt$ is an equivalence have appeared along the way and can be extracted from it:

**Proposition 3.4.** *For any translation $F$, and any a,b:*
*i. if $a$ is positive, $Fa = Pt(F)(\uparrow a)$;*
*ii. $aFb$ iff $Pos(a) \rightarrow b \,\epsilon\, Pt(F)(\uparrow a)$.*

Now we can prove that $Pt$ is a faithful and full functor, i.e. for any information bases $\mathcal{S}$ and $\mathcal{T}$, $Pt : Hom(\mathcal{S}, \mathcal{T}) \rightarrow Hom(Pt(\mathcal{S}), Pt(\mathcal{T}))$ is one-one and onto. Injectivity is immediate because $Pt(F) = Pt(G)$ means that for all positive elements $a$ of $S$, $Pt(F)(\uparrow a) = Pt(G)(\uparrow a)$, hence by i. above $Fa = Ga$ and hence $F = G$. To show surjectivity, for any $f : Pt(\mathcal{S}) \rightarrow Pt(\mathcal{T})$ we have to find a translation $F$ such that $Pt(F) = f$; ii. above tells that there is no other choice than putting

$$aFb \equiv Pos(a) \rightarrow b \,\epsilon\, f(\uparrow a).$$

It is straightforward to check that $F$ so defined is indeed a translation. In fact, for any positive $a$, condition i. (that is $Fa$ is a concept) holds because $Fa \equiv f(\uparrow a)$, and ii. holds because $a \lhd c$ implies $f(\uparrow c) \subseteq f(\uparrow a)$, which means $Fc \subseteq Fa$, because $f$ is monotone. Finally condition iii., i.e. $aFb$ iff $Pos(a) \rightarrow aFb$, holds by intuitionistic logic.

Hence we have proved that

**Theorem 3.5.** *The functor $Pt$ is an equivalence between the category of the information bases* **InfBas** *and the category of Scott domains* **ScDom**.

It is interesting to note that any monotonic function $f$ from the domain $Pt(\mathcal{S})$ into the domain $Pt(\mathcal{T})$ gives rise to a translation between $\mathcal{S}$ and $\mathcal{T}$ by putting

$$aF_f b \equiv Pos(a) \rightarrow b \,\epsilon\, f(\uparrow a)$$

which defines $F_f$ by means of the value of $f$ on the compact elements of $Pt(\mathcal{S})$; that is, to *define* a translation it is not required that $f$ should respect the suprema of the directed families of $Pt(\mathcal{S})$. Moreover $Pt(F_f)$ coincides with $f$ on all the compact elements of the domain $Pt(\mathcal{S})$. This fact should not be surprising since, by lemma 1.6, any domain morphism is completely determined by its behaviour on the compact elements, i.e. among all the monotonic functions between two domains which coincide on all the compact elements there is only one domain morphism. So preservation of suprema is needed to *prove* that the correspondence is biunivocal.

**Appendix 1: Connections with intuitionistic pointfree topology.**

The idea of pointfree topology is to study those properties of a topological space $\langle X, \Omega X \rangle$, where $\Omega X$ is the collection of open subsets of the set $X$, which can be expressed without any mention to points, that is elements of $X$. The basic idea is to consider opens instead of points as primitive entities; the notion of point can be recovered as a "suitable" set of opens, as we see below. Since any topology $\Omega X$ can be described by specifying a base, abstracting from the fact that basic opens are subsets, the structure underlying a pointfree topology is, from an algebraic point of view, just a semilattice $\mathcal{A} \equiv \langle A, \cdot, 1_{\mathcal{A}} \rangle$, where $A$ corresponds to the base, $\cdot$ to intersection and $1_{\mathcal{A}}$ to the whole $X$. Every open subset is obtained as union of elements of the base, but union does not make sense in absence of points; thus all we are left is a subset of $A$, that is a *formal open*. As we want to recover the properties of union, we introduce the relation of cover $a \lhd_{\mathcal{A}} U$ between an element $a$ and a subset $U$ of $A$; intuitively, if $a$ corresponds to a basic neighbourhood $O$ and $U$ to a subset of basic neighbourhoods $\{O_i : i \in I\}$, then $a \lhd_{\mathcal{A}} U$ corresponds to $O \subseteq \cup\{O_i : i \in I\}$. Then the properties we require on $\lhd_{\mathcal{A}}$ are:

$$\text{reflexivity: } \frac{a \,\epsilon\, U}{a \lhd_{\mathcal{A}} U} \qquad \text{transitivity: } \frac{a \lhd_{\mathcal{A}} U \qquad (\forall b \,\epsilon\, U) b \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} V}$$

$$\cdot\text{-left: } \frac{a \lhd_{\mathcal{A}} U}{a \cdot b \lhd_{\mathcal{A}} U} \quad \frac{a \lhd_{\mathcal{A}} U}{b \cdot a \lhd_{\mathcal{A}} U} \qquad \cdot\text{-right: } \frac{a \lhd_{\mathcal{A}} U \qquad a \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} \{b \cdot c : b \,\epsilon\, U, c \,\epsilon\, V\}}$$

Obviously two subsets $\{O_i : i \in I\}$ and $\{O_j : j \in J\}$ of basic neighbourhoods of a base give the same open subset if $\cup\{O_i : i \in I\} = \cup\{O_j : j \in J\}$ or equivalently if $O_i \subseteq \cup\{O_j : j \in J\}$ for all $i \in I$ and $O_j \subseteq \cup\{O_i : i \in I\}$ for all $j \in J$. Thus for all subsets $U$ and $V$ of $A$ we put

$$U =_{\mathcal{A}} V \equiv (\forall b \,\epsilon\, U)(b \lhd_{\mathcal{A}} V) \,\&\, (\forall c \,\epsilon\, V)(c \lhd_{\mathcal{A}} U)$$

that is we consider $U$ and $V$ to be equal formal opens if they cover each other; $=_{\mathcal{A}}$ is obviously an equivalence relation[9].

In order to recover points in a constructive approach, it is convenient to introduce a predicate $Pos_{\mathcal{A}}(a)$ on the elements of $A$ whose intended meaning is that (the basic neighbourhood corresponding to) $a$ is inhabited. The predicate $Pos_{\mathcal{A}}$ is required to satisfy:

$$\text{Monotonicity: } \frac{Pos_{\mathcal{A}}(a) \qquad a \lhd_{\mathcal{A}} U}{(\exists b \,\epsilon\, U) Pos_{\mathcal{A}}(b)} \qquad \text{Positivity: } \frac{Pos(a) \to a \lhd_{\mathcal{A}} U}{a \lhd_{\mathcal{A}} U}$$

A structure $\mathcal{A} \equiv \langle A, \cdot, 1, \lhd, Pos \rangle$ satisfying the above requirements is called a *formal*, or *pointfree*, *topology*.

We have seen in section 2. that for any Scott domain $\mathcal{D}$ the subsets $O_U = \{x \in D : U \leq x\}$, for $U \in \mathcal{P}_{Fin}(K(D))$, form a base; now it is easy to prove that, whenever $O_U$ is inhabited, $O_U \subseteq \cup_{i \in I} O_{U_i}$ if and only if $(\exists i \in I) O_U \subseteq O_{U_i}$. This is the property of Scott topologies which is taken as a definition in the pointfree approach: a pointfree topology $\mathcal{A}$ is called *Scott* if its cover $\lhd_{\mathcal{A}}$ satisfies

$$a \lhd_{\mathcal{A}} U \text{ iff } Pos_{\mathcal{A}}(a) \to (\exists b \,\epsilon\, U)(a \lhd_{\mathcal{A}} \{b\}).$$

[9]It is easy to show that formal opens of $\mathcal{A}$ can be given the structure of a locale (cf. [Sambin 87]).

Thus as for Scott pointfree topologies, the content of a cover $\lhd_{\mathcal{A}}$ is completely determined by its *trace*, namely the binary relation $\lhd^t$ between elements of $A$ defined by putting

$$a \lhd^t b \equiv a \lhd_{\mathcal{A}} \{b\}$$

(for a full proof see [Sambin 88]).

The definition of information base is obtained by characterising Scott topologies, or rather their traces, in the simplest way, that is by giving up to the semilattice structure in favour of one more axiom on $\lhd$, namely $a \lhd \Delta$, and by adding the assumption of properness, i.e. $Pos(\Delta)$.

Now it remains to explain when a subset of $A$ is "suitable" to be called a formal point. Both formal opens and formal points are subsets of $A$, but their characterization is different, since their intended meanings are opposite: while an open $U$ corresponds to a union, a point $\alpha$ corresponds, quite loosely speaking, to the point which is (in) the intersection of all the (basic neighbourhoods corresponding to) elements in $\alpha$. Moreover the formal definition below forces two formal points, intuitively corresponding to the same "concrete" point, to be equal. Thus we say that a subset $\alpha$ of $A$ is a *formal point* if:

i. 1. $1_{\mathcal{A}} \epsilon \alpha$ 2. $\dfrac{a \epsilon \alpha \qquad b \epsilon \alpha}{a \cdot b \epsilon \alpha}$ 3. $\dfrac{a \lhd_{\mathcal{A}} U \qquad a \epsilon \alpha}{(\exists b \epsilon U) b \epsilon \alpha}$

ii. $\dfrac{a \epsilon \alpha}{Pos_{\mathcal{A}}(a)}$

While i.1, i.2 and ii. are obvious, the intuitive meaning of i.3, which is formally just a condition binding points with $\lhd_{\mathcal{A}}$, is that if a point is in a basic neighbourhood $a$ contained in an open subset $U$, then it is also contained in one of the basic neighbourhoods of $U$, i.e. a point can not be "split" by basic neighbourhoods. We call $Pt(\mathcal{A})$ the collection of points of $\mathcal{A}$. We can provide $Pt(\mathcal{A})$ with a topology $\Omega Pt(\mathcal{A})$ by taking

$$\phi(a) \equiv \{\alpha \in Pt(\mathcal{A}) : a \epsilon \alpha\}$$

for any $a \in \mathcal{A}$, as basic neighbourhoods.

Even if in the Scott case the intuition is a little different, it is easily seen that points become exactly what we have called concepts in section 2. It can be proved that, when $\mathcal{A}$ is Scott, then $\Omega Pt(\mathcal{A})$ coincides with the Scott topology (see section 2.) on $\langle Pt(\mathcal{A}), \subseteq\rangle$; hence, by appealing also to theorem 2.9, the Scott topology on a domain is intrinsically characterised by the property of having a base $\{O_i : i \in I\}$, of *super-compact* opens, i.e. such that, for any $J \subseteq I$, $O_i \subseteq \cup\{O_j : j \in J\}$ iff $(\exists j \in J) O_i \subseteq O_j$, which is precisely the property we chose in order to characterize pointfree Scott topology (a detailed proof is in preparation).

It is natural to define a morphism between pointfree topologies $\mathcal{A}$ and $\mathcal{B}$, intuitively corresponding to the topological spaces $\langle X, \Omega X\rangle$ and $\langle Y, \Omega Y\rangle$ respectively, as the formal counterpart of a continuous function, namely a map of the points of $X$ into points of $Y$ such that the inverse image of a basic neighbourhood of $\Omega Y$ is an open in $\Omega X$. Thus, due to the absence of points, a morphism from the pointfree topology $\mathcal{B}$ into $\mathcal{A}$ is a map $f$ from $\mathcal{B}$ into subsets of $\mathcal{A}$ respecting the given structure[10]

$$f(1_\mathcal{B}) =_\mathcal{A} \{1_\mathcal{A}\} \qquad\qquad f(b \cdot_\mathcal{B} d) =_\mathcal{A} \{a \cdot_\mathcal{A} c : a \,\epsilon\, f(b), c \,\epsilon\, f(d)\}$$

$$\frac{b \lhd_\mathcal{B} U}{(\forall x \,\epsilon\, f(b))(x \lhd_\mathcal{A} \cup\{f(u) : u \,\epsilon\, U\})} \qquad\qquad \frac{(\exists x \,\epsilon\, f(b))Pos_\mathcal{A}(x)}{Pos_\mathcal{B}(b)}$$

With any morphism between pointfree topologies, $f : \mathcal{B} \to \mathcal{A}$, a continuous function $f^* : \langle Pt(\mathcal{A}), \Omega Pt(\mathcal{A})\rangle \to \langle Pt(\mathcal{B}), \Omega Pt(\mathcal{B})\rangle$ between topological spaces is associated by putting

$$f^*(\alpha) \equiv \{b \in B : (\exists a \,\epsilon\, \alpha) a \,\epsilon\, f(b)\}$$

for any $\alpha \in Pt(\mathcal{A})$.

We can modify the definition of morphism between pointfree topologies so that it has the same direction of the associated continuous function[11]. The idea is to think of $f$ from $B$ into subsets of $A$ as a binary relation from $B$ into $A$, and then to take its inverse, that is consider $aFb$ in place of $a \,\epsilon\, f(b)$. Such relation $F$ has a clear and independent topological meaning: in fact it can be shown that, if $\mathcal{A}$ "has enough points", $aFb$ holds iff, for any point $\alpha \in Pt(\mathcal{A})$, the point $f^*(\alpha)$ is in $b$ (i.e. $b \,\epsilon\, f^*(\alpha)$) whenever $\alpha$ is in $a$ (i.e. $a \,\epsilon\, \alpha$). Working out such remarks, we are led to define a *continuous relation* from $\mathcal{A}$ into $\mathcal{B}$ as a binary relation $F$ satisfying the following conditions for any $a \in A$ and $b, d \in B$:

1. $aF1_\mathcal{B}$

2. $\dfrac{aFb \qquad aFd}{aFb \cdot d}$

3. $\dfrac{a \lhd_\mathcal{A} W \qquad (\forall w \,\epsilon\, W)wFb}{aFb}$

4. $\dfrac{aFb \qquad b \lhd_\mathcal{B} V}{a \lhd_\mathcal{A} \{w \in A : (\exists v \,\epsilon\, V)wFv\}}$

5. $\dfrac{Pos_\mathcal{A}(a) \qquad aFb}{Pos_\mathcal{B}(b)}$

It is not difficult to check that putting $aF_f b \equiv a \,\epsilon\, f(b)$ and $f_F(b) \equiv \{a \in A : aFb\}$ defines a bijective correspondence between morphisms and continuous relations (for a proof see [Virgili 90]). Now the definition of translation is obtainable as a characterization of continuous relations between Scott pointfree topologies. In fact, 1., 2. and 5. above are respectively identical with i.1, i.2 and i.4 of the definition of translation, while ii. is a special case of 3. above when $W = \{c\}$. Also iii. comes from 3.; in fact, since trivially $(\forall w \,\epsilon\, \emptyset)wFb$, for $W = \emptyset$, 3. gives

$$\frac{a \lhd_\mathcal{A} \emptyset}{aFb}$$

[10] It is not difficult to see that, by associating a morphism $f : \mathcal{B} \to \mathcal{A}$ with $\bar{f}(U) \equiv \cup\{f(a) : a \,\epsilon\, U\}$, one obtains a biunivocal correspondence between morphisms of pointfree topologies and locale morphisms from opens of $\mathcal{B}$ into opens of $\mathcal{A}$.

[11] That is, so that $Pt$ becomes a co-variant rather than a contravariant functor.

or equivalently, since $a \lhd_{\mathcal{A}} \emptyset$ iff $\neg Pos(a)$, $\neg Pos(a) \rightarrow aFb$; we already know that iii. is a uniform way to express this fact. Finally, i.3 follows from 3. and 4. above: in fact, if $aFb$ and $b \lhd_{\mathcal{B}} d$ then, by 4., $a \lhd_{\mathcal{A}} \{w \in A : wFd\}$ which, in the Scott case, implies that there is $w'$ such that $a \lhd_{\mathcal{A}} w'$ and $w'Fd$ and so $aFd$ by 3.

**Appendix 2: Categorical equivalence between InfBas and other presentations of domains.**

In section 2 we have shown how neighbourhood systems and information systems can be seen as special cases of information bases. Now we show that actually the category of neighbourhood systems with approximable mappings (cf. [Scott 81a]), here called **NeighSys**, is equivalent to **InfBas**. To this aim we call $I(\mathcal{N})$ the information base associated, as in section 2, with the neighbourhood system $\mathcal{N} = \langle \Delta, (X_i)_{i \in I} \rangle$ and now we see how to extend $I$ to a functor $I$ : **NeighSys** $\rightarrow$ **InfBas**.

So let $R : \mathcal{N} \rightarrow \mathcal{M}$ be an approximable mapping between two neighbourhood systems; then for any $\{i_1, i_2, \ldots, i_n\} \in I(\mathcal{N})$ and $\{j_1, j_2, \ldots, j_m\} \in I(\mathcal{M})$, put

$$\{i_1, \ldots, i_n\} I(R) \{j_1, \ldots, j_m\} \equiv$$
$$Pos(\{i_1, \ldots, i_n\}) \rightarrow Pos(\{j_1, \ldots, j_m\}) \,\&\, (X_{i_1} \cap \cdots \cap X_{i_n} R X_{j_1} \cap \cdots \cap X_{j_m})$$

Then it is immediate to see that $I(R)$ satisfies the conditions i.1-3 and ii. of the definition of translation since $R$ is an approximable mapping. A quick and instructive way to prove that $I(R)$ is a translation, i.e. it satisfies also 1.4 and iii., is based on the following remark. For any translation $F$,

$$aFb \text{ iff } Pos(a) \rightarrow Pos(b) \,\&\, aFb$$

and so, by improving the similar argument used in section 3, any relation $F_0$ between positive elements satisfying i.1-3 and ii. can be uniquely extended to a translation $F$, which coincides with $F_0$ on positive elements, by putting

$$aFb \equiv Pos(a) \rightarrow Pos(b) \,\&\, aF_0b.$$

So the relation $F$ defined by

$$\{i_1, \ldots, i_n\} F \{j_1, \ldots, j_m\} \equiv$$
$$Pos(\{i_1, \ldots, i_n\}) \rightarrow Pos(\{j_1, \ldots, j_m\}) \,\&\, \{i_1, \ldots, i_n\} I(R) \{j_1, \ldots, j_m\}$$

is a translation between $I(\mathcal{N})$ and $I(\mathcal{M})$, but $F = I(R)$ by intuitionistic logic.

It is easy now to check that indeed $I$ : **NeighSys** $\rightarrow$ **InfBas** is a functor. Moreover, according to its definition on morphisms, it is immediate to see that $I$ is *faithful*, that is $I(R) = I(S)$ implies $R = S$ for any approximable mapping $R$, $S$, since $R$ and $S$ essentially coincide with $I(R)$ and $I(S)$ respectively on all positive elements.

Similarly, $I$ is *full* since any translation $F : I(\mathcal{N}) \to I(\mathcal{M})$ can be seen as the image under $I$ of the approximable mapping which binds $X_i \in \mathcal{N}$ with $X_j \in \mathcal{M}$ iff $\{i\}F\{j\}$.

Finally, given an information base $\mathcal{S}$, we put $\downarrow a \equiv \{c \in S : c \lhd a\}$, and consider $N(\mathcal{S}) = \langle S, (\downarrow a)_{a\epsilon Pos}\rangle$; it is routine to check that $N(\mathcal{S})$ is a neighbourhood system such that its image under $I$ is isomorphic to $\mathcal{S}$, and hence $I$ is *dense*.

Thus $I$ is a categorical equivalence betwwen **NeighSys** and **InfBas**.

In a similar way one can prove the equivalence between the category **InfSys** of information systems and approximable relations and **InfBas**. In fact, in section 2 we gave the definition of the suitable functor on the objects and it is an easy exercise to carry on a complete proof of the equivalence.

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