Rings and categories of modules

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Abstracts of talks

Baer modules over tame hereditary algebras

Lidia Angeleri Hügel

In 1979, Ringel discovered a striking analogy between tame hereditary algebras and Dedekind domains. In particular, over a tame hereditary algebra there is a torsion theory induced by the finite dimensional regular modules which allows to speak about torsion and torsion-free modules and about Baer modules, that is, the modules B such that $\mathrm{Ext}_R(B, T) = 0$ for any torsion module T.

With Silvana Bazzoni and Dolors Herbera, we have recently shown that all Baer modules over a domain are projective. Over a tame hereditary algebra, Baer modules have a much more complex structure. They have been studied since 1980, notably by Okoh and Lukas. In my talk, I will report on recent joint work with Dolors Herbera and Jan Trlifaj. We will see that, on one hand, Baer modules over tame hereditary algebras are locally projective, in the sense that their universal localizations at tubes are projective. On the other hand, there are infinite dimensional Baer modules without indecomposable preprojective summands. Using set-theoretic methods together with infinite dimensional tilting theory, we will obtain some structure results on Baer modules, torsion-free modules and Mittag-Leffler modules.

An application of Mittag-Leffler conditions to modules of finite projective dimension

Silvana Bazzoni

We apply the relative Mittag-Leffler condition to obtain information on the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ generated by the class of modules of projective dimension at most one.

For rings with classical ring of quotients we give a sufficient condition for the finite type of the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$.

As a consequence, we show that for all commutative domains, or more generally for orders in semisimple artinian rings, and for noetherian commutative rings with total ring of quotients of Krull dimension 0, the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.

We exhibit an example of a commutative noetherian domain showing that the same result cannot be extended to the case of projective dimension 2.

This is a joint work with Dolors Herbera.

Late remarks concerning early collaborations with Kent Fuller

Robert R. Colby

The talk is mostly about things related to work on the exactness of the double dual functors that we began in 1979. That is related to 1960's results that Kent obtained about QF-3 rings so I'll be able to "bragg him up" about those results. It will be nearly all expository. I do have a new proof of an old theorem with a couple of immediate corollaries which I have not seen before.

Projective Schur algebras and radical algebras

Ángel Del Río Mateos

A projective Schur algebra over a field K is a central simple K-algebra which is generated by a group of units which is finite modulo K^*. A radical algebra over K is a crossed product $A = (L/K, \alpha)$ where L is radical over K (i.e., generated by elements which are finite modulo K^*) and the values taken by the cocycle α are finite modulo K^*. If moreover the field extension L/K is abelian then one says that A is radical abelian.

Obviously every radical algebra over K is a projective Schur algebra over K. Aljadeff and Sonn conjectured in 1995 that any projective Schur algebra over a field K is equivalent in $\mathrm{Br}(K)$ to a radical (abelian) algebra. The conjecture was known for algebras over fields of positive characteristic. In characteristic zero the conjecture was known for algebras over fields with an Henselian valuation over a local or global field of characteristic zero. We give a proof the the conjecture in full generality. As a consequence we obtain a characterization of the projective Schur group by means of Galois cohomology.

This is a joint work with Eli Aljadeff.

Looking for short complexes and small modules

Gabriella D'Este

We investigate tilting-type modules, regarded as bounded complexes of projective modules, and small "deformations" of these complexes.

Counting monogeny classes

Alberto Facchini

We describe a way of counting the monogeny classes of the direct summands of a module. We study the category obtained factoring out the ideal of the homomorphisms that factors through a module of value zero.

Decomposition theorems for generalized rings

Cristina Flaut and Mirela Stefanescu

We apply the Peirce decomposition to generalized rings as: nearrings, infranearrings, nonassociative algebras. So one can see that very few assumptions are necessary in order to get such theorems.

Every quasitilted algebra is a quasitilted ring

Enrico Gregorio

An artin algebra A is (right) *quasitilted* (Happel and Ringel) if there is a split torsion pair $(\mathcal{X}_0, \mathcal{Y}_0)$ in mod-A such that

(1) $\operatorname{pd} Y \le 1$ for all $Y \in \mathcal{Y}_0$;
(2) $A \in \mathcal{Y}_0$.

A ring is called (right) *quasitilted* (Colpi and Fuller) if there exists a split torsion pair $(\mathcal{X}, \mathcal{Y})$ in Mod-R such that

(1) $\operatorname{pd} Y \le 1$ for all $Y \in \mathcal{Y}$;
(2) $R \in \mathcal{Y}$.

It is clear that an artin algebra A which is also a quasitilted ring is a quasitilted algebra: it suffices to take $\mathcal{X}_0 = \mathcal{X} \cap \text{mod-}A$ and $\mathcal{Y}_0 = \mathcal{Y} \cap \text{mod-}A$. Colpi and Fuller asked whether the converse is also true.

We will prove that this is indeed true, using results by Crawley-Boevey and Buan and Krause.

Mittag-Leffler conditions for modules

Dolors Herbera

Raynaud and Gruson [RG] studied the right modules M over a ring R having the property that the canonical map

$$\rho\colon M \underset{R}{\otimes} \prod_{i\in I} Q_i \to \prod_{i\in I} (M \underset{R}{\otimes} Q_i)$$

is injective for any family of left R-modules $\{Q_i\}_{i\in I}$. They showed that this is the case if and only if M is the direct limit of a direct system $(F_\alpha, f_{\beta\alpha})_{\beta,\alpha\in\Lambda}$ of finitely presented modules such that the inverse system

$$(\operatorname{Hom}_R(F_\alpha, B), \operatorname{Hom}_R(f_{\beta\alpha}, B))_{\beta,\alpha\in\Lambda}$$

satisfies the Mittag-Leffler condition for any right R-module B. Therefore such modules M are said to be Mittag-Leffler modules.

We study relative versions of these properties by restricting the choice of the families of left R-modules $\{Q_i\}_{i\in I}$ to modules in a class $\mathcal{Q}$ and the choice of the right R-module B to a module in a class $\mathcal{B}$. We thus consider the notions of $\mathcal{Q}$-Mittag-Leffler module and of $\mathcal{B}$-stationary module, respectively.

We give characterizations of $\mathcal{Q}$-Mittag-Leffler modules which include and complement the ones given by Rothmaler in [R]. On the other hand, we study the $\mathcal{B}$-stationary property and its relation with the $\mathcal{Q}$-Mittag-Leffler property following closely the work of Raynaud and Gruson [RG] and the work of Zimmermann [Z].

After [BH], our main examples of $\mathcal{B}$-stationary modules appear in the context of modules Ext-orthogonal to a class of modules $\mathcal{B}$ that is closed under direct sums. This allows us to fit our results and apply them in the framework of cotorsion pairs of finite type.

As a consequence of our work, we show that every countably generated Baer module over an arbitrary commutative domain is Mittag-Leffler. This yields another proof of the fact that Baer modules over commutative domains are projective [ABH].

The results presented are based on an ongoing joint work with Lidia Angeleri-Hügel.

References

[ABH] L. Angeleri Hügel, S. Bazzoni, D. Herbera, *A solution to the Baer splitting problem*, to appear in Trans. Amer. Math. Soc.

[BH] S. Bazzoni and D. Herbera, *One dimensional tilting modules are of finite type*, to appear in Algebras and Rep. Theory.

[RG] M. Raynaud et L. Gruson, *Critères de platitude et de projectivité*, Invent. Math. **13** (1971), 1–89.

[R] P. Rothmaler, *Mittag-Leffler modules and positive atomicity*. Habilitationsschrift, Kiel, 1994.

[Z] W. Zimmermann, *Modules with chain conditions for finite matrix subgroups*, J. Algebra **190** (1997), 68–87.

Cluster tilted algebras of rank three

Otto Kerner

Let H be a connected (wild) hereditary algebra with n simple modules and T be a finite dimensional tilting H-module. The endomorphism ring of T in the cluster category $D^b(H\text{-mod})/(\tau^-[1])$, where $[1]$ denotes the shift functor in the derived category, and τ^- the Auslander-Reiten translation, is a cluster tilted algebra of rank n. One knows that all cluster tilted algebras of rank 2 are hereditary. The case $n = 3$ will be discussed.

Nakayama-Fuller rings

Surender K. Jain

Nakayama (Ann. Math. 42 (1941)) and Fuller (Pac. J. Math. 29, 1 (1969)) showed that over an artinian serial ring every module is a direct sum of uniserial quasi-injective modules. Hence artinian serial rings have the property that each right ideal is a finite direct sum of quasi-injective right ideals. A ring with the property that each right ideal is a finite direct sum of quasi-injective right ideals will be called a right Nakayama-Fuller ring. For example, commutative self-injective rings are Nakayama-Fuller rings. In this talk, various classes of these rings that include local, simple, prime, right non-singular right artinian, and right serial will be discussed. Prime right self-injective right Nakayama-Fuller rings are shown to be simple artinian. Right artinian right non-singular right Nakayama-Fuller rings are upper triangular block matrix rings over rings which are either zero rings or division rings. The Nakayama-Fuller ring is not left-right symmetric nor it is Morita invariant.

Algebras and coalgebras in monoidal categories

Claudia Menini

We would like to present some methods of dealing with algebras and coalgebras in monoidal categories giving also some meaningful examples and results.

Recent developments in the study of Butler groups

Claudia Metelli

In the following, *group* means *torsionfree Abelian group of finite rank.*

Once rank 1 groups (i.e., additive subgroups of $\mathbb{Q}$) are known, the next step is the study of their finite sums: they constitute the class of finite rank Butler groups, so named because Richard Butler, in his 1967 paper, gave two equivalent characterizations, which promised rich developments, and generated a vaste literature. Structure theorems—though—were scarce: besides the classical completely decomposable groups (direct sums of rank 1 groups, classified by Baer in 1937) the other well-researched subclass is the class of $B(1)$-groups, which originated more than a hundred papers by many Authors, in particular Arnold, Vinsonhaler, Goeters, and De Vivo, Metelli.

A Butler group is defined by its representation: G is Butler if it coincides with the sum of a finite number of its rank 1 subgroups, i.e., if it can be represented as the quotient X/K of a direct sum X of rank one groups over a pure subgroup K. Completely decomposable groups, i.e., $B(0)$-groups, are those (characterized by Baer) for which $K = 0$; $B(n)$-groups are those for which rank $K = n$. The class $B(1)$ turned out to be much more complex than initially expected: this is due to the intricate relations it hosts between linear, order-theoretic and combinatoric structures, inducing the introduction of new tools to deal with them and of new ways of representing its members.

Recently some work has started on $B(2)$-groups, and this has caused a re-thinking of the tools introduced in the $B(1)$-case, and the introduction of new tools, to be used in the general $B(n)$-case.

Prüfer modules

Claus M. Ringel

Let R be a ring. An R-module M will be said to be a Prüfer module provided there exists a locally nilpotent, surjective endomorphism of M which has kernel of finite length. In the lecture we want to outline the relevance of Prüfer modules.

Nontrivial finitely injective modules

Luigi Salce

An old construction originally performed by Paul Hill for abelian p-groups in 1969, and generalized to algebraically compact modules by B. Zimmermann-Huisgen in 1992, is adapted to divisible modules over non-Noetherian Matlis valuation domains to produce finitely injective modules which fail to be direct sums of injectives. An application to locally pure-injective torsionfree modules via the Matlis equivalence is derived. It is also proved, as a consequence of a recent result by Salce-Vamos, that all torsionfree complete modules over a domain R are locally pure-injective if and only if R is an almost maximal Prüfer domain. This implies that the semi-Dedekind domains, recently introduced by S. B. Lee, are Dedekind.

Telescope conjecture for modules and cotorsion pairs of countable type

Jan Šaroch

I shall speak about recent joint work with Jan Šťovíček concerning hereditary cotorsion pairs with the right-hand class closed under directed unions. Let such a cotorsion pair, $\mathfrak{C} = (\mathcal{A}, \mathcal{B})$, in Mod-$R$, where R is an arbitrary ring, be given. We are able to prove that $\mathfrak{C}$ is of countable type in this setting; that is $\mathcal{B} = \operatorname{Ker}\operatorname{Ext}^1_R(C, -)$ for some class C of modules possessing a projective resolution consisting of countably generated projective modules. This result immediately implies completeness of cotorsion pairs under consideration. As another consequence, using a theorem by Silvana Bazzoni and Dolors Herbera, we get also that $\mathcal{B}$ is a definable class. Assuming in addition that $\mathcal{A}$ is closed under direct limits, there is an open question—so-called Telescope conjecture for module categories—whether it is true that $\mathcal{A} = \varinjlim(\mathcal{A} \cap \text{mod-}R)$. Our result shows that to prove this conjecture, one needs to show that each countably presented module from $\mathcal{A}$ is a direct limit of finitely presented modules in $\mathcal{A}$.

Koszul duality for multigraded algebras

Haohao Wang

We discuss the BGG correspondence for polynomial algebras with a multigrading.

Walter Burgess, **The Cartan determinant conjecture since Kent Fuller's 1992 survey**
José Luis Gómez Pardo, **Some problems and results on module decompositions related to Kent Fuller's work**
Pedro Antonio Guil Asensio, **On the cotorsion envelope of a ring**
Birge Huisgen-Zimmermann, **Generic representation theory**
Idun Reiten, **Calabi-Yau categories**
Jan Trlifaj, **Abstract elementary classes of the roots of Ext**
Dan Zacharia, **Representations of Koszul algebras**