# Krivine Intuitionistic Proof Of Classical Completeness (for countable languages)

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**Abstract**

In 1996, Krivine applied Friedman's *A-translation* in order to get a constructive version of Gödel Completeness result for first order classical logic. Such result is known to be intuitionistically underivable, but Krivine managed to constructively derive a weak form of it. In this paper, we want to analyze the ideas Krivine's remarkable result relies on, ideas which where somehow hidden by the heavy formal machinery used in the original proof. We show that the ideas in Krivine's proof can be used to intuitionistically derive crucial mathematical results, which were supposed to be purely classical up to now: the Ultrafilter Theorem in Boolean Algebra Theory, and the Maximal Ideal Theorem in Ring Theory.

## 1 Intuitionistic Model Theory

We have first to explain what we mean by "intuitionistic proof of first order classical completeness". Thus, in this section we outline an intuitionistic version of first order model theory, following the ideas in [1]. At the end of this section we will be able to state Krivine's result, namely, the first goal of this paper.

Let $\mathcal{L}$ be a first order countable language over some subset of the set

$$\{\forall, \rightarrow, \bot, \neg, \&, \vee, \exists\}$$

of the first order connectives. We feel free to use similar denotations for the corresponding metalinguistic connectives. The context will always clarify if we are speaking about a connective of $\mathcal{L}$ or about a meta-connective.

First order logical rules are the intuitionistic rules of introduction and elimination for each connective, plus the $\mathsf{DN}$ axiom schemata, that is, for each formula $A$, $\neg\neg A \to A$.

A classical theory on $\mathcal{L}$ is any set of formulas of $\mathcal{L}$ closed under classical logical rules. In the metatheory, we will use the same rules, except $\mathsf{DN}$, that is, we will reason intuitionistically.

Now, we will develop an intuitionistic version of classical model theory. We will identify a structure $\mathcal{M}$ with the set of statements that it satisfies, so there will be no longer difference between structures and theories. We will denote "$A$ is true in $\mathcal{M}$" by $\mathcal{M} \Vdash A$, as usual. If the structure is a theory $\mathcal{T}$, then $\mathcal{T} \Vdash A$ will in fact means $T \vdash A$, that is, "$T$ derives $A$". The elements of the structure $\mathcal{M}$ will be the closed terms of the language $\mathcal{L}$, modulo Leibnitz equality, that is $a = b$ if and only if $\mathcal{M} \Vdash A[a] \Leftrightarrow \mathcal{M} \Vdash A[b]$ for every $A[x]$ of $\mathcal{L}$ with only $x$ free.

Let $\mathsf{c}$ be a first order connective and $\mathsf{c}'$ the corresponding metalinguistic connective. We say that a structure $\mathcal{M}$ is $\mathsf{c}$-*sound* if and only if

$$\mathcal{M} \Vdash \mathsf{c}(A_1, \ldots, A_n) \Rightarrow \mathsf{c}'(\mathcal{M} \Vdash A_1, \ldots, \mathcal{M} \Vdash A_n)$$

Let us spell out $\mathsf{c}$-soundness for $\mathsf{c} \equiv \to, \forall, \bot$

1. $\mathcal{M} \Vdash A \to B \Rightarrow [(\mathcal{M} \Vdash A) \Rightarrow (\mathcal{M} \Vdash B)]$
   for any $A$, $B$ closed formulas in $\mathcal{L}(\mathcal{M})$
2. $\mathcal{M} \Vdash \forall x.A \Rightarrow (\mathcal{M} \Vdash A[t])$ for all closed terms $t$
   for any $\forall x.A$ closed in $\mathcal{L}(\mathcal{M})$
3. $\mathcal{M} \Vdash \bot \Rightarrow \mathsf{False}$

A set of formulas $\mathcal{M}$ is $\to$-sound if and only if it is closed under $\to$-elimination, namely, modus ponens. Moreover, it is $\forall$-sound if and only if it is closed under $\forall$-elimination. Any classical theory is both $\to$-sound and $\forall$-sound.

We say that a structure $\mathcal{M}$ is $\mathsf{c}$-*complete* if and only if the reverse of $\mathsf{c}$-soundness holds, that is,

$$\mathcal{M} \Vdash \mathsf{c}(A_1, \ldots, A_n) \Leftarrow \mathsf{c}'(\mathcal{M} \Vdash A_1, \ldots, \mathcal{M} \Vdash A_n)$$

We can spell out what $\mathsf{c}$-completeness is for $\mathsf{c} \equiv \to, \forall, \bot$, by just reversing arrows in the previous table.

We say that $\mathcal{M}$ is a Tarski structure for $\mathsf{c}$ if $\mathcal{M}$ is both $\mathsf{c}$-sound and $\mathsf{c}$-complete, assuming that the connective $\mathsf{c}$ is interpreted by $\mathsf{c}'$ in the metalanguage. We say that $\mathcal{M}$ satisfies $\mathsf{DN}$ if and only if $\mathcal{M} \Vdash \neg\neg A \to A$ for all $A$.

Let us fix a complete set of first order connectives: $\{\forall, \to, \bot\}$. In our terminology, $\mathcal{M}$ is a Tarsky model for a first order language $\mathcal{L}$ over $\forall$, $\to$, $\bot$ if and only if $\mathcal{M}$ is a Tarski structure for $\forall$, $\to$, $\bot$, assuming these three connectives are interpreted by the corresponding ones in the metalanguage, $\mathcal{M}$ has language $\mathcal{L}$, plus at least one extra constant, and $\mathcal{M}$ satisfies $\mathsf{DN}$. In the case the metatheory is classical, DN is in fact a consequence of being a Tarski structure for $\forall$, $\rightarrow$, $\bot$. Tarski did not need the condition on DN in his definition of model.

Fix a theory $\mathcal{T}$ on the language $\mathcal{L}$. We say that a class K of structures for $\mathcal{L}$, and some extra constants, is complete for $\mathcal{T}$ if and only if for all $\mathcal{M} \in$ K, $\mathcal{M} \Vdash A \Rightarrow \mathcal{T} \vdash A$.

# 2 Proving Completeness

We are now ready to state Krivine's completeness result. We cannot prove intuitionistically the completeness of Tarksi models for $\mathcal{L}$: this fact was first shown by Gödel and a recent proof can be found in [1]. Krivine shown that:

- we can still prove completeness in the weaker form stating the "every consistent theory has a model";
- we can even prove completeness in its original form, provided we drop the third condition above, that is, provided we accept one model more, the *all-true* model $\mathcal{M}_0$.[1]

Classically, the notion of validity, that is, being true in all models, does not change by adding $\mathcal{M}_0$, because this extra model adds no extra condition on a formula since it satisfies everything. Thus, we can intuitionistically prove

$$\text{derivability} \Leftrightarrow \text{validity}$$

for a notion of validity classically equivalent, and intuitionistically very close, to the original one proposed by Tarski.

## 2.1 The intuitionistic Completeness Proof

In this section we re-organize the proof of Krivine's completeness result, in order to stress the principle it relies on. We need to define formally some well-known concepts. We first sum up some trivialities about the $\bot$ constant and negation.

**Definition 2.1 (consistency)** *Let $\mathcal{M}$ be any structure. Then*

1. *$\mathcal{M}$ is consistent if and only if $\mathcal{M} \nVdash \bot$, inconsistent if $\mathcal{M} \Vdash \bot$;*
2. *$\mathcal{M}$, $\mathcal{M}'$ are equiconsistent if and only if $\mathcal{M} \Vdash \bot \Leftrightarrow \mathcal{M}' \Vdash \bot$;*
3. *$\mathcal{M}$ satisfies intuitionistic $\bot$-rule if and only if $\mathcal{M} \Vdash \bot \Rightarrow \mathcal{M} \Vdash A$.*

Consistency, that is, $\mathcal{M} \nVdash \bot$, can be written as $(\mathcal{M} \Vdash \bot) \Rightarrow \mathsf{false}$, and it is nothing but $\bot$-soundness. On the other hand, $\bot$-completeness means $\mathsf{false} \Rightarrow \mathcal{M} \Vdash \bot$, and it always holds. Any classical theory satisfies intuitionistic $\bot$-rule, that is, if it is inconsistent then it derives all formulas.

By $\neg A$, we will mean the standard definition of $\neg A$ in our language. For instance, with the set $\{\forall, \rightarrow, \bot\}$ of connectives we fixed, we set $\neg A \equiv (A \rightarrow \bot)$.

[1] $\mathcal{M}_0$ is defined by the condition that for all closed formula $A$ of $\mathcal{L}$, $\mathcal{M}_0 \Vdash A$.

**Definition 2.2 (Meta-$\mathsf{DN}$)** *Let $\mathcal{M}$, $\mathcal{M}'$ be any structures, $A$ any closed formula. Then*

1. *$\mathcal{M}$ satisfies metalinguistic double negation for $A$ (meta-$\mathsf{DN}$ for $A$, for short) if and only if $[(\mathcal{M} \Vdash \neg A) \Rightarrow (\mathcal{M} \Vdash \bot)] \Rightarrow (\mathcal{M} \Vdash A)$.*

   *$\mathcal{M}$ satisfies meta-$\mathsf{DN}$ if and only if $\mathcal{M}$ satisfies meta-$\mathsf{DN}$ for all closed formula $A$.*

2. *$\mathcal{M}$ is complete for $A$, or equivalently, $\mathcal{M}$ decides $A$, if and only if $(\mathcal{M} \Vdash A) \vee (\mathcal{M} \Vdash \neg A)$. $\mathcal{M}$ is complete if and only if it is complete for all closed $A$.*

If $\mathcal{M} \Vdash A$, then $\mathcal{M}$ trivially satisfies meta-$\mathsf{DN}$ and completeness for $A$. Any inconsistent (all-true) structure satisfies trivially meta-$\mathsf{DN}$ and completeness.

We can think of completeness as a kind of meta-linguistic excluded middle for $\mathsf{M}$. Classically, meta-$\mathsf{DN}$ for a closed $A$, and completeness for $A$ amount to the same for any classical theory. Indeed, if $\mathcal{M}$ is inconsistent then it is an all-true structure. Thus, $\mathcal{M}$ is both meta-$\mathsf{DN}$ and complete for any closed $A$. On the other hand, if $\mathcal{M}$ is consistent, then meta-$\mathsf{DN}$ for $A$ means that if $\neg A$ does not hold in $\mathcal{M}$, then $A$ does; that is, classically, either $\neg A$ or $A$ hold in $\mathcal{M}$.

The key property in Gödel's classical completeness proof was:

**Lemma 2.3 (Gödel's Lemma)** *Let $\mathcal{T}$ be any classical theory and $A$ any closed formula. Then*

1. *there is a classical theory $\mathcal{T}'$ extending $\mathcal{T}$, complete for $A$, that is, deciding $A$, and equiconsistent with $\mathcal{T}$.*

2. *being complete for $A$ is a monotonic property, that is, if $\mathcal{T}$ is complete for $A$ and $\mathcal{T}'$ extends $\mathcal{T}$, then $\mathcal{T}'$ is complete for $A$.*

*Proof (classic)*

1. If $\mathcal{T}$ derives $\neg A$, we set $\mathcal{T}' \equiv \mathcal{T}$. Then $\mathcal{T}'$ decides $A$ and it is equiconsistent with $\mathcal{T}$. If $\mathcal{T}$ does not derive $\neg A$, we set $\mathcal{T}' \equiv \mathcal{T} \cup \{A\}$. Both $\mathcal{T}$ and $\mathcal{T}' = \mathcal{T} \cup \{A\}$ are consistent, and $\mathcal{T}'$ decides $A$.

2. Obvious

This lemma is not derivable without double negation since we cannot decide in general whether $\mathcal{T}$ derives $\neg A$. Still, for the intuitionistic counterpart of completeness, that is, meta-$\mathsf{DN}$, Gödel's lemma holds. Switching between completeness for $A$ and meta-$\mathsf{DN}$ for $A$ will be enough in order to constructivize Gödel's proof.

**Lemma 2.4 (meta-$\mathsf{DN}$ and monotonicity)** *Let $\mathcal{T}$ be any classical theory and $A$ any closed formula. Then*

1. *there is a classical theory $\mathcal{T}'$ extending $\mathcal{T}$, equiconsistent with $\mathcal{T}$ and meta-*DN *for $A$.*
2. *being meta-$\mathcal{DN}$ for $A$ is a monotonic property for equiconsistent structures.*

*Proof.*

1. Let $\mathcal{T}'$ be the classical theory with axioms set $\mathcal{T}_0$ defined by:

   $$B \in \mathcal{T}_0 \text{ if and only if } (B \in \mathcal{T}) \text{ or } (B = A \ \& \ (\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot))$$

   We have to prove that $\mathcal{T}'$ is equiconsistent with $\mathcal{T}$ and that $\mathcal{T}'$ satisfies meta-DN for $A$.

   *Equiconsistency.* By induction over proofs in $\mathcal{T}'$, we can prove that each proof in $\mathcal{T}'$ is either a proof in $\mathcal{T}$, or it is a proof in $\mathcal{T} \cup \{A\}$, in which case $(\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$ holds. Assume we have a proof of $\bot$ in $\mathcal{T}'$. Then either we have a proof of $\bot$ in $\mathcal{T}$, and we are done, or we have a proof of $\bot$ in $\mathcal{T} \cup \{A\}$, and $[(\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)]$ holds. From $\mathcal{T} \cup \{A\} \vdash \bot$ we get $\mathcal{T} \vdash \neg A$; from this latter and $(\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$ we conclude again $\mathcal{T} \vdash \bot$.

   *Meta*-DN *for $A$.* Assume $(\mathcal{T}' \vdash \neg A) \Rightarrow (\mathcal{T}' \vdash \bot)$. Then $(\mathcal{T}' \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$ by equiconsistency, and hence $(\mathcal{T} \vdash \neg A) \Rightarrow (\mathcal{T} \vdash \bot)$, since $\mathcal{T}$ included in $\mathcal{T}'$ yields that if $\mathcal{T} \vdash \neg A$ then $\mathcal{T}' \vdash \neg A$. Thus, $\mathcal{T}' = \mathcal{T} \cup \{A\}$ and $\mathcal{T}'$ derives $A$.

2. Assume $\mathcal{M}' \Vdash \bot \Leftrightarrow \mathcal{M} \Vdash \bot$. Note that the statements $\mathcal{M} \Vdash \neg A$, $\mathcal{M}' \Vdash \neg A$ and $\mathcal{M} \Vdash A$, $\mathcal{M}' \Vdash A$ are both positive subformulas in the definition of meta-DN for $\mathcal{M}$, $\mathcal{M}'$ and $A$. Thus if, for each $B$, $(\mathcal{M} \Vdash B)$ yields $(\mathcal{M}' \Vdash B)$, then meta-DN for $\mathcal{M}$ and $\mathcal{A}$ implies meta-DN for $\mathcal{M}'$ and $A$.

The rest of the proof works essentially like in Gödel, but for a metalinguistic property saying that for classical theories, being $\rightarrow$-complete is equivalent to being meta-DN. Gödel used completeness in place of meta-DN, of course.

**Lemma 2.5 (meta-DN and completeness)** *Let $\mathcal{M}$ be any classical theory. Then $\mathcal{M}$ satisfies meta-*DN *if and only if it is $\rightarrow$-complete.*

*Proof.* From left to right. Assume $[(\mathcal{M} \Vdash A) \Rightarrow (\mathcal{M} \Vdash B)]$ in order to prove $\mathcal{M} \Vdash (A \rightarrow B)$. By meta-DN, it is enough to prove $[(\mathcal{M} \Vdash \neg(A \rightarrow B) \Rightarrow (\mathcal{M} \Vdash \bot)]$, that is, the inconsistency of $\mathcal{M}$ under the extra assumption $\mathcal{M} \Vdash \neg(A \rightarrow B)$. From $\mathcal{M} \Vdash \neg(A \rightarrow B)$ and classical logic within $\mathcal{M}$, we get $\mathcal{M} \Vdash A$ and $\mathcal{M} \Vdash \neg B$. From the assumption $[(\mathcal{M} \Vdash A) \Rightarrow (\mathcal{M} \Vdash B)]$, we obtain $\mathcal{M} \Vdash B$. Eventually, from $\mathcal{M} \Vdash B$ and $\mathcal{M} \Vdash \neg B$ we get $\mathcal{M} \Vdash \bot$, as wished. From right to left. If $\mathcal{M}$ is $\rightarrow$-complete, then DN axiom schema in $\mathcal{M}$ is equivalent to meta-DN.

**Lemma 2.6 (Completeness Lemma)** *For* all *classical theories $\mathcal{T}$, there is some classical theory $\mathcal{U}$ extending $\mathcal{T}$, which is both $\rightarrow$-complete and equiconsistent with $\mathcal{T}$.*

*Proof.* Let $A_0$, $A_1$, $A_2$, ... be the list of closed formulas of $\mathcal{L}$. As Gödel did, we define a sequence $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, ... of theories, with $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{T}_n$. The theory $\mathcal{T}_{n+1}$ is chosen as a theory including $\mathcal{T}_n$, equiconsistent with it, and meta-$\mathsf{DN}$ for $A_n$. By construction, all theories $\mathcal{T}_n$ are equiconsistent each other. $\mathcal{U}$ is a classical theory because it is union of a chain of classical theories. $\mathcal{U}$, being the union of an equiconsistent family, is equiconsistent with each $\mathcal{T}_n$. For each $A_n$, $\mathcal{U}$ includes some equiconsistent theory meta-$\mathsf{DN}$ for $A_n$. By monotonicity and equiconsistency, $\mathcal{U}$ is meta-$\mathsf{DN}$ for all $A_n$. $\mathcal{U}$ is a classical theory; thus, $\mathcal{U}$ is $\rightarrow$-complete.

**Theorem 2.7 (Completeness Theorem (Krivine))** *If $\mathcal{T}$ is consistent, then there is a Tarski model $\mathcal{M}$ of $\mathcal{T}$.*

*Proof (intuitionistic).* For each $\mathcal{T}$, we can constructively define a conservative (hence equiconsistent) extension $\mathcal{H}$ of $\mathcal{T}$, having infinitely many new constants, and which is Henkin-complete, that is it contains an Henkin axiom $\neg A[c] \rightarrow \forall x.\neg A[x]$ for each universal formula in $\mathcal{L}(\mathcal{H})$. Such a (constructive) proof can be found in any textbook. By the completeness Lemma 2.6, there is some $\rightarrow$-complete extension $\mathcal{U}$ of $\mathcal{H}$ which is equiconsistent with $\mathcal{H}$ and hence with $\mathcal{T}$. $\mathcal{U}$ is still Henkin-complete since it includes all Henkin axioms in $\mathcal{H}$, and therefore ($\mathcal{U} \vdash \forall x.A$) if and only if ($\mathcal{U} \vdash A[t]$ for all closed terms $t$ in $\mathcal{L}(\mathcal{U})$). Then $\mathcal{U}$ is a Tarski model. Indeed, it is $\rightarrow$-complete by construction, $\forall$-complete because it is an Henkin theory, $\bot$-complete because $\mathsf{false} \Rightarrow \mathcal{U} \Vdash \bot$ always holds. It is a consistent classical theory, thus it is $\forall$-sound, $\rightarrow$-sound, $\bot$-sound (consistent means this), and it enjoys $\mathsf{DN}$.

## 2.2 A note on strong completeness

Strong completeness, that is, $\mathsf{K}(\mathcal{T}) \Vdash A$ if and only if $\mathcal{T} \vdash A$ ("$\mathcal{T}$ derives $A$") can be proved only if we drop the consistency assumption on models of first order logic. In this case, assume $\mathsf{K}(\mathcal{T}) \Vdash A$ in order to prove $\mathcal{T} \vdash A$ (the reverse is trivial). Let $\mathcal{T}' = \mathcal{T} \cup \{\neg A\}$. By repeating the construction of the previous theorem, there is a model $\mathcal{M}$ in $\mathsf{K}(\mathcal{T}')$ equiconsistent with $\mathcal{T}'$. $\mathcal{M}$ is a model of $\mathcal{T}'$, hence $\mathcal{M} \Vdash \neg A$. Since $\mathcal{M}$ is in $\mathsf{K}(\mathcal{T})$, too, we have also $\mathcal{M} \Vdash A$. Then $\mathcal{M} \Vdash \bot$ because $\mathcal{M}$ is a model, and therefore we have $\mathcal{T} \cup \{\neg A\} \vdash \bot$ by equiconsistency. Thus, $\mathcal{T} \vdash \neg\neg A$, and since $\mathcal{T}$ is a classical theory, $\mathcal{T} \vdash A$.

Classically, what we just proved is nothing but the usual completeness result, because validity over a class of model extended with the *all-true* model is equivalent with Tarski validity.

## 2.3 About completeness for logical connectives

The model we built is sound and complete for all connectives but disjunction. In fact, it is sound and complete for $\forall$, $\rightarrow$, $\perp$ by construction. Moreover, it is sound and complete for $\neg$ because $\neg A \equiv A \rightarrow \perp$, for & because all classical theories are and for $\exists$ because it is an Henkin model. Indeed, $\exists x.A[x]$ stays for $\neg\forall x.\neg A[x]$; if $\mathcal{M} \Vdash \exists x.A[x]$, that is, if $\mathcal{M} \Vdash \neg\forall x.\neg A[x]$, then $\mathcal{M} \Vdash A[c]$, where $c$ is the Henkin constant for the closed Henkin axiom $\neg A[c] \rightarrow \forall x.\neg A[x]$ and $\mathcal{M} \Vdash \neg\forall x.\neg A[x]$ implies $\mathcal{M} \Vdash \neg\neg A[c]$, that is, $\mathcal{M} \Vdash A[c]$.

Our model is not complete for disjunction, otherwise from $\mathcal{M} \Vdash A \vee \neg A$ we would get $\mathcal{M} \Vdash A$ or $\mathcal{M} \Vdash \neg A$. The realizer of such constructive result would define a recursive complete extension of the original consistent theory $\mathcal{T}$. When $\mathcal{T}$ is Peano Arithmetic, this is in contradiction with Gödel Incompleteness theorem.

# 3 The Ultrafilter Theorem

We will try to isolate the constructive principle which makes a constructive proof of completeness possible, with the hope of applying them to other results as well. The main principle we used seems to be that every filter over a countable boolean algebra can be extended to an ultrafilter (we applied it to a classical theory, which is a particular case of boolean algebra). This theorem is well-known, but it was considered purely classical up to now. Let us spell out the ultrafilter theorem in details.

**Definition 3.1** *Let $\mathcal{B} \equiv (B, \wedge, 1_B, \vee, 0_B, \neg)$ be any boolean algebra. Then*

1. *a filter $F$ over $\mathcal{B}$ is any non-empty subset of $B$ closed upwards and by finite intersection.*
2. *a filter $F$ is consistent if $F$ does not include $0_B$, inconsistent when it does. Note that any inconsistent filter is equal to $B$.*
3. *a filter $F$ is complete if and only if, for all $x \in B$, if $x \in F$ yields $F$ inconsistent then $\neg x \in F$.*
4. *two filters $F$ and $G$ are equiconsistent if and only if $F$ is inconsistent if and only if $G$ is inconsistent.*
5. *a filter $F$ is an ultrafilter if and only if $F$ is consistent and complete.*

All definitions are taken from classical mathematics, but for completeness for a filter $F$, which is usually stated as: for all $x$, either $x \in F$ or $\neg x \in F$. The definition we chosen is classically equivalent, but intuitionistically weaker. The fact that it is weaker is essential in order to prove constructively the ultrafilter theorem for countable boolean algebra (it cannot be proved with the original definition).

**Theorem 3.2** *Let $F$ be any filter over a countable boolean algebra $\mathcal{B}$. Then*

1. *$F$ can be extended to a complete filter $Z$ equiconsistent with $F$*
2. *In particular: if $F$ is consistent filter then it can be extended to an ultra-filter.*

*Proof (constructive).*

1. Let $x_n$ be any enumeration of elements of $B$. Define a filter chain $F_n$, by:

$$\begin{cases} F_0 & = & F \\ F_{n+1} & = & \uparrow (F_n \cup \{\neg x_n | \ (x_n \in F_n) \Rightarrow (F_n \text{ inconsistent})\}) \end{cases}$$

   where $\uparrow (X)$ denotes the filter generated from $X$. Note that the set $\{\neg x_n | \ (x_n \in F_n) \Rightarrow (F_n \text{ inconsistent})\}$ is at most a singleton. Our thesis now is that $Z \equiv \bigcup_{n \in \omega} F_n$ is complete and equiconsistent with $F$.

   *Equiconsistency.* We prove first by induction on $n$ that all $F_n$ are equiconsistent with $F$ (and hence also one each other). For $n = 0$ this follows from $F_0 = F$. Suppose $F_n$ be equiconsistent with $F$. Assume that $F_{n+1}$ is inconsistent, that is, that $(y_1 \wedge \ldots \wedge y_k) = 0_B$ for some $y_1, \ldots, y_k$ generators of $F_{n+1}$. We have to prove that $0_B$ is in $F_n$, and hence in $F$. By induction we can prove that any intersection of generators of $F_{n+1}$ is either in $F_n$, or it has the shape $y \wedge \neg x_n$ for some $y \in F_n$. In the first case, $0_B$ is in $F_n$ and we are done. In the latter case we also know that $(x_n \in F_n) \Rightarrow (F_n \text{ inconsistent})$. Now, $y \wedge \neg x_n = 0_B$ for some $y \in F_n$, therefore we obtain $y \leq \neg\neg x_n = x_n$ since $\mathcal{B}$ is a boolean algebra. But $F_n$ is a filter and thus $x_n$ is in $F_n$. Thus, by using $(x_n \in F_n) \Rightarrow (F_n \text{ inconsistent})$, we conclude again that $F_n$ is inconsistent. $Z$, being an union of equiconsistent filters, is equiconsistent with each of them.

   *Completeness.* Take any $x \in B$, and assume that $(x \in Z) \Rightarrow (Z \text{ inconsistent})$ holds in order to prove that $\neg x$ is in $Z$. By definition of $Z$, for some $n \in \omega$, we have $x = x_n$. Assume $x_n \in F_n$. Then $x$ is in $Z$, since $F_n \subseteq Z$, and hence, by $(x \in Z) \Rightarrow (Z \text{ inconsistent})$, $Z$ is inconsistent. By equiconsistency, also $F_n$ is inconsistent. We thus obtained $(x_n \in F_n) \Rightarrow (F_n \text{ inconsistent})$ and hence, by definition of $F_{n+1}$, we get $\neg x_n \in F_{n+1}$. We conclude that $\neg x_n$, that is $\neg x$, is in $Z$, as wished.

2. Let $F$ and $Z$ be as above. If $F$ is consistent, then also $Z$ is: therefore $Z$ is an ultrafilter.

## 3.1 Some reflections from a constructive viewpoint

Consider the case $F$ is consistent. The main limit of the definition of the ultrafilter $Z$, from a constructive viewpoint, is that, for all $x \in B$, $\neg x \in Z$ if and only if $x \notin Z$. This means that $Z$ is a non-informative predicate since it is equivalent to a negation. So, if we build any witness $w$ for an assumption of the form $a \in Z$, then $w$ does not depend on $a$. Thus we wonder if the ultrafilter theorem could be, from a constructive standpoint, no more than a (nice) curiosity.

Still, if we are trying to prove constructively that a filter $F$ is inconsistent, to extend it to an ultrafilter could be the right way to do it. This is how Krivine proved constructively completeness for the class of Tarski models plus the *all-true* model.

Notice that the statement "$F$ is inconsistent" means that $0_B \in F$. Thus "$F$ is inconsistent" can be an informative statement, in spite of appearances: it can be associated to a witness. In Krivine, inconsistency of the filter associated to the classical theory $\mathcal{T} \cup \{\neg A\}$ carried, in fact, a witness, in form of a syntactic object, that is, a proof of $\mathcal{T} \cup \{\neg A\} \vdash \bot$. From it, a classical first order proof of $\mathcal{T} \vdash A$ could be obtained.

# 4 The Maximal Ideal Theorem

Some results, which are in classical logic easy consequences of the ultrafilter theorem for countable boolean algebras, can be constructivized as they are. Some others require more detailed constructions. We will devote this section to an example of the first kind, the maximal ideal theorem for countable rings.

## 4.1 The Maximal Ideal Construction

Let us begin with some definitions about ideals in a ring. An ideal $I$ in a ring $\mathcal{A} \equiv (A, +, \cdot, -, 0_A, 1_A)$ is any non-empty subset closed under $0_A$, opposite, sum, and such that $x \in I$, $a \in A$ imply $ax \in I$. Let us call inconsistent any ideal $I$ of $\mathcal{A}$ including $1_A$, namely $I$ is inconsistent if and only if $I = A$. Let consistent mean "not inconsistent", that is, not including $1_A$. Let equiconsistent for two ideals $I$ and $J$ means $(1_A \in I)$ if and only if $(1_A \in J)$. Denote by $(X)$ the minimal ideal including a set X, and $(I, x)$ the minimal ideal including an ideal $I$ and $x$, that is, $(I, x) = (I \cup \{x\})$. The following is a well-known result.

**Lemma 4.1 (construction of $(X)$)** *If $X$ is any subset of the ring $\mathcal{A}$, then*

$$(X) = \{a_1y_1 + \ldots + a_ky_k |\ y_1, \ldots, y_k \in X, a_1, \ldots, a_k \in A\}$$

*Proof.* Any ideal $J$ including $X$ must include all expressions $a_1y_1 + \ldots + a_ny_k$, for any $y_1, \ldots, y_k \in X$ and $a_1, \ldots, a_k \in A$. Moreover $\{a_1y_1 + \ldots + a_ky_k |\ y_1, \ldots, y_k \in X, a_1, \ldots, a_k \in A\}$ is an ideal. In fact, it is closed under $0_A$, opposite and sum and for all $b \in A$, $b(a_1y_1 + \ldots + a_ny_k) = ba_1y_1 + \ldots + ba_ny_k$.

Call *complete for* $x$ any ideal $I$ such that if $(I, x)$ is consistent, then $x$ is in $I$. Call *complete* any ideal complete for any $x \in A$. If $M$ is complete, then "we already added to $M$ everything we could, without producing inconsistency". Any inconsistent ideal is trivially complete. Define "maximal ideal" any consistent ideal maximal by inclusion.

**Lemma 4.2 (maximality)** *An ideal $M$ is maximal if and only if it is complete and consistent.*

*Proof.* From left to right. Let $M$ be maximal. Then $M$ is consistent by definition. For any $x \in A$, $(M, x)$ includes $M$; if $(M, x)$ is consistent, then, by definition of maximality, $M = (M, x)$ and hence $x$ is in $M$. Thus $M$ is complete. From right to left. Assume $M$ is consistent and complete. For any consistent $I$ including $M$, we have to prove $I = M$, that is, $I \subseteq M$. Let $x \in I$, we have to prove that $x \in M$. $(M, x)$, being included in I, is consistent. By completeness, $x$ is in $M$, as wished.

An ideal $M$ is maximal, that is, complete and consistent, if and only if for all $x \in A$, $(M, x)$ is consistent if and only if $x \in M$. Indeed the implication from left to right is completeness and the one from right to left is consistency. Moreover we can prove that completeness is monotone.

**Lemma 4.3 (monotonicity)** *Let $I$ and $J$ be ideals, with $I \subseteq J$. Then, for any $x \in A$, if $I$ is complete for $x$ then also $J$ is.*

*Proof.* The condition "$(I, x)$ consistent" is antimonotonic because if $(J, x)$ is consistent, then $(I, x)$, being included in it, is also consistent. The condition "$x \in I$" is obviously monotone. Completeness spells out as "$(I, x)$ consistent yields $x \in I$", with the antimonotonic condition occurring negatively. Thus completeness is a monotonic condition.

**Theorem 4.4 (Maximal Ideal)** *Let $\mathcal{A}$ be any countable ring. Assume $I$ is any proper ideal of $\mathcal{A}$, that is, $1_A \notin I$. Then there is a maximal ideal $M$ in $\mathcal{A}$ such that $I \subseteq M$.*

*Proof.* We will prove that any ideal $I$ can be extended to a complete one $M$, equiconsistent with $I$. Let us define $M$ as follows. Let $x_n$ be any enumeration (with possibly repetitions) of $A$. Define $M_0 = I$. Suppose now we already defined $M_n$ and set $M_{n+1} \equiv (X_{n+1})$ where

$$X_{n+1} = \{x | \ (x \in M_n) \vee ((M_n, x_n) \text{ consistent } \& \ x = x_n)\}$$

Define $M \equiv \bigcup_{n \in \omega} M_n$.

Any element $m'$ of $M_{n+1}$ has the shape $a_1 y_1 + \ldots + a_k y_k$, with $y_1, \ldots, y_k \in X_{n+1}$. By induction over $k$, we can prove that either $m' \in M_n$, or it has the form $m + a x_n$, for some $m \in M_n$ and $a \in A$; and in the latter case $M_{n+1} = (M_n, x_n)$ and $(M_n, x_n)$ is consistent. Moreover we can prove that each $M_n$ satisfies both equiconsistency with $I$ and completeness for $x_{n-1}$.

*Equiconsistency.* Assume $1_A \in M_n$. Then either $1_A$ is in $M_n$, and we are done, or $1_A = m + a x_n$ for some $m \in M_n$ and $a \in A$, and $M_{n+1} = (M_n, x_n)$, and $(M_n, x_n)$ is consistent. But $1_A = m + a \cdot x_n$ means that $(M_n, x_n)$ is inconsistent.

*Completeness.* $M_{n+1}$ is complete for $x_n$. Assume $(M_{n+1}, x_n)$ be consistent. Then $(M_n, x_n)$ is consistent by antimonotonicity. Thus, by definition of $M_{n+1}$, we conclude $x_n \in M_{n+1}$.

By induction over $n$, we conclude that all ideals $M_n$ are equiconsistent with $I$. $M$, being the union of a chain of ideals, is also an ideal. Being the union of a family equiconsistent with $I$, it is equiconsistent with $I$. By monotonicity, it is complete over each $x_n$, because it includes $M_{n+1}$ which is complete over $x_n$.

Thus $M$ is a complete ideal equiconsistent with $I$.

### 4.2 A note about constructivism

Membership to $M$ is a negated predicate. Thus, also this construction is non-informative. We proved no more than the maximal ideal $M$ exists, and that we can use it in constructive reasoning. From the statement $x \in M$ we will get no witness. And no extra information about $M$ is available.

## References

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