Extensionality versus Constructivity*

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Abstract

We will analyze some extensions of Martin-Löf's constructive type theory by means of extensional set constructors and we will show that often the most natural requirements over them lead to classical logic or even to inconsistency.

1 Introduction

The main purpose of the paper is to show that by adding extensional set-constructors to a constructive set theory like Martin-Löf's constructive type theory (see [ML84]) can lead to classical logic, or even to inconsistency, if a great attention is not used.

The need for the addition of extensional set constructors to Martin-Löf's type theory appears immediately as soon as one tries to develop within such a framework a piece of actual mathematics, as we did with constructive topology (see the introductory sections in [CSSV]). For instance, one would like to be able to work with power-sets, in order to quantify on the elements of the collection of the subsets of a set, or with quotient-sets, in order to be able to construct in a straightforward way some well known sets, or, at least, with the collection of the finite subsets of a set, in order, for instance, to be able to express easily statements like Tychonoff's theorem on the compactness of the product of compact spaces.

Of course, the main point is to be able to deal with such constructions and still work in a set-theory with such a clear computational semantics as Martin-Löf's type theory has. The bad news is that there is no possibility to be sure that this is possible at all, as we will see in this paper.

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2 The power-set constructor

In this section we will consider the problems that arise when the power-set constructor is added to Martin-Löf's type theory. In particular we will show that by adding to Martin-Löf's type theory a power-set constructor which satisfies some very natural conditions we obtain classical logic, that is, for any proposition A, $A \vee \neg A$ holds. Here we will show only a sketch of the proof since a fully detailed proof of this result can be found in [MV99].

In order to be able to deal with the collection of all the subsets of a given set, we have first to state what a subset of a set is. One possibility, which up to now seems to work well, is to identify a subset U of a set S with a propositional function over S, that is, to put

$$U \subseteq S \equiv U(x) \ \mathsf{prop}\ [x : S]$$

or, equivalently,

$$U : (x : S)\ \mathsf{prop}$$

This is a good approach since most of subset theory can be recovered in this way and it has been sufficient till now for developing constructive topology [SV98].

If we follow this approach, then the most natural introduction rule for the power-set is

$$\frac{U(x)\ \mathsf{prop}\ [x : S]}{\{x \in S |\ U(x)\} \in \mathcal{P}(S)}$$

where $\{x \in S|\ U(x)\}$ is just a nice notation for a subset of S.

The real problems come with the equality rule. Indeed, it is meaningless to put

$$\frac{U(x) = V(x)\ [x : S]}{\{x \in S|\ U(x)\} = \{x \in S|\ V(x)\} \in \mathcal{P}(S)}$$

since $U(x) = V(x)$ means that $U(x)$ and $V(x)$ have the same *proofs* while what we want is to state that the subsets U and V are equal when they have the same *elements*. Thus, we have first to define the membership relation between an element and a subset of S by putting

$$a \varepsilon U \equiv U(a)$$

and then we will be able to define the extensional equality between two subsets U and V of S by putting

$$U =^S V \equiv (\forall x \in S)\ (x \varepsilon U) \leftrightarrow (x \varepsilon V)$$

So the equality rule could be something like

$$\frac{U(x) \leftrightarrow V(x)\ \mathsf{true}\ [x : S]}{\{x \in S|\ U(x)\} = \{x \in S|\ V(x)\} \in \mathcal{P}(S)}$$

But this rule does not fit with the general idea that the judgement of equality should be definitional. Thus a reasonable choice is to require only that, whatever is the equality rule that one can choose, the weaker condition

$$\text{If } U =^S V \ \mathsf{true} \text{ then } \{x \in S|\ U(x)\} =_{\mathcal{P}(S)} \{x \in S|\ V(x)\} \ \mathsf{true}$$

holds, where $=_{\mathcal{P}(S)}$ is the intensional identity proposition over the elements of $\mathcal{P}(S)$. Since we are going to show that this condition leads to classical logic we will propose no formal rule of equality, otherwise an Heyting semantics for classical logic would be possible.

Before going on, it is worth noting that the previous equality condition, even if reasonable, can already be pretty dangerous. In fact, if we try to follow the same approach within a calculus which allows to quantify over the collection of all the propositions, then we would obtain *proof-unicity.* And note that to allow to quantify over the collection of all the propositions is not really different than to allow to quantify over the collection $\mathcal{P}(\top)$ of the subsets of the one-element set $\top$.

To prove this result first observe that the following lemma holds.

Lemma 2.1 *The intensional equality proposition is equivalent to the Leibniz equality proposition.*

Proof. Let us first recall the formation, introduction and elimination rules for the intensional equality proposition [NPS90].

$$\frac{A \ \mathsf{set} \quad a \in A \quad b \in A}{a =_A b \ \mathsf{prop}} \qquad\qquad \frac{A \ \mathsf{set} \quad a \in A}{\mathsf{r}(a) \in a =_A a}$$

$$\frac{A \ \mathsf{set} \quad c \in a =_A b \quad \begin{array}{c}[x, y : A, z : x =_A y]_1 \\ \vdots \\ Q(x, y, z) \ \mathsf{prop}\end{array} \quad \begin{array}{c}[x : A]_1 \\ \vdots \\ d(x) \in Q(x, x, \mathsf{r}(x))\end{array}}{\mathsf{K}(c, d) \in Q(a, b, c)}\ 1$$

It can be useful to recall also that the Leibniz equality is defined by using a quantification over the collection of the propositional functions.

$$\mathsf{Eq}_L(A, a, b) \equiv (\forall P \in A \to \mathsf{prop})\ P(a) \to P(b)$$

that is, *those things which cannot be distinguished by means of a proposition are equal.*

Now, we want to show that, for any set A, and elements a and b in A, $a =_A b$ if and only if $\mathsf{Eq}_L(A, a, b)$. Thus, let us suppose that $c \in a =_A b$, $P : A \to \mathsf{prop}$, $w : P(a)$, and put $Q(x, y, z) \equiv P(x) \to P(y)$.

Then, supposing x is any element in A, $Q(x, x, \mathsf{r}(x)) \equiv P(x) \to P(x)$ and hence, $\lambda y.\ y \in Q(x, x, \mathsf{r}(x))$. So, by using the elimination rule for intensional equality, we obtain $\mathsf{K}(c, \lambda y.\ y) \in P(a) \to P(b)$ and hence, by discharging the assumptions $P : A \to \mathsf{prop}$ and $w : P(a)$,

$$\lambda P.\ \lambda w.\ \mathsf{K}(c, \lambda y.\ y)(w) \in \mathsf{Eq}_L(A, a, b)$$

On the other hand, suppose $c \in \mathsf{Eq}_L(A, a, b)$, that is

$$c \in (\forall P \in A \to \mathsf{prop})\ P(a) \to P(b),$$

and put $P \equiv (x : A)\ a =_A x$. Then $c(P) \in (a =_A a) \to (a =_A b)$. But we know that $\mathsf{r}(a) \in a =_A a$ and hence

$$c(P)(\mathsf{r}(a)) \in a =_A b$$

Now, we can prove the following theorem.

Theorem 2.2 *Extensionality yields proof-unicity, that is, if*

$$\textit{(extensionality)} \qquad (\forall P, Q \in \mathsf{prop})\ (P \leftrightarrow Q) \to P =_{\mathsf{prop}} Q$$

then

$$\textit{(proof-unicity)} \qquad (\forall P \in \mathsf{prop})(\forall x, y \in P)\ x =_P y$$

Proof. Let P be any proposition and put $Q \equiv \top$, that is Q is the one-element set inductively generated with the following introduction and elimination rules.

$$* \in \top \qquad\qquad \frac{c \in \top \qquad C(x)\ \mathsf{prop}\ [x : \top] \qquad d \in C(*)}{\mathsf{R}_1(c, d) \in C(c)}$$

Suppose now that x and y are elements in P. Then P is not empty and hence $P \leftrightarrow \top$ holds (for instance one can consider the proof-elements $\lambda z.\ * \in P \to \top$ and $\lambda w.\ x \in \top \to P$). Then extensionality yields $P =_{\mathsf{prop}} \top$. Consider now the following property on propositions:

$$\mathsf{OneEl}(A) \equiv (\forall x, y \in A)\ x =_A y$$

which states that A has at most one proof-element. It is easy to show that $\mathsf{OneEl}(\top)$ holds. In fact, let us put $C(x) \equiv x =_\top *$ and use the $\top$-elimination rule in order to obtain, for any x and y in $\top$, that $\mathsf{R}_1(x, \mathsf{r}(*)) \in x =_\top *$ and $\mathsf{R}_1(y, \mathsf{r}(*)) \in y =_\top *$. Hence it is sufficient to use the fact that for the intensional equality proposition symmetry and transitivity hold to obtain that $x =_\top y$ holds.

But then $P =_{\mathsf{prop}} \top$ yields, by the previous lemma, that P and $\top$ satisfy the same propositions and hence also $\mathsf{OneEl}(P)$ holds, that is, we obtained proof-unicity.

Now we have an immediate corollary.

Corollary 2.3 *Extensionality is not consistent with inductive types.*

Proof. The proof is straightforward since it is possible to prove that there are inductive types with more then one element.

After the previous corollary one can wonder weather the equality condition that we proposed is consistent at all. The answer is positive if we assume to work within Martin-Löf's type theory instead that within an impredicative type system, since in this case a model can be found within Zermelo-Frankel set theory with the axiom of choice (see [MV99]).

Let us go back to our main topic. Another reasonable condition that one could require on the power-set $\mathcal{P}(S)$ is that for any element $c \in \mathcal{P}(S)$ and $a \in S$ there exists a proposition $a\varepsilon c$. Indeed, $\mathcal{P}(S)$ is supposed to be the collection of the propositional functions over S, even if it was equipped with an extensional equality, and hence it should be possible to obtain back for each of its elements a suitable propositional function. But then it is natural to require that the following condition holds:

$$\text{if } U(x)\ \mathsf{prop}\ [x : S] \text{ and } a \in S \text{ then } a\varepsilon\{x : S|\ U(x)\} \leftrightarrow U(a)\ \mathsf{true}.$$

It is interesting to note that also this kind of extensionality is dangerous if we are working within a framework which allows to quantify over the collection of all the propositions. In fact in this case we can argue like in [Coq90] (or also [Jac89]) where it is proved that if there exists a type B and two elements $\mathsf{code} \in \mathsf{prop} \to B$ and $\mathsf{decode} \in B \to \mathsf{prop}$ such that

$$(\forall A \in \mathsf{prop})\ A \leftrightarrow \mathsf{decode}(\mathsf{code}(A))$$

then we obtain an inconsistent theory since the inconsistent lambda-calculus λ_U (see [Bar92] for its definition) can be embedded into it.

Now, note that the possibility to build the power-set of the one-element set, together with the required logical equivalence condition for the membership proposition, allow us to define a type B and the necessary elements by putting

$$\begin{array}{l} B \equiv \mathcal{P}(\top) \\ \mathsf{code}(U) \equiv \{w \in \top|\ U\} \\ \mathsf{decode}(c) \equiv *\varepsilon c \end{array}$$

In fact, the required equivalence condition yields that, for any proposition A,

$$*\varepsilon\{w \in \top|\ A\} \leftrightarrow A$$

that is

$$\mathsf{decode}(\mathsf{code}(A)) \leftrightarrow A$$

Thus, it is clear that we have to avoid to use this kind of power-set constructor together with impredicative quantification. One can now wonder what is going to happen if we add this power-set constructor to an intuitionistic and predicative set theory like Martin-Löf's type theory. In this case the situation is not such a big disaster since we can still interpret the resulting theory into Zermelo-Frankel set theory with the axiom of choice, and the latter theory is supposed to be consistent. But it is possible to show that the resulting theory is not so good. In fact if we have the possibility to quantify over the elements of the collection $\mathcal{P}(\top)$ then we can formalize within type theory the proof that the axiom of choice, which is provable within Martin-Löf's type theory, implies classical logic. What we will formalize here within Martin-Löf's type theory is one of the many re-writing of the original result by Diaconescu in [Dia75] that we found in [Bel88] (in order to see a fully detailed proof of this formalization inside Martin-Löf's intensional type theory with a power-set constructor see [MV99]).

First note that for any $U, V \in \mathcal{P}(\top)$, if $\mathsf{decode}(U) \vee \mathsf{decode}(V)$ holds then there exists an element $x \in \mathsf{Boole}$ such that

$$(x =_{\mathsf{Boole}} \mathsf{true} \rightarrow \mathsf{decode}(U)) \wedge (x =_{\mathsf{Boole}} \mathsf{false} \rightarrow \mathsf{decode}(V))$$

because $\neg(\mathsf{true} =_{\mathsf{Boole}} \mathsf{false})$ is provable in Martin-Löf's type theory when we add to it the power-set of $\top$.

Then, by the axiom of choice, there exists a function f such that, for any $U, V \in \mathcal{P}(\top)$ such that $\mathsf{decode}(U) \vee \mathsf{decode}(V)$ holds,

$$(f(\langle U, V \rangle) =_{\mathsf{Boole}} \mathsf{true} \rightarrow \mathsf{decode}(U)) \wedge (f(\langle U, V \rangle) =_{\mathsf{Boole}} \mathsf{false} \rightarrow \mathsf{decode}(V))$$

Now, let A be any proposition. Then $\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle$ and $\langle \mathsf{code}(\top), \mathsf{code}(A) \rangle$ are two couples such that

$$\mathsf{decode}(\mathsf{code}(A)) \ \vee \ \mathsf{decode}(\mathsf{code}(\top))$$

and

$$\mathsf{decode}(\mathsf{code}(\top)) \ \vee \ \mathsf{decode}(\mathsf{code}(A))$$

hold because $\mathsf{decode}(\mathsf{code}(\top)) \leftrightarrow \top$. Then, with a bit of intuitionistic logic, we can obtain both $f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{true} \rightarrow \mathsf{decode}(\mathsf{code}(A))$ and $f(\langle \mathsf{code}(\top), \mathsf{code}(A) \rangle) =_{\mathsf{Boole}} \mathsf{false} \rightarrow \mathsf{decode}(\mathsf{code}(A))$ and hence

$$f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{true} \rightarrow A$$

and

$$f(\langle \mathsf{code}(\top), \mathsf{code}(A) \rangle) =_{\mathsf{Boole}} \mathsf{false} \rightarrow A$$

because $\mathsf{decode}(\mathsf{code}(A)) \leftrightarrow A$.

But we know that the set Boole is decidable (see [NPS90]) and hence

$$f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{true} \vee f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{false}$$

holds. Thus we can argue by $\vee$-elimination as follows. If

$$f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{true}$$

then $f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{true} \rightarrow A$ yields A and hence

$$A \vee f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{false}$$

holds. On the other hand $f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{false}$ yields directly

$$A \vee f(\langle \mathsf{code}(A), \mathsf{code}(\top) \rangle) =_{\mathsf{Boole}} \mathsf{false}$$

Similarly we can obtain

$$A \vee f(\langle \mathsf{code}(\top), \mathsf{code}(A)\rangle) =_{\mathsf{Boole}} \mathsf{true}$$

Hence, by distributivity, we have

$$A \ \vee \ (f(\langle \mathsf{code}(A), \mathsf{code}(\top)\rangle) =_{\mathsf{Boole}} \mathsf{false} \ \wedge \ f(\langle \mathsf{code}(\top), \mathsf{code}(A)\rangle) =_{\mathsf{Boole}} \mathsf{true})$$

and we can argue by $\vee$-elimination to prove that $A \ \vee \ \neg A$ holds.

In fact, assuming that A holds yields directly that $A \ \vee \ \neg A$ holds.

On the other hand, let us suppose that $f(\langle \mathsf{code}(A), \mathsf{code}(\top)\rangle) =_{\mathsf{Boole}} \mathsf{false}$ and $f(\langle \mathsf{code}(\top), \mathsf{code}(A)\rangle) =_{\mathsf{Boole}} \mathsf{true}$ and assume that A is true. Then $A \leftrightarrow \top$ holds, and hence $\mathsf{code}(A) =_{\mathcal{P}(\top)} \mathsf{code}(\top)$ by equality introduction. Thus

$$\langle \mathsf{code}(A), \mathsf{code}(\top)\rangle =_{\mathcal{P}(\top)\times\mathcal{P}(\top)} \langle \mathsf{code}(\top), \mathsf{code}(A)\rangle$$

and hence

$$\mathsf{false} =_{\mathsf{Boole}} f(\langle \mathsf{code}(A), \mathsf{code}(\top)\rangle) =_{\mathsf{Boole}} f(\langle \mathsf{code}(\top), \mathsf{code}(A)\rangle) =_{\mathsf{Boole}} \mathsf{true}$$

So we are arrived to a contradiction starting from the assumption that A holds, and thus $\neg A$ holds which gives also in this case that $A \ \vee \ \neg A$ holds.

Thus we proved that classical logic is yielded by our two conditions. Of course we did not furnish a proof element for $A \vee \neg A$ since we only required that for any $a \in S$ and $U \subseteq S$, $a \varepsilon \{x \in S | \ U(x)\} \leftrightarrow U(a)$ holds but we could not furnish a proof element for this judgement, that is, we destroyed the correspondence between the judgements A true and the fact that there exists a proof-term a such that $a \in A$ (see [Val98]).

The structure of the proof above is surely well known but we wanted to write it down again in order to make it explicit where the various conditions on the power-set constructor that we required play a main role. Let us analyze what we did. First of all we had the possibility to use a quantification over the elements of $\mathcal{P}(\top)$ only because we assumed that it is a set. Moreover, in order to formalize the condition that $\mathsf{decode}(U) \ \vee \ \mathsf{decode}(V)$ holds for any U and V in $\mathcal{P}(\top)$ we could write

$$\mathsf{code}(\mathsf{decode}(\mathsf{code}(U)) \ \vee \ \mathsf{decode}(\mathsf{code}(V))) =_{\mathcal{P}(\top)} \mathsf{code}(\top)$$

since the extensionality condition yields that, for any proposition A, A holds if and only if $\mathsf{code}(A) =_{\mathcal{P}(\top)} \mathsf{code}(\top)$ holds. In this way the choice function f that we used in the proof above can no more know why the proposition $\mathsf{decode}(\mathsf{code}(U)) \ \vee \ \mathsf{decode}(\mathsf{code}(V))$ holds but only that it has to behave in the same way on any pair $\langle U, V\rangle$ such that $\mathsf{decode}(\mathsf{code}(U)) \vee \mathsf{decode}(\mathsf{code}(V))$ holds.

3 The collection of the finite subsets

It is interesting to observe that all of the proofs in the previous section can be redone also if we assume a set-constructor which seems to be much more safe than the power-set constructor, namely the set $\mathcal{P}_\omega(S)$ of the finite subsets of the set S.

In order to define $\mathcal{P}_\omega(S)$ we have first to state what a finite subset is. The first idea is probably to identify a finite subset of S with a list of elements of S, but if extensionality is required we have to force two lists to be equal if they are formed with the same elements, that is, we have to force a quotient over the set of the lists on S and this quotient is going to be based on an equivalence relation which will be defined in terms of the equality over S which can be non-decidable. And we will see that also quotient sets over non decidable equivalence relations are not safe set-constructions.

The good news is that we have a possible way out. In fact, we can find a suitable condition which allows to state which ones are finite among the standard subsets. To this aim, let $\mathsf{N}(k)$ prop $[k : \mathsf{N}]$ be the family of sets defined over the natural numbers by using the following inductive definition

$$\begin{cases} \mathsf{N}(0) & = & \emptyset \\ \mathsf{N}(k+1) & = & \mathsf{S}(\mathsf{N}(k)) \end{cases}$$

where the type $\mathsf{S}(A)$ is the set-constructor which, given any set A, allows to obtain a new set with one element more than A. So, for any $k \in \mathsf{N}$, the set $\mathsf{N}(k)$ contains exactly k elements.

Now, given any subset U of S, that is any propositional function over S, we can put

$$\mathsf{Fin}(U) \equiv (\exists k \in \mathsf{N})(\exists f \in \mathsf{N}(k) \to S)\ U \subseteq \mathsf{Im}(f)$$

where $\mathsf{Im}(f) \equiv \{x \in S |\ (\exists n \in \mathsf{N}(k))\ x =_S f(n)\}$ is the image of the function f. The previous definition states that a subset U is finite whenever it is a subsets of a subset which is surely finite, namely the image of a finite set.

This is a good definition of "finite subset" because it follows our intuition about what a finite subset should be and meanwhile it does not force us to know how many elements the subset should contains but it only require that there is a finite upper bound to the number of elements that it can contain. This fact allows to prove some expected result on finite subsets which were not provable with the approach in [SV98]. For instance, here not only the union of two finite subsets is finite but also the intersection of a finite subset with any other subset is finite and this fact would have been difficult to prove if the notion of finite subset would have required to know the number of its elements.

A consequence of the previous definition is that we can state the introduction rule for the set $\mathcal{P}_{\mathsf{fin}}(S)$ by putting

$$\frac{U(x)\ \mathsf{prop}\ [x : S] \quad \mathsf{Fin}(U)\ \mathsf{true}}{\{x \in S |\ U(x)\} \in \mathcal{P}_{\mathsf{fin}}(S)}$$

But then we obtain that all of the subsets of the one-element set are finite and hence $\mathcal{P}_{\mathsf{fin}}(\top)$ and $\mathcal{P}(\top)$ are (almost) the same set and hence all of the proof in the previous section can be redone by using the set $\mathcal{P}_{\mathsf{fin}}(\top)$.

4 The quotient set constructor

We observed above that it would have been possible to define the set of the finite subsets of a given set S also considering the set of the list over S and then forcing a quotient over it in order to obtain the wished extensionality. But it is possible to prove that also a quotient set constructor is not safe from a constructive point of view.

Indeed, the quotient set-constructor can be defined by using the following formation and introduction rules

$$\frac{A \ \mathsf{set} \quad R(x,y) \ \mathsf{prop} \ [x,y:A] \quad \mathsf{EqRel}(R) \ \mathsf{true}}{A_R \ \mathsf{set}}$$
$$\frac{a \in A \quad R(x,y) \ \mathsf{prop} \ [x,y:A] \quad \mathsf{EqRel}(R) \ \mathsf{true}}{[a]_R \in A_R}$$

where $\mathsf{EqRel}(R)$ is any proposition which formalizes the standard conditions requiring that R is an equivalence relation.

Now, in order to obtain a quotient set, we should require a suitable equality rule, but one can show that this is not possible if he wants to avoid to obtain classical logic and still obtain a set which is a real quotient set. In fact, if we require that, whatever equality rule one can use, the following condition is satisfied, for any $a, b \in A$,

$$R(a,b) \ \mathsf{true} \text{ if and only if } [a]_R =_{A_R} [b]_R \ \mathsf{true}$$

then we can construct a proof of $A \vee \neg A$, for any small set A, by arguing as in section 2 but using, instead that $\mathcal{P}(\top)$, the quotient set V obtained by using the first universe U_0, which contains the codes of all the small sets, and the equivalence relation of equi-provability between small types (for a detailed proof see [Mai99]).

5 The extensional two-subset set

In the previous sections we proved that by adding extensional set-constructors to Martin-Löf's type theory one can obtain classical logic. The obvious question is: where is the problem? Here we will prove that even a very weak form of extensionality seems not to fit well with constructive type theory. Indeed, it is possible to obtain classical logic even considering like a set the collection of *two* subsets, if an extensional equality is required on the elements of this set. In fact, let us suppose to add to Martin-Löf's type theory the following formation rule

$$\frac{U(x) \ \mathsf{prop} \ [x:S] \qquad V(x) \ \mathsf{prop} \ [x:S]}{\{U,V\} \ \mathsf{set}}$$

and then suitable introduction and elimination rules in such a way that the following very natural conditions are satisfied

1. (pair axiom) If $W \subseteq S$ then

$$W \in \{U, V\} \text{ if and only if } (W =^S U) \ \vee \ (W =^S V) \ \mathsf{true}$$

2. (extensionality) If $W_1, W_2 \in \{U, V\}$ then

$$\text{if } W_1 =^S W_2 \ \mathsf{true} \text{ then } W_1 =_{\{U,V\}} W_2 \ \mathsf{true}$$

Then, we can formalize within this extended type theory a proof by J. Bell in [Bel88] which allows to prove $A \vee \neg A$ for any proposition A. In fact, let

$$\begin{aligned} V_0 &\equiv \{x \in \mathsf{Boole} | \ (x =_{\mathsf{Boole}} \mathsf{false}) \ \vee \ A\} \\ V_1 &\equiv \{x \in \mathsf{Boole} | \ (x =_{\mathsf{Boole}} \mathsf{true}) \ \vee \ A\} \end{aligned}$$

Then we can prove that

$$(\forall W \in \{V_0, V_1\})(\exists y \in \mathsf{Boole}) \ y \varepsilon W$$

In fact, by the pair axiom, $W \in \{V_0, V_1\}$ yields $(W =^S V_0) \ \vee \ (W =^S V_1)$ and hence we can argue by $\vee$-elimination. Now, supposing $W =^S V_0$, the fact that $\mathsf{false} \varepsilon V_0$ yields that $\mathsf{false} \varepsilon W$ and then $(\exists y \in \mathsf{Boole}) \ y \varepsilon W$ and, similarly, supposing $W =^S V_1$, the fact that $\mathsf{true} \varepsilon V_1$ yields that $\mathsf{true} \varepsilon W$ and then also in this case we obtain $(\exists y \in \mathsf{Boole}) \ y \varepsilon W$.

Then, by the axiom of choice, we obtain

$$(\exists f \in \{V_0, V_1\} \rightarrow \mathsf{Boole}) \ (\forall W \in \{V_0, V_1\}) \ f(W) \varepsilon W$$

and hence

$$(f(V_0) \varepsilon V_0) \ \wedge \ (f(V_1) \varepsilon V_1)$$

Note now that $f(V_0) \varepsilon V_0$ holds if and only if $(f(V_0) =_{\mathsf{Boole}} \mathsf{false}) \ \vee \ A$ and $f(V_1) \varepsilon V_1$ holds if and only if $(f(V_1) =_{\mathsf{Boole}} \mathsf{true}) \ \vee \ A$; hence we using a bit of intuitionistic logic we obtain that

$$(f(V_0) =_{\mathsf{Boole}} \mathsf{false} \ \wedge \ f(V_1) =_{\mathsf{Boole}} \mathsf{true}) \ \vee \ A$$

and thus we can argue by $\vee$-elimination. If A holds then we obtain directly that $A \ \vee \ \neg A$ holds. On the other hand, if we assume that A holds, then $V_0 =^S V_1$, and hence, by extensionality, we obtain that $V_0 =_{\{V_0,V_1\}} V_1$. Thus, by one of the property of the intensional equality proposition, we obtain that $f(V_0) =_{\mathsf{Boole}} f(V_1)$ which together with $(f(V_0) =_{\mathsf{Boole}} \mathsf{false}) \wedge (f(V_1) =_{\mathsf{Boole}} \mathsf{true})$ gives $\mathsf{false} =_{\mathsf{Boole}} \mathsf{true}$ that leads to a contradiction when we work within a theory with the universe of the small types. Thus we can conclude that the assumption that A holds lead to a contradiction, that is $\neg A$ holds and hence also in this case $A \ \vee \ \neg A$ holds by $\vee$-introduction.

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