ON A FINITISTIC COTILTING-TYPE DUALITY

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ABSTRACT. Let R and S be arbitrary associative rings. Given a bimodule ${}_R W_S$, we denote by $\Delta_?$ and $\Gamma_?$ the functors $\text{Hom}_?(-, W)$ and $\text{Ext}_?^1(-, W)$, where $? = R$ or S . We say that ${}_R W_S$ is a finitistic weakly cotilting bimodule (briefly FWC) if for each module M cogenerated by W , finitely generated or homomorphic image of a finite direct sum of copies of W , $\Gamma M = 0 = \text{Ext}^2(M, W)$. We are able to describe, on a large class of finitely generated modules, the cotilting-type duality induced by a FWC-bimodule.

INTRODUCTION

Let R and S be arbitrary associative rings with unity and ${}_R W_S$ be a R - S -bimodule. We denote by Δ_R and Δ_S the contravariant functors

$$\text{Hom}_R(-, W) : R\text{-Mod} \rightarrow \text{Mod-}S \quad \text{and} \quad \text{Hom}_S(-, W) : \text{Mod-}S \rightarrow R\text{-Mod}$$

and by Γ_R and Γ_S the contravariant functors

$$\text{Ext}_R^1(-, W) : R\text{-Mod} \rightarrow \text{Mod-}S \quad \text{and} \quad \text{Ext}_S^1(-, W) : \text{Mod-}S \rightarrow R\text{-Mod}.$$

In this paper we are interested to describe a cotilting-type duality on the categories of finitely generated modules. The notion of *cotilting module* appears, in the context of finitely generated modules over artin algebras, as dual of the concept of *tilting module*, introduced in the early eighties by Brenner and Butler [2] and Happel and Ringel [9]. In 1989 Colby in [3] extended the notion of cotilting to finitely generated modules over noetherian rings which are noetherian over their endomorphism ring. Next, the notion has been once more extended and studied for arbitrary modules over arbitrary rings (see [4, 6, 8, 5, 7]).

In this paper we reconsider the original spirit of the theory concentrating on objects with finiteness conditions, but over arbitrary associative rings. The idea was inspired by reading the paper [1] of Angeleri Hügel. She studied dualities induced by a balanced *Colby bimodule* ${}_R W_S$ (see [1, section 2]), i.e. a bimodule satisfying the following conditions:

1. $\text{Ext}_?^2(X, W) = 0$ for all finitely presented modules X ,
2. $\text{Ext}_?^1(W, W) = 0$,
3. if $\text{Hom}_?(X, W) = 0 = \text{Ext}_?^1(X, W)$ and X is finitely presented, then $M = 0$,
4. ${}_R W$ and W_S are finitely generated and the functors $\Delta_?$ carry finitely generated modules to finitely generated modules,

for $? = R$ and S . For such a bimodule, the classes $\mathcal{Y}_?$ of finitely generated modules cogenerated by W coincide with the classes of finitely presented modules belonging to $\text{Ker } \Gamma_?$ (see [1, Proposition 2.6, 4.3]); the functors $\Delta_?$ induce a duality between the classes $\mathcal{Y}_?$ and the functors $\Gamma_?$ induce a duality between the classes $\mathcal{X}'_?$ of finitely presented modules in $\text{Ker } \Delta_?$ (see [1, Theorem 4.4]).

A bimodule ${}_R W_S$ is called *weakly cotilting* (see [10]) if the left R - and right S -modules cogenerated by W are contained in $\text{Ker } \Gamma_R$ and $\text{Ker } \Gamma_S$, respectively, and the injective dimension of ${}_R W$ and W_S is less than or equal to 1. We say that ${}_R W_S$ is a *finitistic weakly cotilting bimodule* (briefly FWC-bimodule) if for each module M cogenerated by W , finitely generated or homomorphic image of a finite direct sum of copies of W , we have $\Gamma M = 0 = \text{Ext}^2(M, W)$. For what we have observed before about

the classes $\mathcal{Y}_?$, any faithfully balanced Colby bimodule ${}_R W_S$ is a faithfully balanced FWC-bimodule.

Notation: we denote by Δ^2 and by Γ^2 both the compositions $\Delta_S \circ \Delta_R$ and $\Delta_R \circ \Delta_S$, and $\Gamma_S \circ \Gamma_R$ and $\Gamma_S \circ \Gamma_R$. Throughout the paper, we will write simply Δ , Γ , $\Delta\Gamma$, $\Gamma\Delta$, ... as well “module” when we are indifferently working with left R - or right S -modules.

In this paper, using an homological approach similar to [10], we define a natural notion of Γ -reflexivity (see Definition 2.5). Then we are able to prove a cotilting type duality (see Theorem 2.8) satisfied by a large class of finitely generated modules: in particular, by Proposition 2.9, if ${}_R W_S$ is a balanced FWC-bimodule, this class contains all finitely presented modules.

1. FIXING THE SETTING

Let ${}_R W_S$ be a bimodule and Δ and Γ be the contravariant functors associated. For each module M , we denote by $\delta_M : M \rightarrow \Delta^2 M$ the evaluation map. These maps define natural transformations δ between the identity functors and Δ^2 , which are the unities of the right adjoint pair (Δ_R, Δ_S) . A module M is said to be Δ -*reflexive* if δ_M is an isomorphism. Since the functors Γ_R and Γ_S are not adjoint, analogous maps to define the notion of Γ -reflexivity do not exist. To give a notion of Γ -reflexivity will be one of the main goals of the next section.

Let X be a module. Consider an exact sequence

$$0 \rightarrow K \xrightarrow{\lambda} P \xrightarrow{\varepsilon} X \rightarrow 0$$

with P projective. Applying Δ two times and considering the evaluation map δ , we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta\Gamma X & \longrightarrow & \Delta^2 K & \xrightarrow{\Delta^2(\lambda)} & \Delta^2 P \longrightarrow \text{Coker } \Delta^2(\lambda) \longrightarrow 0 \\ & & & & \delta_K \uparrow & & \delta_P \uparrow \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & X \longrightarrow 0 \end{array}$$

Therefore there exists a unique map $\xi_{P,X} : X \rightarrow \text{Coker } \Delta^2(\lambda)$ such that $\text{coker } \Delta^2(\lambda) \circ \delta_P = \xi_{P,X} \circ \varepsilon$.

Proposition 1.1. *The module $\text{Coker } \Delta^2(\lambda)$ and the map $\xi_{P,X}$ do not depend on the particular exact sequence $0 \rightarrow K \xrightarrow{\lambda} P \xrightarrow{\varepsilon} X \rightarrow 0$ considered.*

Proof. Let us consider another exact sequence $0 \rightarrow H \xrightarrow{\mu} Q \xrightarrow{\eta} X \rightarrow 0$. By the projectivity of P and Q there exist morphisms f_i and g_i such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & X \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \parallel \\ 0 & \longrightarrow & H & \xrightarrow{\mu} & Q & \xrightarrow{\eta} & X \longrightarrow 0 \\ & & \downarrow g_1 & & \downarrow g_0 & & \parallel \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & X \longrightarrow 0 \end{array}$$

commutes. Denote by C and C' the cokernels of $\Delta^2(\lambda)$ and $\Delta^2(\mu)$, applying the functor Δ two times, we obtain the following diagram, whose solid part is commutative

and with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Delta\Gamma X & \longrightarrow & \Delta^2 K & \xrightarrow{\Delta^2(\lambda)} & \Delta^2 P \xrightarrow{\partial} C \longrightarrow 0 \\
& & \parallel & & \downarrow \Delta^2(f_1) & & \downarrow \Delta^2(f_0) \\
0 & \longrightarrow & \Delta\Gamma X & \longrightarrow & \Delta^2 H & \xrightarrow{\Delta^2(\mu)} & \Delta^2 Q \xrightarrow{\partial'} C' \longrightarrow 0 \\
& & \parallel & & \downarrow \Delta^2(g_1) & & \downarrow \Delta^2(g_0) \\
0 & \longrightarrow & \Delta\Gamma X & \longrightarrow & \Delta^2 K & \xrightarrow{\Delta^2(\lambda)} & \Delta^2 P \xrightarrow{\partial} C \longrightarrow 0
\end{array}$$

φ (vertical arrow from C to C')
 ψ (vertical arrow from C to C')

The morphisms φ and ψ are the only ones making the diagram commute. We have

$$\psi \circ \varphi \circ \partial = \partial \circ \Delta^2(g_0) \circ \Delta^2(f_0) = \partial \circ \Delta^2(g_0 \circ f_0).$$

Since there exists $\sigma : P \rightarrow K$ such that $g_0 \circ f_0 - 1_P = \lambda \circ \sigma$, we have $\partial \circ \Delta^2(g_0 \circ f_0) = \partial \circ 1_{\Delta^2 P} = \partial$ and hence $\psi \circ \varphi = 1_C$. Analogously we prove $\varphi \circ \psi = 1_{C'}$ and hence C and C' are isomorphic. Moreover, considering the diagram

$$\begin{array}{ccccccc}
\Delta^2 P & \xleftarrow{\delta_P} & P & \xrightarrow{f_0} & Q & \xrightarrow{\delta_Q} & \Delta^2 Q \\
\downarrow \partial & & \downarrow \varepsilon & & \downarrow \eta & & \downarrow \partial' \\
C & \xleftarrow{\xi_{P,X}} & X & \xlongequal{\quad} & X & \xrightarrow{\xi_{Q,X}} & C' \\
& & & \searrow \varphi & & &
\end{array}$$

we have

$$\varphi \circ \xi_{P,X} \circ \varepsilon = \varphi \circ \partial \circ \delta_P = \partial' \circ \Delta^2(f_0) \circ \delta_P = \partial' \circ \delta_Q \circ f_0 = \xi_{Q,X} \circ \eta \circ f_0 = \xi_{Q,X} \circ \varepsilon$$

and hence $\varphi \circ \xi_{P,X} = \xi_{Q,X}$; analogously $\psi \circ \xi_{Q,X} = \xi_{P,X}$. ■

In the sequel the map $\xi_{P,X}$ and the module $\text{Coker } \Delta^2(\lambda)$ will be denoted simply with ξ_X and ΞM . It is easy to see that for each morphism of modules $f : M \rightarrow N$ there is a uniquely determined morphism $\Xi(f) : \Xi M \rightarrow \Xi N$ such that $\Xi(f) \circ \xi_M = \xi_N \circ f$. Therefore Ξ is a covariant functor and ξ a natural map between $\mathbf{1}$ and Ξ . The functor Ξ is not left, nor right, nor half exact, but it preserves epimorphisms. It follows easily by the construction that on the subcategory $\text{Ker } \Delta$ (resp. $\text{Ker } \Gamma$) Ξ is a subfunctor of Γ^2 (resp. Δ^2); we denote by κ (resp. ζ) the natural embedding between the restrictions to $\text{Ker } \Delta$ (resp. $\text{Ker } \Gamma$) of Ξ and Γ^2 (resp. Δ^2).

Denote by $\text{gen}(W)$ the class of modules which are homomorphic images of a finite direct sum of copies of W .

Definition 1.2. We say that a bimodule ${}_R W_S$ is *finitistic weakly cotilting* (briefly *FWC-bimodule*) if for each module M cogenerated by W , finitely generated or in $\text{gen}(W)$, we have $\Gamma M = 0 = \text{Ext}^2(M, W)$.

In the sequel of the paper ${}_R W_S$ will be always a FWC-bimodule.

Let M be a finitely generated module. Consider an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ with P projective and finitely generated. Since ΔP is a direct summand of W^n , $n \in \mathbb{N}$, $\Gamma \Delta P = 0$. In general we know nothing about $\Gamma \Delta K$.

Lemma 1.3. *Let M be a finitely generated module. Let us consider the exact sequences $0 \rightarrow H \xrightarrow{\mu} Q \xrightarrow{\eta} M \rightarrow 0$ and $0 \rightarrow K \xrightarrow{\lambda} P \xrightarrow{\varepsilon} M \rightarrow 0$ with P, Q projective and P finitely generated. Then*

1. $\Delta^2(\eta) : \Delta^2 Q \rightarrow \Delta^2 M$ is surjective;
2. $\Gamma \Delta K = 0$ if and only if, denoting by J the image of $\Delta(\mu)$, the morphism $\partial : \Delta J \rightarrow \Gamma^2 M$ is surjective.

Proof. Let us consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & H & \xrightarrow{\mu} & Q & \xrightarrow{\eta} & M & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \parallel & & \\
0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \xrightarrow{\varepsilon} & M & \longrightarrow & 0
\end{array}$$

with P, Q projective and P finitely generated. Denote by I and J the images of $\Delta(\lambda)$ and $\Delta(\mu)$. Since ΔP is a direct summand of W^n , $n \in \mathbb{N}$, the module I belongs to $\text{gen}(W)$; since $I \leq \Delta K \in \text{Cogen } W$, $\Gamma I = 0$. Let us apply the functor Δ two times.

1 follows from the commutativity of the following diagram with exact rows:

$$\begin{array}{ccccc}
\Delta^2 P & \longrightarrow & \Delta^2 M & \longrightarrow & 0 \\
\downarrow & & \parallel & & \\
\Delta^2 Q & \xrightarrow{\Delta^2(\eta)} & \Delta^2 M & \longrightarrow & \Gamma J.
\end{array}$$

2. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\Delta I & \longrightarrow & \Gamma^2 M & \longrightarrow & \Gamma \Delta K & \longrightarrow & 0 \\
\downarrow & & \parallel & & & & \\
\Delta J & \xrightarrow{\partial} & \Gamma^2 M & \longrightarrow & \Gamma \Delta H & \longrightarrow & \Gamma J \\
\downarrow & & \parallel & & & & \\
\Delta I & \longrightarrow & \Gamma^2 M & \longrightarrow & \Gamma \Delta K & \longrightarrow & 0.
\end{array}$$

Then ∂ is an epimorphism if and only if $\Gamma \Delta K = 0$. ■

We denote by $\mathcal{K}$ the class of finitely generated modules M admitting an exact sequence $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ with P projective and finitely generated, and with $\Gamma \Delta K = 0$. The class $\mathcal{K}$ is not small:

Proposition 1.4. *Each finitely presented module M and each finitely generated module N in $\text{Ker } \Gamma$ belong to $\mathcal{K}$.*

Proof. Let us consider a projective resolution

$$\dots \rightarrow P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon=d_{-1}} M \rightarrow 0$$

with P_0 and P_1 finitely generated. Denote by K_i the image of d_i and $\lambda_i : K_i \rightarrow P_i$ the morphism induced by d_i . Let us prove that $\Gamma \Delta K_0 = 0$. We indicate with I the image of $\Delta(\lambda_1)$. Since ΔP_1 is a direct summand of a finite direct sum of copies of W , $I \in \text{gen}(W)$ and it is cogenerated by W ; therefore $\text{Ext}^2(I, W) = 0$. Applying Δ to $0 \rightarrow \Delta K_0 \rightarrow \Delta P_1 \rightarrow I \rightarrow 0$ we conclude from the exactness of

$$\Gamma \Delta P_1 = 0 \rightarrow \Gamma \Delta K_0 \rightarrow \text{Ext}^2(I, W) = 0.$$

If N is finitely generated, there exists an exact sequence $0 \rightarrow H \rightarrow Q \rightarrow N \rightarrow 0$ with Q projective and finitely generated. Since $\Gamma N = 0$, applying Δ we obtain the short exact sequence

$$0 \rightarrow \Delta N \rightarrow \Delta Q \rightarrow \Delta H \rightarrow 0.$$

Therefore, since ΔQ is a direct summand of a finite direct sum of copies of W , and ΔH is cogenerated by W , we have $\Gamma \Delta H = 0$. ■

The finitely presented modules and the finitely generated modules in $\text{Ker } \Gamma$ are exactly the modules involved in the duality theorem [1, Theorem 4.4].

Proposition 1.5. *On finitely presented modules the functor Ξ is naturally isomorphic to the 0-th left derived functor $L_0\Delta^2$.*

Proof. Given a finitely presented module M , let us consider the projective resolution

$$\dots \rightarrow P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} M \rightarrow 0$$

with P_0 and P_1 finitely generated. Denoted by K_1 the kernel of d_0 and by K_0 the kernel of ε , let $\mu_0 : P_1 \rightarrow K_0$ and $\lambda_0 : K_0 \rightarrow P_0$ be the morphisms induced by d_0 . By Lemma 1.3, 1., $\Delta^2(\mu_0)$ is surjective. Then

$$(L_0\Delta^2)M = [\Delta^2 P_0]/[\text{Im } \Delta^2(d_0)] = [\Delta^2 P_0]/[\text{Im } \Delta^2(\lambda)] \cong \text{Coker } \Delta^2(\lambda). \blacksquare$$

Lemma 1.6 (Lemma 1.1, [10]). *Given the solid part of the commutative diagram*

$$\begin{array}{ccccccc} L & \xrightarrow{\psi} & M & \longrightarrow & A & \longrightarrow & 0 \\ \parallel & & \downarrow \vartheta & & \downarrow \alpha & & \\ L & \xrightarrow{\varphi} & N & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \beta & & \\ & & C & \xlongequal{\quad} & C & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns, there are unique maps α and β such that the diagram commutes. With these maps the second column is exact; moreover, if ϑ is monic, then so is α .

Proof. It follows by diagram chasing. $\blacksquare$

The following theorem describes the relationships, inside the subcategory $\mathcal{K}$, among the functors Γ^2 , Ξ and Δ^2 .

Theorem 1.7. *There exist, in the subcategory $\mathcal{K}$, natural maps α and β between the restrictions of the functors Γ^2 and Ξ and of the functors Ξ and Δ^2 , such that*

$$\begin{array}{ccccccc} & & & & 1_{\mathcal{K}} & & \\ & & & & \downarrow \xi & \searrow \delta & \\ 0 & \longrightarrow & \Gamma^2 & \xrightarrow{\alpha} & \Xi & \xrightarrow{\beta} & \Delta^2 \longrightarrow 0 \end{array}$$

is a commutative diagram with exact row of functors and natural maps. In particular, on the subcategory $\text{Ker } \Gamma_R \cap \mathcal{K}$ (resp. $\text{Ker } \Delta_R \cap \mathcal{K}$) β (resp. α) is a natural isomorphism.

Proof. Given a module $M \in \mathcal{K}$, let us consider an exact sequence $0 \rightarrow K \xrightarrow{\lambda} P \xrightarrow{\varepsilon} M \rightarrow 0$ with P projective. Denote by I the image of $\Delta(\lambda)$ and by $i : I \rightarrow \Delta K$, $p : \Delta P \rightarrow I$ the morphisms factorizing $\Delta(\lambda)$. Applying the functor Δ two times we obtain, by Lemma 1.3, after an application of Lemma 1.6, the commutative diagram

with exact rows and columns

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
0 & \longrightarrow & \Delta \Gamma M & \longrightarrow & \Delta^2 K & \xrightarrow{\Delta(i)} & \Delta I \longrightarrow \Gamma^2 M \longrightarrow 0 \\
& & \parallel & & \parallel & & \downarrow \Delta(p) \\
0 & \longrightarrow & \Delta \Gamma M & \longrightarrow & \Delta^2 K & \xrightarrow{\Delta^2(\lambda)} & \Delta^2 P \xrightarrow{\rho} \Xi M \longrightarrow 0 \\
& & & & & & \downarrow \Delta^2(\varepsilon) \\
& & & & & & \Delta^2 M = \Delta^2 M \\
& & & & & & \downarrow \\
& & & & & & 0
\end{array}$$

$\downarrow \alpha_M$ $\downarrow \beta_M$ $\downarrow$

Analogously to the proof of [10, Theorem 1.2], it is possible to prove the naturality of the maps α_M and β_M . Finally let us prove that $\beta \circ \xi = \delta$ on $\mathcal{K}$. Looking at the above diagram and at the initial exact sequence $0 \rightarrow K \xrightarrow{\lambda} P \xrightarrow{\varepsilon} M \rightarrow 0$, we have

$$\beta_M \circ \xi_M \circ \varepsilon = \beta_M \circ \rho \circ \delta_P = \Delta^2(\varepsilon) \circ \delta_P = \delta_M \circ \varepsilon.$$

Then we conclude, since ε is an epimorphism. ■

2. THE FINITISTIC COTILTING THEOREM

Lemma 2.1. *Let us consider, for a finitely generated module M , the exact sequence $0 \rightarrow \text{Rej}_W M \xrightarrow{\iota} M \xrightarrow{\pi} [M/\text{Rej}_W M] \rightarrow 0$. Then the maps $\Gamma(\iota)$ and $\Delta(\pi)$ are isomorphisms, $\Delta \text{Rej}_W M = 0$ and $\Gamma[M/\text{Rej}_W M] = 0$.*

Proof. Applying Δ to the above exact sequence, we have the long exact sequence

$$\begin{aligned}
0 &\rightarrow \Delta[M/\text{Rej}_W M] \xrightarrow{\Delta(\pi)} \Delta M \rightarrow \Delta \text{Rej}_W M \rightarrow \\
&\rightarrow \Gamma[M/\text{Rej}_W M] \rightarrow \Gamma M \xrightarrow{\Gamma(\iota)} \Gamma \text{Rej}_W M \rightarrow \text{Ext}^2([M/\text{Rej}_W M], W).
\end{aligned}$$

Since $[M/\text{Rej}_W M]$ is finitely generated and cogenerated by W , we have

$$\Gamma[M/\text{Rej}_W M] = 0 = \text{Ext}^2([M/\text{Rej}_W M], W).$$

Then $\Gamma(\iota)$ is an isomorphism. Since by construction $\Delta(\pi)$ is an isomorphism, $\text{Rej}_W M$ belongs to $\text{Ker } \Delta$. ■

Proposition 2.2. *For each M in $\mathcal{K}$, we have the following commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Rej}_W M & \xrightarrow{\iota} & M & \xrightarrow{\pi} & [M/\text{Rej}_W M] \longrightarrow 0 \\
& & \downarrow \xi_{\text{Rej}_W M} & & \downarrow \xi_M & \nearrow \Xi[M/\text{Rej}_W M] & \downarrow \delta_{[M/\text{Rej}_W M]} \\
& & \Xi \text{Rej}_W M & \xrightarrow{\Xi(\iota)} & \Xi M & \xrightarrow{\beta_{[M/\text{Rej}_W M]} \circ \Xi(\pi)} & \Delta^2[M/\text{Rej}_W M] \\
& & \downarrow \Gamma^2(\iota) \circ \kappa_{\text{Rej}_W M} & & \parallel & & \uparrow \Delta^2(\pi) \cong \\
0 & \longrightarrow & \Gamma^2 M & \xrightarrow{\alpha_M} & \Xi M & \xrightarrow{\beta_M} & \Delta^2 M \longrightarrow 0
\end{array}$$

with the first and the third rows exact, $\Xi(\iota)$ injective and $\Xi(\pi)$ surjective.

Proof. First of all, observe that by Proposition 1.4 $M/\text{Rej}_W M$ belongs to $\mathcal{K}$ and $\beta_{[M/\text{Rej}_W M]}$ is an isomorphism by Theorem 1.7 and Lemma 2.1. Next, the second row of the diagram is essentially obtained by applying Ξ to the first row;

since Ξ preserves the epimorphisms, $\Xi(\pi)$ is surjective. By Lemma 2.1, $\Delta^2(\pi)$ is an isomorphism. Clearly $\delta_{[M/\text{Rej}_W M]}$ is a monomorphism. The third row is the exact sequence obtained in Theorem 1.7. The commutativity of the squares on the top of the diagram follows by the naturality of ξ . We have to verify only that $\alpha_M \circ \Gamma^2(\iota) \circ \kappa_{\text{Rej}_W M} = \Xi(\iota)$ and $\Delta^2(\pi) \circ \beta_M = \beta_{[M/\text{Rej}_W M]} \circ \Xi(\pi)$. Let us prove the first equality; the second one is obtained in a similar way. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{\mu} & Q & \longrightarrow & \text{Rej}_W M \longrightarrow 0 \\ & & \downarrow & & \downarrow \varphi & & \downarrow \iota \\ 0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \longrightarrow & M \longrightarrow 0 \end{array}$$

where Q and P are projective, P is finitely generated and $\Gamma \Delta K = 0$. Applying Δ , by Lemma 2.1 we get the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 = \Delta \text{Rej}_W M & \longrightarrow & \Delta Q & \xrightarrow{\Delta(\mu)} & \Delta H & \longrightarrow & \Gamma \text{Rej}_W M \longrightarrow 0 \\ & & \uparrow \Delta(\varphi) & & \uparrow & & \uparrow \Gamma(\iota) \\ 0 & \longrightarrow & \Delta M & \longrightarrow & \Delta P & \xrightarrow{\Delta(\lambda)} & \Delta K \longrightarrow \Gamma M \longrightarrow 0 \end{array}$$

Denote by I the image of $\Delta(\lambda)$, by $p : \Delta P \rightarrow I$ the canonical projection and by $\rho : I \rightarrow \Delta Q$ the induced morphism such that $\rho \circ p = \Delta(\varphi)$. Applying Δ to the last diagram and Ξ to $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ we have, proceeding as in the proof of Theorem 1.7, the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta \Gamma \text{Rej}_W M & \longrightarrow & \Delta^2 H & \longrightarrow & \Delta^2 Q \longrightarrow \Xi \text{Rej}_W M \longrightarrow 0 \\ & & \downarrow \Delta \Gamma(\iota) & & \downarrow & & \downarrow \Delta(\rho) \\ 0 & \longrightarrow & \Delta \Gamma M & \longrightarrow & \Delta^2 K & \longrightarrow & \Delta I \longrightarrow \Gamma^2 M \longrightarrow 0 \\ & & & & \downarrow \Delta(p) & & \downarrow \alpha_M \\ & & & & \Delta^2 P & \longrightarrow & \Xi M \longrightarrow 0. \end{array}$$

Looking at the first and the third rows of the diagram, since $\Delta(p) \circ \Delta(\rho) = \Delta(\rho \circ p) = \Delta^2(\varphi)$, we have $\alpha_M \circ \Gamma^2(\iota) \circ \kappa_{\text{Rej}_W M} = \Xi(\iota)$. In particular, since $\Gamma^2(\iota)$ is an isomorphism and $\kappa_{\text{Rej}_W M}$ is injective, we obtain that $\Xi(\iota)$ is a monomorphism. ■

Corollary 2.3. *Let M be a module in $\mathcal{K}$. The following assertions are equivalent*

1. ξ_M is an isomorphism,
2. $\kappa_{\text{Rej}_W M}$, $\xi_{\text{Rej}_W M}$ and $\xi_{[M/\text{Rej}_W M]}$ are isomorphisms.

In such a case $\xi_{\Gamma^2 M}$ and $\xi_{\Delta^2 M}$ are isomorphisms.

Proof. 1. $\Rightarrow$ 2. Clearly $\delta_{[M/\text{Rej}_W M]}$ is an isomorphism. Considering only the first and the third rows of the diagram of Proposition 2.2, by their exactness we have that $\Gamma^2(\iota) \circ \kappa_{\text{Rej}_W M} \circ \xi_{\text{Rej}_W M}$ is an isomorphism. Since $\Gamma^2(\iota)$ is an isomorphism and $\kappa_{\text{Rej}_W M}$ is a monomorphism, $\kappa_{\text{Rej}_W M}$ and $\xi_{\text{Rej}_W M}$ are isomorphisms. Finally, since $\xi_{[M/\text{Rej}_W M]} \circ \pi = \Xi(\pi) \circ \xi_M$ is an epimorphism, also $\xi_{[M/\text{Rej}_W M]}$ is an isomorphism.

2. $\Rightarrow$ 1. Since $\xi_{[M/\text{Rej}_W M]}$ is an isomorphism, by Theorem 1.7 also $\delta_{[M/\text{Rej}_W M]}$ is an isomorphism. Considering again only the first and the third rows of the diagram of Proposition 2.2, by their exactness ξ_M is also an isomorphism. ■

Proposition 2.4. *A module M in $\mathcal{K}$ is Δ -reflexive if and only if ξ_M and β_M are isomorphisms. In such a case $\Gamma M = 0$.*

Proof. Since $\delta_M = \beta_M \circ \xi_M$ the sufficiency is clear. Suppose $\delta_M = \beta_M \circ \xi_M$ an isomorphism; looking at the diagram of Proposition 2.2, this happen if and only if

$\delta_{[M/\text{Rej}_W M]} \circ \pi$ is an isomorphism. Now, since π is surjective, $\delta_{[M/\text{Rej}_W M]} \circ \pi$ is an isomorphism if and only if both π and $\delta_{[M/\text{Rej}_W M]}$ are isomorphisms: in particular, in such a case, $\text{Rej}_W M = 0$. Then, by Theorem 1.7, $\beta_{[M/\text{Rej}_W M]} = \beta_M$ is an isomorphism and hence ξ_M is also an isomorphism. Finally, since $M \cong \Delta^2 M$, it is finitely generated and cogenerated by W : therefore $\Gamma M = 0$. ■

On the subcategory $\text{Ker } \Gamma$, the evaluation map δ is equal to the composition of the natural map ξ and the natural embedding ζ between the restrictions of Ξ and Δ^2 . In particular a module M in $\text{Ker } \Gamma$ is Δ -reflexive if and only if both ξ_M and ζ_M are isomorphisms.

Definition 2.5. We say that a module M in $\text{Ker } \Delta$ is Γ -reflexive if and only if ξ_M and κ_M are isomorphisms.

Proposition 2.6. *A module M in $\mathcal{K}$ belongs to $\text{Ker } \Delta$ and it is Γ -reflexive if and only if ξ_M and α_M are isomorphisms.*

Proof. It is an easy consequence of Proposition 2.2. ■

Let us denote by $\mathcal{M}$ the classes of left R - and right S - modules in $\mathcal{K}$ such that ξ_M is an isomorphism.

For each module M in $\mathcal{M}$ we define a morphism $\gamma_M : \Gamma^2 M \rightarrow M$, setting $\gamma_M = \xi_M^{-1} \circ \alpha_M$. It is a natural transformation between Γ^2 and the identity functor restricted to $\mathcal{M}$. The module M is Γ -reflexive if and only if γ_M is an isomorphism.

Lemma 2.7. *Let M be a module in $\mathcal{M}$. If $\Delta \Gamma M = 0$, then $\Gamma(\alpha_M)$ is an isomorphism and*

$$\Gamma(\xi_M) \circ [\Gamma(\alpha_M)]^{-1} \circ \kappa_{\Gamma M} \circ \xi_{\Gamma M} = 1_{\Gamma M}.$$

Proof. Since $M \cong \Xi M$ are finitely generated, $\Delta^2 M$ is finitely generated and cogenerated by W : thus, since $\Gamma \Delta^2 M = 0$, $\Gamma(\alpha_M)$ is an isomorphism. Next, consider a short exact sequence $0 \rightarrow K \xrightarrow{i} P \xrightarrow{p} M \rightarrow 0$ with P finitely generated and projective, and $\Gamma \Delta K = 0$. Denote by I the image of $\Delta(i)$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{i} & P & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow \delta_K & & \downarrow \delta_P & & \downarrow \xi_M & & \\ 0 & \longrightarrow & \Delta^2 K & \longrightarrow & \Delta^2 P & \longrightarrow & \Xi M & \longrightarrow & 0 \\ & & \parallel & & \uparrow & & \uparrow \alpha_M & & \\ 0 & \longrightarrow & \Delta^2 K & \longrightarrow & \Delta I & \longrightarrow & \Gamma^2 M & \longrightarrow & 0. \end{array}$$

Applying Δ to it and Ξ to $\Delta K \xrightarrow{\partial_1} \Gamma M \rightarrow 0$ we get the following diagram

$$\begin{array}{ccc}
 & \Delta K & \xrightarrow{\partial_1} \Gamma M \\
 \Delta(\delta_K) \uparrow & & \uparrow \Gamma(\xi_M) \\
 & \Delta^3 K & \xrightarrow{\partial_2} \Gamma \Xi M \\
 \parallel & & \downarrow \Gamma(\alpha_M) \\
 & \Delta^3 K & \xrightarrow{\partial_3} \Gamma^3 M \\
 \zeta_{\Delta K} \uparrow & & \uparrow \kappa_{\Gamma M} \\
 \Xi \Delta K & \xrightarrow{\Xi(\partial_1)} & \Xi \Gamma M \\
 \xi_{\Delta K} \uparrow & & \uparrow \xi_{\Gamma M} \\
 \Delta K & \xrightarrow{\partial_1} & \Gamma M
 \end{array}$$

(#)

Its solid part is commutative; let us prove that the whole diagram is commutative. Given an exact sequence $0 \rightarrow H_1 \rightarrow Q \xrightarrow{q} \Delta K \rightarrow 0$ with Q projective, we can construct the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_1 & \longrightarrow & H_2 & \longrightarrow & I \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H_1 & \longrightarrow & Q & \xrightarrow{q} & \Delta K \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \partial_1 \\
 0 & \longrightarrow & H_2 & \longrightarrow & Q & \longrightarrow & \Gamma M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & I & \longrightarrow & \Delta K & \xrightarrow{\partial_1} & \Gamma M \longrightarrow 0
 \end{array}$$

where $H_2 = q^{-1}(I)$. Since $I \in \text{gen}(W)$ and it is cogenerated by W , $\Gamma I = 0$. We have also $\Delta \Gamma^2 M = 0$: indeed, looking at the diagram of Proposition 2.2, since ξ_M is an isomorphism we have that $\delta_{M/\text{Rej}_W M}$ is an isomorphism; then, since $\Gamma^2(\iota)$ is an isomorphism, by diagram chasing we get that $\xi_{\text{Rej}_W M}$ and $\kappa_{\text{Rej}_W M}$ are isomorphisms, and hence $\Gamma^2 M \cong \Gamma^2 \text{Rej}_W M \cong \text{Rej}_W M$ belong to $\text{Ker } \Delta$. Therefore, applying Δ two times we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Delta^2 H_1 & \longrightarrow & \Delta^2 H_2 & \longrightarrow & \Delta^2 I \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Delta^2 H_1 & \longrightarrow & \Delta^2 Q & \longrightarrow & \Delta^3 K \\
 & & \downarrow & & \parallel & & \downarrow \zeta_{\Delta K} \\
 & & & & & & \Xi \Delta K \\
 & & & & & & \downarrow \text{dotted} \\
 & & & & & & \Xi \Gamma M \\
 & & & & \nearrow & & \searrow \kappa_{\Gamma M} \\
 0 & \longrightarrow & \Delta^2 H_2 & \longrightarrow & \Delta^2 Q & \longrightarrow & \Gamma^3 M \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & \Delta^2 I & \longrightarrow & \Delta^3 K & \xrightarrow{\partial_3} & \Gamma^3 M
 \end{array}$$

The dotted arrow $\Xi\Delta K \dashrightarrow \Xi\Gamma M$ is the only one such that the diagram commutes. On one side it is, by construction, $\Xi(\partial_1)$; on the other side, looking at the commutative right bottom square, it must be the restriction of ∂_3 to $\Xi\Delta K$ and $\Xi\Gamma M$. Therefore, the whole diagram $(\#)$ commutes. Now the requested identity follows by

$$\begin{aligned} \Gamma(\xi_M) \circ [\Gamma(\alpha_M)]^{-1} \circ \kappa_{\Gamma M} \circ \xi_{\Gamma M} \circ \partial_1 &= \Gamma(\xi_M) \circ [\Gamma(\alpha_M)]^{-1} \circ \kappa_{\Gamma M} \circ \Xi(\partial_1) \circ \xi_{\Delta K} = \\ &= \Gamma(\xi_M) \circ [\Gamma(\alpha_M)]^{-1} \circ \partial_3 \circ \zeta_{\Delta K} \circ \xi_{\Delta K} = \Gamma(\xi_M) \circ \partial_2 \circ \zeta_{\Delta K} \circ \xi_{\Delta K} = \\ &= \partial_1 \circ \Delta(\delta_K) \circ \zeta_{\Delta K} \circ \xi_{\Delta K} = \partial_1 \circ \Delta(\delta_K) \circ \delta_{\Delta K} = \partial_1 \end{aligned}$$

and the fact that ∂_1 is epic. ■

Theorem 2.8 (Cotilting Theorem). *For each module $M \in \mathcal{M}$ the sequence*

$$(*) \quad 0 \rightarrow \Gamma^2 M \xrightarrow{\gamma_M} M \xrightarrow{\delta_M} \Delta^2 M \rightarrow 0$$

is exact, ΔM and $\Delta^2 M$ are Δ -reflexive and $\Gamma^2 M$ is Γ -reflexive. If $\Delta\Gamma M = 0$, then also ΓM is Γ -reflexive.

Proof. Since $\xi_M : M \rightarrow \Xi M$ is an isomorphism, and $\beta_M \circ \xi_M = \delta_M$ and $\xi_M^{-1} \circ \alpha_M = \gamma_M$, we obtain the sequence $(*)$ from Theorem 1.7. Looking at Proposition 2.2, we have that $\delta_{[M/\text{Rej}_W M]}$ is an isomorphism and hence $\Delta^2 M \cong [M/\text{Rej}_W M]$ is Δ -reflexive. Analogously, as observed in the proof of Lemma 2.7, we get that $\Gamma^2 M \cong \text{Rej}_W M$ is Γ -reflexive. In particular $\Delta\Gamma^2 M = 0$; therefore, applying Δ to the exact sequence $(*)$, we obtain that $\Delta(\delta_M)$ is an isomorphism. Since the natural maps δ are the unities of the adjunction between the functors Δ_R and Δ_S , we have that $\delta_{\Delta M}$ is an isomorphism and hence ΔM is Δ -reflexive. Finally, suppose $\Delta\Gamma M = 0$. By Lemma 2.7, $\kappa_{\Gamma M} \circ \xi_{\Gamma M}$ is an isomorphism. Since $\kappa_{\Gamma M}$ is injective, both $\kappa_{\Gamma M}$ and $\xi_{\Gamma M}$ are isomorphisms. ■

Proposition 2.9. *If the FWC-bimodule ${}_R W_S$ is balanced, then the finitely presented modules belong to $\mathcal{M} \cap \text{Ker}(\Delta\Gamma)$; a finitely generated module M , such that $\Gamma M = 0$, belongs to $\mathcal{M} \cap \text{Ker}(\Delta\Gamma)$ if and only if it is cogenerated by W .*

Proof. Let M be a finitely presented module. In Proposition 1.4 we have seen that any finitely presented module M belongs to $\mathcal{K}$. We have to prove that ξ_M is an isomorphism and that $\Delta\Gamma M = 0$. Consider an exact sequence $0 \rightarrow K \xrightarrow{\lambda} P \rightarrow M \rightarrow 0$ with P projective and finitely generated and $\Gamma\Delta K = 0$. Since M is finitely presented, K is finitely generated. Since the rings R and S are Δ -reflexive, δ_K is surjective and the module P is Δ -reflexive. Therefore, by the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\lambda} & P & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \delta_K & & \downarrow \cong & & \downarrow \xi_M \\ & & \Delta^2 K & \xrightarrow{\Delta^2(\lambda)} & \Delta^2 P & \longrightarrow & \Xi M \longrightarrow 0 \end{array}$$

we obtain that ξ_M and δ_K are isomorphisms, and $\Delta^2(\lambda)$ is injective. Applying Δ to $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ we have exact sequences

$$0 \rightarrow \Delta M \rightarrow \Delta P \rightarrow C \rightarrow 0 \quad \text{and} \quad 0 \rightarrow C \rightarrow \Delta K \rightarrow \Gamma M \rightarrow 0;$$

and applying Δ again we obtain exact sequences

$$0 \rightarrow \Delta C \xrightarrow{\varphi} \Delta^2 P \rightarrow \Delta^2 M \quad \text{and} \quad 0 \rightarrow \Delta\Gamma M \rightarrow \Delta^2 K \xrightarrow{\psi} \Delta C.$$

Since $\varphi \circ \psi = \Delta^2(\lambda)$, we have $\Delta\Gamma M = 0$.

If M is finitely generated, then M is an homomorphic image of a Δ -reflexive module and hence δ_M is surjective. If M is also cogenerated by W , then δ_M is an isomorphism. Since $\Gamma M = 0$, by Proposition 1.4 M belongs to $\mathcal{K}$: therefore, by Proposition 2.4, ξ_M is an isomorphism. Conversely, if M is finitely generated and $\Gamma M = 0$ and ξ_M is an

isomorphism, by Proposition 2.2 we have $\text{Rej}_W M = 0$ and hence M is cogenerated by W . ■

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